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MECHANICAL SYSTEMS CALCULATIONS OF THE MARINE FACILITIES

Textbook

The textbook is recommended by the Far Eastern Regional Methodology Centre of Russian Ministry of Education for multilevel training within the group 26.00.00 «Engineering and technology of shipbuilding and water transport» for training bachelors, masters and engineers, for the postgraduates, as well for the retraining and skills improving of the professional directions: 26.04.02 «Shipbuilding, ocean engineering and marine infrastructure systems engineering», 26.05.01 «Design and construction of ships and ocean engineering objects» and 26.05.05 «Navigation» as the additional material for training graduate students, as well as for advanced training of considered issues of educated experts

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The textbook Mechanical systems calculations of the marine facilities is a methodological material for the training at universities of engineers involved in the creation and maintenance of mechanical systems of marine structures and their elements, including the construction of ships, the organization of the manufacture of parts, assemblies, sections and blocks of ships under construction, their devices and mechanisms, systems and elements as part of structures, devices and mechanisms used in shipbuilding and ship repair (frames, trusses, beams, shafts), including structures and equipment in shipyards, as well as including the creation of structures used in industrial sectors related to the development of sea resources. The material of the textbook is organized in a methodological sequence convenient for the study of tasks characteristic for considering area in the programs of mechanics and theory of elasticity. The book also includes the theoretical sections used in solving these problems. The preliminary knowledge of mathematics and physics (the mechanics), corresponding to the initial university courses is required to study the material.

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Introduction

The material in the book is organized in a methodological sequence of characteristic issues in the solid mechanics and elasticity of systems and their elements widely used in marine engineering, as part of the structures of sea ships, of the shipbuilding enterprises equipment, as well as of the hydraulic structures, the other structures, devices and mechanisms used for the development of resources of the reservoirs, seas and their shores, etc. The textbook includes the exercises and theoretical sections used in solving of these problems.

The book discusses the mechanical characteristics and patterns, methods, sequences and general algorithms for calculating mechanical systems and their elements, relevant for the professional training in mechanical issues, in particular the following thematic sections for specialists involved in the technology of ships constructing, of the marine engineering equipment and the machinery of the ships, of the offshore rigs, of shipbuilding plants and yards, as well as those carrying out maintenance of these objects:

1. Forces, moments of forces, the systems of forces, the pairs of forces, the coordinate systems, the balance of a system, the mechanical systems in marine structures, connections in mechanical systems.

2. Static calculations of elements of the mechanical systems (the frames and trusses), finding of the reactions and creating the force diagrams, the three–moment method, the moving loads and lines of influences.

3. Static indefinability of mechanical systems. The canonical equations of the force method and the displacements one used to determine the reactions under static indefinability.

4. Using the computer algebra system Mathcad and the Microsoft EXCEL program for mathematical operations on matrices and vectors, to determine the displacements of nodes of mechanical system, using the Mohr's integral, and to find the reactions, forces and displacements using the finite element method.

5. The mechanical sustainability of compressed elastic elements of the marine structures, the influence of material plasticity on the sustainability of system.

6. The shear and torsion of the mechanical system elements.

7. Stress–strain state of the elastic elements of mechanical systems, the hypotheses of plasticity and destruction of elements, the main areas and main stresses. The use of the plasticity and destruction concepts to assess the strength of the mechanical system or the body.

8. The methods for specifying the motion of point and kinematics of elements in mechanical systems, the coordinate systems, the curvature of the motion trajectory, the instantaneous centers of the velocities and accelerations.

9. Forces acting on elements during the movement of mechanical system (the mechanism) and the mechanical patterns of motion under the influence of forces. The momentum, moment of momentum, the kinetic energy.

10. The vibrations of mechanical systems and their elements under the influence of external forces. The impact of the mechanical system weight, inertia and elasticity on th vibration.

11. The patterns of ship rolling and motion, the characteristics of sustainability of ship motion in water and of it controllability.

The mechanical properties of structural elements used in industries related to the development of marine resources, including the shipbuilding, are definitely universal, relating to systems of different types and their elements, along with the special properties inherent in them, such as their functional performances, for example, the suitability of structures and mechanical systems for storing or transporting of dry or liquid substances, transporting goods, the ability of the structures and systems to withstand the external environment and its temperature influences, creating certain conditions in enclosed spaces, the functionality of mechanical systems and their elements, the manufacturability of the systems and elements construction and suitability for maintenance and recycling, the controllability of mechanical systems during safe and efficient operation, etc.

Despite the specified availability of the special performances of mechanical systems and their elements – the functional and technological properties that characterize the ability of these systems to be used as part of structures and equipment in industries related to the development of marine resources, including the shipbuilding, ship repair, and industrial development of resources of the seas and coasts, i.e. those special performances that are studied, defined and assessed by the appropriate approaches outside the framework of mechanics, these systems and elements also have the universal mechanical properties and the elastic ones, which methods of the assessment are discussed in this book in the sequence and volume, according to the opinion of authors, that contribute to the understanding of the methodological material, and also allow them to streamline the learning process and to help students navigate the subject, assimilate concepts, characteristics and methods of calculating of mechanical systems, including in the process of independent study of the book subject in the sequence of sections included in it with the sufficient initial preparation, if students also have access to other methodological works and professional publications, and also have the access to the information sources and the ability to use the necessary programs in some of the issues considered.

The object in each section of the book is, to a certain extent, an abstract mechanical system consisting of elements, the properties of which are studied and the characteristics are determined from the standpoint of mechanics (in relation to a rigid system) or from the point of view of the theory of elasticity, and the choice of object in each of those

discussed in the book exercises is associated with assessing the extent to which the selected object allows one to reveal the applied approach or method of mechanics and theory of elasticity.

The book covers, in the appropriate sequence, both the methodological material necessary to solve the problems set out in the book, and modern methods of calculating of mechanical systems and their elements, including the methods of matrix and vector algebra using the EXCEL program in the standard Microsoft Office application package and using the Mathcad computer algebra system, as well as numerical methods, including the finite element one, which is widely used in calculating elements of mechanical systems. These methods are shown in the process of their use in calculating of elements of mechanical systems that can be used as part of structures and equipment in industries related to the development of marine resources, including the shipbuilding, ship repairing, and industrial development of sea and shelf resources.

The recommended choice of universal programs for calculating mechanical systems and their elements and for mathematical operations with matrices and vectors, for example, using the finite element method, or when calculating statically indefinable mechanical systems, as well as for solving canonical equations, in particular, the choice of the EXCEL program and computer algebra system Mathcad is associated both with their general availability and with exercise of these programs when used for these purposes.

It would be mentioned that there are professional programs for use in mechanical calculations of certain projects of offshore structures, transmitted in the form of executable modules as products of their developers, often with simple instructions for use and a visual presentation of the results for offshore structures of these projects, including graphical presentations. At the same time, the other programs, transmitted in the form of executable files closed to users, i.e. in the form of a «black box», usually have a narrow purpose, and they could not be used if are not intended directly for marine structures of other projects, as well as for the independent development of algorithms and mathematical models for calculating the elements of any mechanical systems, except for the structures of marine equipment of that project, for which such a program was developed. Created by their developers as a product, these professional programs for use in the calculations of certain projects of marine structures would be of little use when teaching methods of calculation of mechanical systems, except when studying the instructions for these programs themselves: entering data and documenting the results – without independently forming a sequence for calculating one or another mechanical system, its algorithm and mathematical model, which is the subject of this book. The narrow focus of programs transmitted in the form of executable files (.exe, etc.) created for specific projects, and the closed nature of the algorithms when transferring such programs to their users, associated both with the commercial interests of the developers and with the assumption that users may be

insufficiently qualified, which in this case would only have access to data entry and documentation of the results, also serve as a prerequisite for recommendations to use the EXCEL program and the Mathcad computer algebra system to apply, on their basis, the methods for calculating mechanical systems and their elements, to develop algorithms and the models for such calculations, in particular, for operations with vectors and matrices when solving the systems of equations.

The material in the book refers to the second of the following two alternatives in the areas of research of mechanical systems and their elements as part of structures and equipment in industries related to the development of marine resources, including shipbuilding, ship repair, and industrial development of sea and coastal resources:

(a) the improvement of the theoretical foundations of mechanics, strength of materials and the theory of elasticity, including the mathematical research and development of professional programs for solving narrow problems of mechanics – for certain structural projects, research of the mechanical properties of new materials and structures, for example, the study of nonlinear elastic or composite materials, creation and development of experimental methods for these studies, etc.;

(b) creation of methodological works that can be used in the process of studying mechanics in specialized departments of universities, as well as in industry organizations in the development of structures, including the use of methods intended for calculations of mechanical systems and their elements, which in terms of professional training complements the above alternative direction of work related to improving the theoretical foundations of mechanics, strength of materials and elasticity theory.

To find the characteristics of mechanical systems and their elements when performing the exercises discussed in the book, the external load is assumed to be given, i.e. the subject is the assessment of the internal stress–strain state of a mechanical system, except of certain tasks considered in the book, for example, if external forces arise from the action of pressure (from water or load), or from the action of a temperature difference, during thermal deformation of the elements of a mechanical system. Physical problems of determining external load (the forces, volume or three–dimensional flow rate, productivity, temperature, amplitude of acting forces, etc.) are usually found in other subject areas and can serve as independent problems for research. If there are analogues (the functional or structural) of mechanical systems and their elements as part of structures and equipment in industries related to the development of marine resources, including shipbuilding, ship repair, and industrial development of sea and coastal resources, the use of these analogues can serve as a practically convenient formal method of finding the external load for the calculated offshore structures. By analogy, the external load can be found by selection, using algorithmic models created for the structures being calculated using the methods discussed in the book. With this approach, in order to determine the external load

withstood by the analogue, which is subsequently used to find the forces, moments, stresses, deformations and displacements for the calculated offshore structure, it is assumed that the calculated mechanical system and its elements will be equal in strength to the analogues, based on the data of which the withstand–able external load was found.

Unlike textbooks by famous authors on mechanics and elasticity theory, this book is aimed at developing practical calculation skills within the framework of modern mathematical modeling capabilities, for which theoretical sections are also included to the required methodological extent. In addition, the calculation methods used in mechanics and the theory of elasticity are shown, based on mathematical operations with matrices and vectors, for example, the finite element method in the calculation of deformations and stresses in structures and in their discrete models, which is carried out on the basis of the available programs EXCEL and Mathcad. It can be said that the development of mechanics, the theory of elasticity and the mathematical apparatus used, which predetermines the teaching methods in these programs, in the post–industrial period, in contrast to the previous stages, is characterized not so much by new information about the laws of mechanics and the properties of elastic materials in the composition of mechanical systems and their elements, but today it could be considered generally known [1–8, 10, 11, 21, 23, 25 and 27] how many new possibilities for performing calculations and mathematical modeling within the framework of certain algorithms for solving problems and processing the digital information, which creates the appearance and direction of improving modern methods of mechanics and serves as the basis for new methodological books, which are necessary, in particular, for specialists in the field of shipbuilding and marine technology. These scientific motives guided the authors when forming the material of this book.

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Chapter 1. Forces, moments of forces, force systems, force pairs, coordinate systems, system equilibrium, mechanical systems in marine structures, connections in mechanical systems

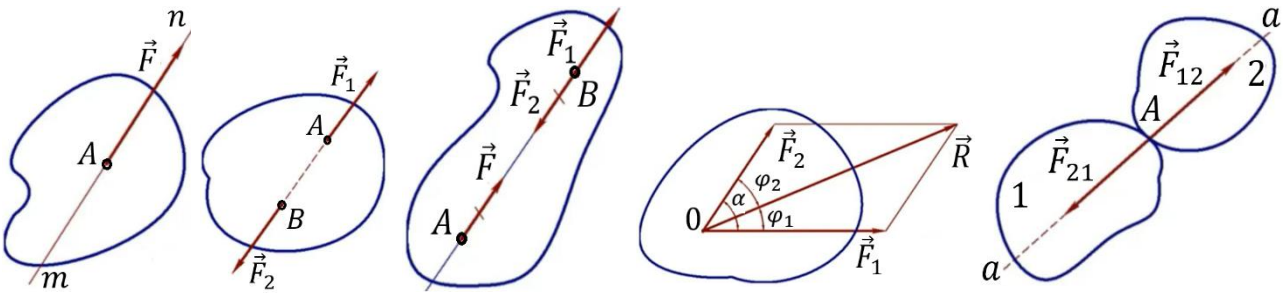
1.1. Mechanical bodies, the action of forces and the interaction of bodies, axioms of the properties of force vectors

The main issues of statics: the issue of reducing complex systems to a simpler form and the issue of the equilibrium conditions for various systems of forces.

The concepts of objects in theoretical mechanics are: a material point, an absolutely rigid body, a system of bodies. An absolutely rigid body can be free or non–free.

The force (Fig. 1.1.1 a) is the measure of the mechanical interaction of bodies. The force is a vector quantity, characterized by magnitude, direction and point of application. The unit of force in the International System of Units (SI) – Newton (N=kg×m/s^2).

Types of forces are: the concentrated and distributed, the active forces and reactions of connections, the external and internal forces, the constant forces and variable ones.



(a) (b) (c) (d) (e)

Fig. 1.1.1. The actions of forces on the mechanic bodies

The system of forces is the set of forces applied to a body $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$.

The systems of forces are: spatial, flat, convergent, parallel.

The resultant of forces' system is: $\vec{R} \sim (\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$.

Equivalent force systems (replaced by the resultant one):

$(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n) \sim (\vec{P}_1, \vec{P}_2, \dots, \vec{P}_k)$.

The balanced system of forces is when: $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n) \sim 0$.

Axiom 1. The axiom about the balance of a system of two forces. For the balance a system of two forces (Fig. 1.1.1 b) applied to points of a rigid body, it is necessary and sufficient that these forces be equal in magnitude and act along one straight line passing through the points of their application in the opposite directions:

$\vec{F}_1 = -\vec{F}_2$, $\vec{F}_1 + \vec{F}_2 = 0$ and $(\vec{F}_1, \vec{F}_2) \sim 0$.

Two forces are denominated directly opposite under this condition.

The corollary 1. Under the influence of a single force, a body cannot be in balance.

Axiom 2. If a rigid body is acted upon by the system of forces (Fig. 1.1.1 c), then when another system of forces equivalent to zero is added (discarded) to this system, the resulting new system of forces is equivalent to the initially given system of forces.

The corollary 2. The force is a vector sliding along the line of its action.

If: $(\vec{F}_1, \vec{F}_2) \sim 0$, and $\vec{F}_1 = -\vec{F}_2 = \vec{F}$, then $\vec{F} \sim (\vec{F}, \vec{F}_1, \vec{F}_2) \sim \vec{F}_1$.

A force acting on an absolutely rigid body can be transferred along the line of action without changing the effect of the action on the rigid body.

Axiom 3. The parallelogram of forces. Two forces applied at one point of a rigid body can be replaced by single resultant force (Fig. 1.1.1 d) equal in magnitude and direction to the diagonal of a parallelogram built on the given forces: $\vec{R} = \vec{F}_1 + \vec{F}_2$.

The opposite is also true. One force can be decomposed into components using the parallelogram rule.

The magnitude of the resultant $\vec{R}$ (the parallelogram diagonal) taking into account the angles between the forces is equal to: $R = \sqrt{F_1^2 + F_2^2 + 2 \times F_1 \times F_2 \times \cos\alpha}$,
where: $\alpha = \varphi_1 + \varphi_2$ – the angle between the force vectors in the parallelogram, equal to the sum of the angles between each of the force vectors: $\vec{F}_1$ and $\vec{F}_2$ and their resultant $\vec{R}$, wherein: $\frac{F_1}{\sin\varphi_2} = \frac{F_2}{\sin\varphi_1} = \frac{R}{\sin\alpha}$.

Axiom 4. On the equality of action and reaction forces (Newton's third law of mechanics). Any force of action has an equal and opposite force of reaction (Fig. 1.1.1 e).

Axiom 5. Axiom of the «solidification». The equilibrium of a deformable body (for example, the equilibrium of a weightless thread loaded with forces at the ends) will not be disrupted if the condition of transforming it into an absolutely rigid body is imposed on it.

Questions and comments:

1.1.1. Models of objects in the mechanics, the force – the magnitude of the mechanical interaction of bodies, the types of forces.

1.1.2. The system of forces. The resultant of system of the forces.

1.1.3. Equivalent system of the forces.

1.1.4. Balanced system of the forces.

1.1.5. Axiom on the balance of the system of forces.

1.1.6. Axiom on the property of force as a vector sliding along the line of it action.

1.1.7. Axiom on vector addition of forces according to the parallelogram rule.

1.1.8. The magnitude of the resultant considering the angle between the lines of acting forces.

1.1.9. Axiom on the equality of action and reaction forces (the Newton's third law of the mechanics).

1.1.10. Mechanical equilibrium of a deformable body and the condition of it «solidification».

1.2. Finding force vector projections

The geometric conditions for finding projections of the force vector.

1. Determination of force vector projections when the force vector and axis belong to the same plane (Fig. 1.2.1).

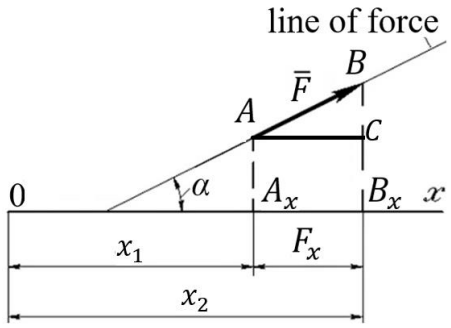


Fig. 1.2.1

The projection of the force vector onto the axis corresponds to the difference between the coordinates of its end and beginning: $F_x = x_2 - x_1$, $F_x > 0$ if $x_2 > x_1$, $F_x < 0$ if $x_2 < x_1$ and $F_x = 0$ if $x_2 = x_1$.

The triangle ABC is right angled, its leg is equal to: $AC = F_x$, $F_x = F \times \cos\alpha$, $F_x > 0$ if $0 < \alpha < 90^0$, $F_x < 0$ if $90^0 < \alpha < 180^0$ and $F_x = 0$ if $\alpha = 90^0$.

An example of determining the projections of forces on a plane (Fig. 1.2.2):

$F_{1X} = F_1 \times \cos\alpha$, $F_{2X} = F_2$, $F_{3X} = F_3 \times \cos\beta$,

$F_{3X} = -F_3 \times \cos\varphi$, $F_{5X} = -F_5$, $F_{4X} = F_{6X}$

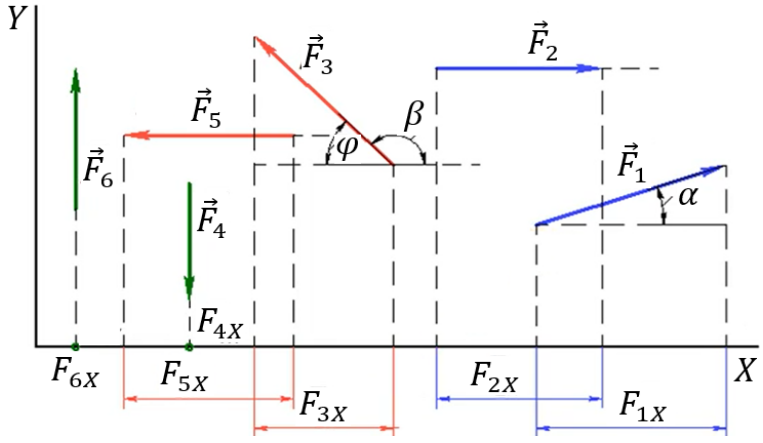


Fig. 1.2.2

2. Determination of projections of the force vector on the axes of the Cartesian coordinate system when the force vector does not belong to any of the coordinate planes.

Decomposition of the force vector into projections: $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$ and $\vec{F} = F_x \times \vec{\imath} + F_y \times \vec{\jmath} + F_z \times \vec{k}$.

The modulus of each projection of force onto the axis (Fig. 1.2.3) is determined as: $F_x = F \times \cos\alpha$, $F_y = F \times \cos\beta$ and $F_z = F \times \cos\gamma$, where α, β and γ – the angles between the force vector and coordinate axes $0XYZ$.

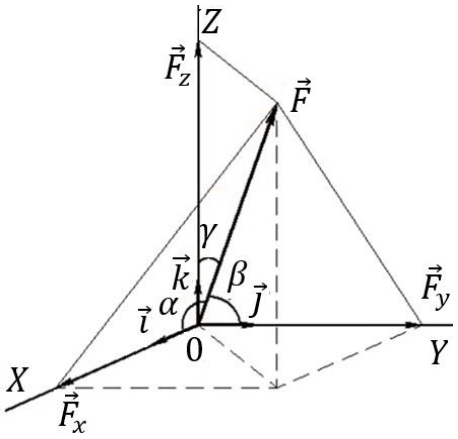


Fig. 1.2.3

Of the three angles between the force vector and the coordinate axes, only two are independent, because: $F_x^2 + F_y^2 + F_z^2 = F^2$, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, therefore it is convenient to use double projection (two vector angles, Fig. 1.2.4), for example, such: φ – the angle between the vector and one of the coordinate planes, for example: $X0Y$, ψ – the angle between the projection of force in this plane and one of the axes, for example: $0X$.

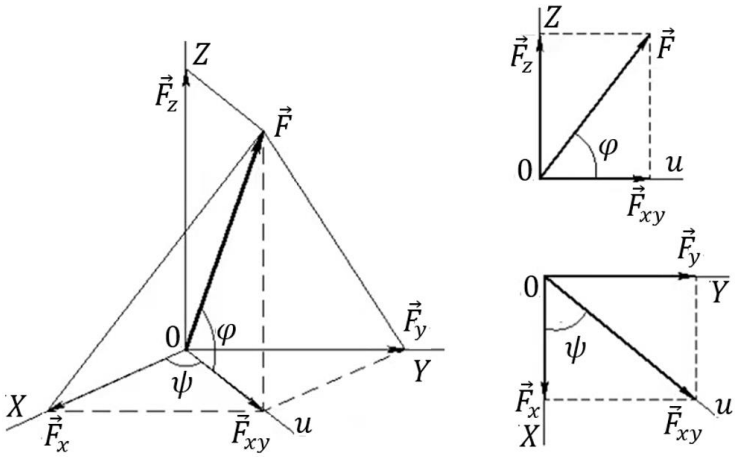


Fig. 1.2.4

With double projection of the force on the axes, taking into account the two independent vector angles used: φ and ψ, the resultant and projections are determined as:

$\vec{F} = \vec{F}_{xy} + \vec{F}_z$, $F_{xy} = F \times \cos\varphi$, $F_x = F_{xy} \times \cos\psi$, $F_y = F_{xy} \times \sin\psi$,

$F_x = F \times \cos\varphi \times \cos\psi$, $F_y = F \times \cos\varphi \times \sin\psi$ and $F_z = F \times \sin\varphi$.

The examples of determining the projections of forces on an axes (Fig. 1.2.5) are:

$F_{1x} = -F_1$, $F_{1y} = 0$ and $F_{1z} = 0$

$F_{2x} = 0$, $F_{2y} = -F_2 \times \cos\alpha$ and $F_{2z} = F_2 \times \sin\alpha$

$F_{3x} = -F_3 \times \cos\beta \times \cos\gamma$, $F_{3y} = F_3 \times \cos\beta \times \sin\gamma$ and $F_{3z} = -F_3 \times \sin\beta$

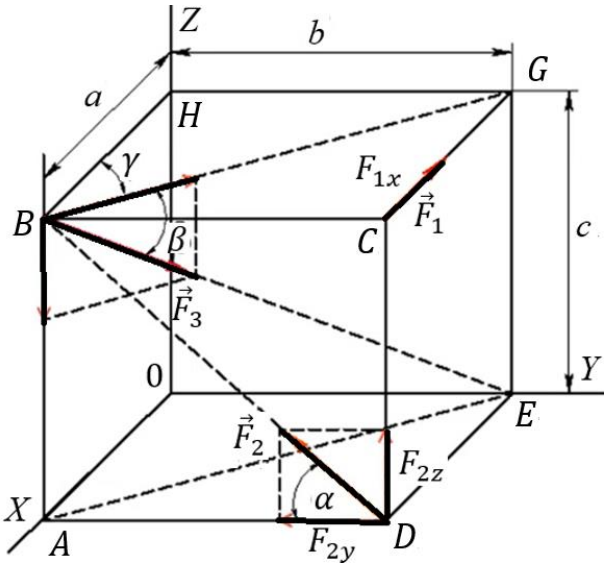


Fig. 1.2.5

Questions and comments:

1.2.1. Finding the projection of the force vector onto an axis, when the force vector and the axis belong to the same plane.

1.2.2. Determination of projections of the force vector on the axes of the Cartesian coordinate system, when the force vector does not belong to any of the coordinate planes.

1.2.3. Using double projection and independent force vector direction angles in a Cartesian coordinate system.

1.3. Vector of the moment of force relative to a point, magnitude of the moment of force relative to the axis

The ability to determine the moment of force relative to a point or relative to an axis (Fig. 1.3.1) is necessary to draw up the balance equation. The algebraic moment $M_0(\vec{F})$ of a force relative to point 0 is equal to the product of the modulus of force by the arm h,

taken with the «plus» sign, if the force $\vec{F}$ tends to rotate the body relative to the moment point 0 counterclockwise (in another case – with the «minus» sign): $M_0(\vec{F}) = \pm F \times h$.

The magnitude of the moment, which is: $|\vec{M}_0| = F \times h$ corresponds to two areas of a triangle $0AB$ (Fig. 1.3.1).

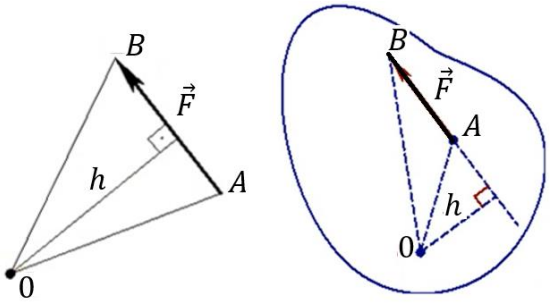


Fig. 1.3.1

The arm h of the force $\vec{F}$ is the shortest distance from the moment point (pole) 0 to the line of action of the force. The vector of moment $\vec{M}_0$ of the force $\vec{F}$ relative to the point 0 is equal to the product of the radius $\vec{r}$ by the force vector $\vec{F}$: $\vec{M}_0 = \vec{r} \times \vec{F}$, the magnitude of the moment is: $|M_0| = |F| \cdot |r| \cdot \sin\varphi = |F| \cdot h$.

The vector of moment $\vec{M}_0$ is applied at pole 0 and is directed perpendicular to the plane in which they lie: the force $\vec{F}$ and the point 0, so that, looking towards the vector, one can see the tendency of the force to rotate the body counterclockwise (Fig. 1.3.2).

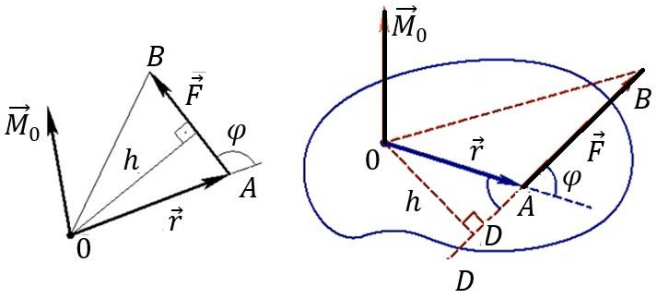


Fig. 1.3.2

The determination of the vector of moment relative to a point through the projection of forces is shown on the vector diagram (Fig. 1.3.3): $\vec{M}_0 = (M_x, M_y, M_z)$,

$$\vec{M}_0 = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \vec{\imath} \times y \times F_z + \vec{\jmath} \times z \times F_x + \vec{k} \times x \times F_y -$$

$$-\vec{\imath} \times z \times F_y - \vec{\jmath} \times x \times F_z - \vec{k} \times y \times F_x = \vec{\imath} \times M_x + \vec{\jmath} \times M_y + \vec{k} \times M_z ,$$

$M_x = y \times F_z - z \times F_y$, $M_y = z \times F_x - x \times F_z$ and $M_z = x \times F_y - y \times F_x$.

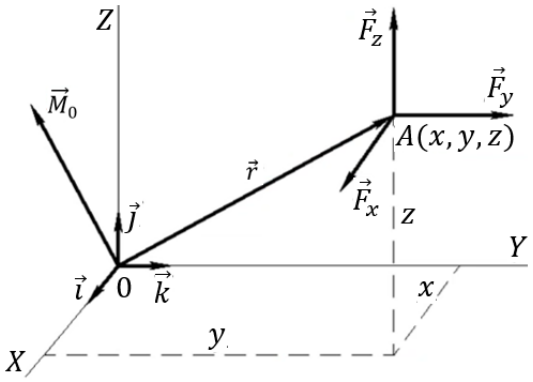


Fig. 1.3.3

When solving spatial issues, the moment equations are used, including sums of moments of forces about the axes. The moment of the force relative to an axis is the algebraic moment of the projection of this force onto a plane perpendicular to the axis, relative to the point of intersection of the axis with this plane. The moment of the force relative to the axis depends on the projection of the force onto the plane perpendicular to the $\vec{F}_1$, axis, which is illustrated by the diagram (Fig. 1.3.4). The moment is considered positive if, looking towards the axis, the tendency of the projection of the force to rotate the body counterclockwise is visible, and negative, if the tendency of the projection of the force is visible to rotate the body clockwise: $M_z(\vec{F}_1) = \pm F_1 \times h_1$, $M_z(\vec{F}_z) = 0$.

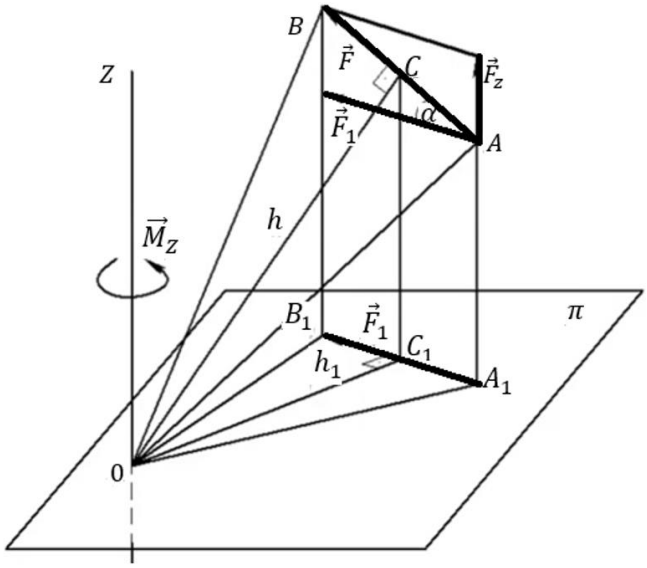


Fig. 1.3.4

The moment of the force $\vec{Q}$ relative to the $0Z$ axis is zero if the line of action of the force intersects the axis or is parallel to the axis, i.e. in variants of the direction of the force ($\vec{P}$ and $\vec{Q}_1$), indicated in the diagram (Fig. 1.3.5).

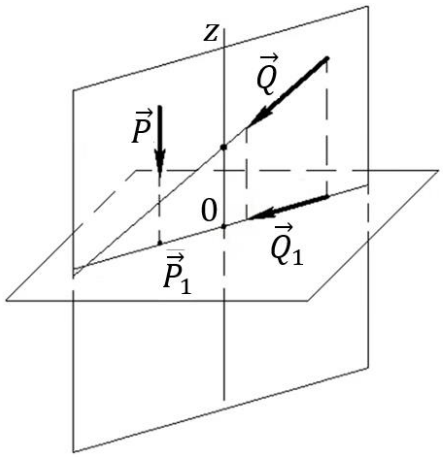


Fig. 1.3.5

In the first case, the line of action of the projection of the force $\vec{Q}_1$ passes through the point of intersection of the axis with the plane perpendicular to it. In the second case, the projection of the force $\vec{P}$ onto this plane is equal to zero.

The correlation of the moment of force relative to an axis and the moment of the force relative to a point on this axis (Fig. 1.3.6): $\vec{M}_0/\vec{M}_z = h/h_1 = 1/\cos\varphi$.

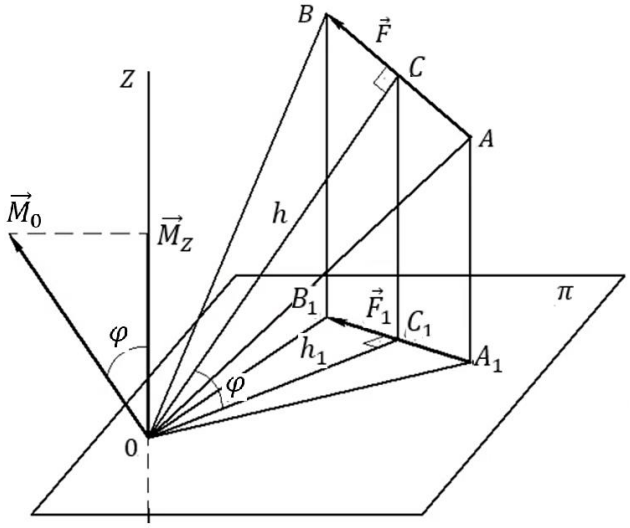


Fig. 1.3.6

The moment of the force relative to the axes (Fig. 1.3.7) is determined as follows:

$M_x(\vec{F}_1) = 0$, $M_y(\vec{F}_1) = -F_1 \times c$ and $M_z(\vec{F}_1) = F_1 \times b$

$M_x(\vec{F}_2) = F_2 \times \sin\alpha \times b$, $M_y(\vec{F}_2) = -F_2 \times \sin\alpha \times a$ and $M_z(\vec{F}_2) = -F_2 \times \cos\alpha \times a$

$M_x(\vec{F}_3) = -F_3 \times \cos\beta \times \sin\gamma \times c$, $M_y(\vec{F}_3) = 0$ and $M_z(\vec{F}_3) = F_3 \times \cos\beta \times \sin\gamma \times a$

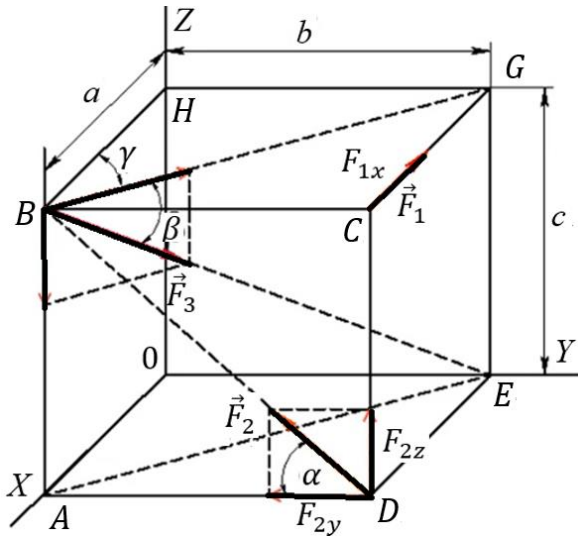


Fig. 1.3.7

Questions and comments:

1.3.1. Algebraic moment of the force relative to a point. The arm of moment of the force relative to the point (pole).

1.3.2. The vector of moment of the force relative to the point. Direction of the vector of moment of force.

1.3.3. Determination of the vector of moment of the force relative to a point using the force projections.

1.3.4. Moment of the force relative to the axis. The dependence of the moment relative to the axis on the direction of the line of action of force relative to the axis.

1.4. Vector transformation of a system of forces acting on a mechanical system

The transformation of the spatial system of forces. In the general case, an arbitrary spatial system of forces is reduced to a force equal to the main vector (the resultant) $\vec{R}$, and to pair of forces, the vector of moment of which is equal to the main moment $\vec{M}_0$.

The system of forces can be simplified depending on the magnitudes of the vectors and on their mutual direction, i.e. on angle α between the vectors, in particular cases.

The particular case 1. Reducing a spatial system of forces to a pair of forces.

If $\vec{R} = 0$ and $\vec{M}_0 \neq 0$, then the system of forces is reduced to one pair of forces, and the main moment of forces in this case does not depend on the choice of the center of reduction.

The particular case 2. Bringing the spatial system of forces to the resultant.

If $\vec{R} \neq 0$ and $\vec{M}_0 = 0$, then the system of forces is reduced to the resultant force $\vec{R}^*$, equal in magnitude and direction to the main vector $\vec{R}^* = \vec{R}$. The line of action of the resultant wherein passes through the center of reduction of the system of forces.

If $\vec{R} \neq 0$, $\vec{M}_0 \neq 0$ and $\alpha = 90^0$, that is if the main vector and the main moment are mutually perpendicular, then the system of forces is also reduced to the resultant force $\vec{R}^*$, equal in magnitude and direction to the main vector: $\vec{R}^* = \vec{R}$.

The particular case 3. Reduction of a system of forces lying on a plane.

Need to determine the point 0_1 of the application of resultant $\vec{R}^*$ (Fig. 1.4.1).

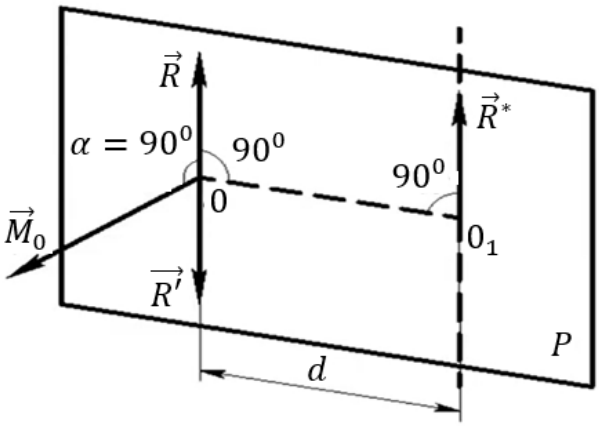


Fig. 1.4.1

The vector of moment $\vec{M}_0$ of the resultant $\vec{R}^*$ can be represented as an equivalent pair $(\vec{R}, \vec{R}^*)$, located in the plane P, perpendicular to the torque vector $\vec{M}_0$. Wherein the leverage of a couple of forces is equal to: $d = M_0/R$.

If the force $\vec{R'}$ and one $\vec{R}$, applied to point 0, lie on the same straight line and act in opposite directions. These two forces ($\vec{R'}$ and $\vec{R}$) make up the balanced system.

The remaining force $\vec{R}^*$ and is the resultant for a given system. Point of the application of the resultant 0_1.

The particular case 4. The spatial system of forces is a dynamic screw (dynamo), when the resultant vector is parallel to the moment vector.

If the force and the moment are not zero: $\vec{R} \neq 0$ and $\vec{M}_0 \neq 0$ and these vectors are parallel to each other, then such a system is called the dynamic screw or dynamo, and the straight line, along which the vector $\vec{R}$ is directed, is denominated the axis of the screw.

In other words, in the particular case under consideration, the system of forces is reduced to a force and a pair, the plane of action of which is perpendicular to the vector of the force (Fig. 1.4.2). Further simplification of such a system of forces is impossible.

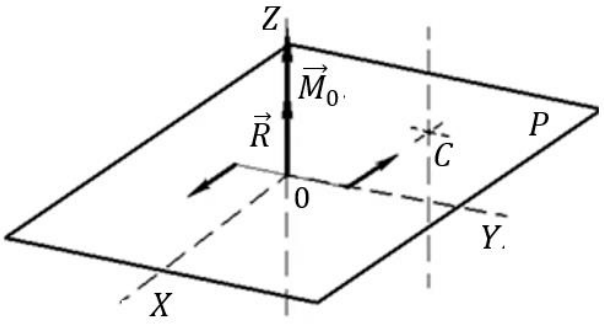


Fig. 1.4.2

If one takes any point as the reduction center, for example, the point C, then the free moment vector $\vec{M}_0$ one can transfer to the point. When transferring of the force $\vec{R}$ to the point C another pair with the moment $\overrightarrow{M'}_C(\vec{R})$ perpendicular to the vector of the force $\vec{R}$ will be added, and, therefore, perpendicular to the vector of moment $\vec{M}_0$.

As a result, the moment of the resulting pair is equal to: $\vec{M}_C = \vec{M}_0 + \overrightarrow{M'}_C$ and the magnitude of the resulting moment $\vec{M}_C$ is greater than the set moment $\vec{M}_0$. Thus, the set moment $\vec{M}_0$ of the resulting pair relative to the center 0 has a minimum value.

The particular case 5. The reduction of a spatial system of forces to the dynamic screw. If the force and moment are not zero: $\vec{R} \neq 0$, $\vec{M}_0 \neq 0$ and these vectors are not perpendicular and not parallel to each other, then such a system can be reduced to the dynamic screw (Fig. 1.4.3). But the axis of the dynamic screw wherein does not pass through the center of the application of the force $\vec{R}$.

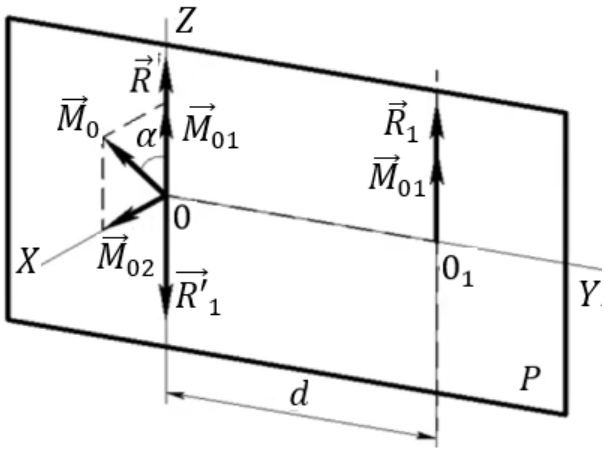


Fig. 1.4.3

At the center of reduction, that is at the point 0 the main vector of force $\vec{R}$ and the main moment $\vec{M}_0$ are applied, directed at an angle α to each other.

One can draw thd plane P through the vector of force $\vec{R}$, perpendicular to the plane in which the vectors are of the force $\vec{R}$ and the moment $\vec{M}_0$.

Let one decompose the moment vector $\vec{M}_0$ into two mutually perpendicular vectors $\vec{M}_{01}$ and $\vec{M}_{02}$, so that vector $\vec{M}_{01}$ coincides in direction with the main vector of the force $\vec{R}$, then the values of the vector $\vec{M}_{01}$ and $\vec{M}_{02}$ are: $M_{01} = M_0 \times \cos\alpha$ and $M_{02} = M_0 \times \sin\alpha$.

Vectors of the force $\vec{R}$ and the moment M_{01} are mutually perpendicular and can be reduced to the same force: $\vec{R}_1 = \vec{R}$. For this transformation of vectors one has to replace (Fig. 1.4.4) the vector of moment $\vec{M}_{01}$ with pair of forces $(\overrightarrow{R'}_1, \vec{R})$, whose leverage is equal to: $d = M_{01}/R$.

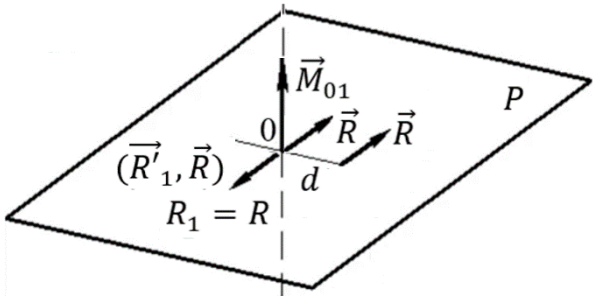


Fig. 1.4.4

To transform one needs to rotate the force pair $(\overrightarrow{R'}_1, \vec{R})$ in the plane P so that vector of the force $\overrightarrow{R'}_1$ is opposite to vector of the force $\vec{R}$.

Considering the equality of force vectors accepted in the presumptions for transformations: $R = R'_1 = R_1$, the forces $\overrightarrow{R'}_1$ and $\vec{R}$ are mutually balanced at the point 0.

The force $\vec{R}_1$ turns out to be applied at the point 0_1 at a distance d from the point 0.

If the free vector $\vec{M}_{01}$ is transferred parallel to itself to the point 0_1, then a pair will be found – the force and moment $(\vec{R}_1, \vec{M}_{01})$ equivalent to the given ones: force and moment $(\vec{R}, \vec{M}_0)$. So, the result of the reduction is a dynamic screw (torsion).

Thus, the following general options for transforming the spatial system of forces to the simplest form are possible:

1. If $\vec{R} = 0$, $\vec{M}_0 \neq 0$, then the system of forces is reduced to one pair of forces, and the main moment in this case does not depend on the choice of the center of the system reduction.

2. If $\vec{R} \neq 0$ and $\vec{M}_0 = 0$, then the system of forces is reduced to the resultant force $\vec{R}^*$ equal in magnitude and direction to the main vector $\vec{R}^* = \vec{R}$. The line of action of the resultant force wherein passes through the center of the system reduction.

3. If $\vec{R} \neq 0$, $\vec{M}_0 \neq 0$ and $\alpha = 90^0$, that is if the main vector and the main moment are mutually perpendicular, then the system of forces is also reduced to the resultant force $\vec{R}^*$ equal in magnitude and direction to the main vector $\vec{R}^* = \vec{R}$.

4. If $\vec{R} \neq 0$, $\vec{M}_0 \neq 0$ and these vectors are parallel to each other, then such a system is a dynamic screw. In this case, the system is reduced to a force and a couple, the plane of action of which is perpendicular to the force vector. Further simplification of such a system of forces is impossible.

5. If $\vec{R} \neq 0$, $\vec{M}_0 \neq 0$ and these vectors are not perpendicular and not parallel to each other, then such a system is also reduced to a dynamic screw. But the axis of the screw in this case does not pass through the point of the application of the force $\vec{R}$.

Exercise 1. The assigned system, consisting of three forces: $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, bring to the center at the point 0, the data indicated in the Table 1.4.1 and on the diagram (Fig. 1.4.5).

Table 1.4.1

F_1, kN	F_2, kN	F_3, kN	a, m	b, m	c, m	d, m
4	3	6	3	4	6	5

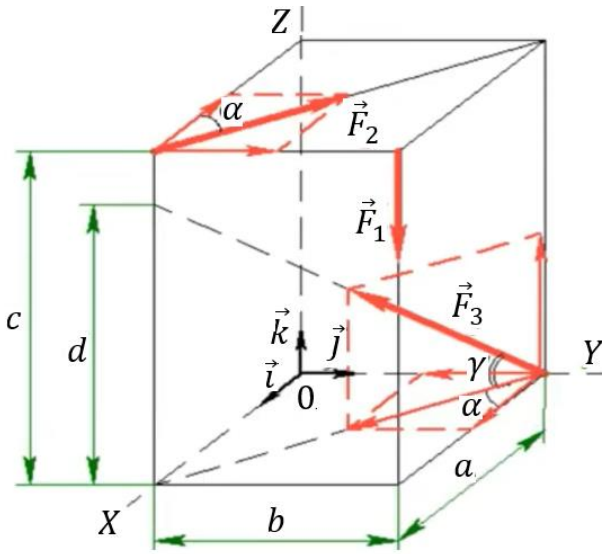


Fig. 1.4.5

The main vector $\vec{R}$ of the force system has the form:

$\vec{R} = R_x \times \vec{i} + R_y \times \vec{j} + R_z \times \vec{k}$.

To determine the projections of the main force vector $\vec{R}$ on the axis, it is necessary to determine the values of the trigonometric functions of the angles:

$\cos\alpha = 0.6$, $\sin\alpha = 0.8$, $\cos\gamma = 0.707$ and $\sin\gamma = 0.707$.

The projections of the main vector of forces $\vec{R}$ on the coordinate axes are:

$R_x = -F_2 \times \cos\alpha + F_3 \times \cos\gamma \times \cos\alpha = -3 \times 0.6 + 6 \times 0.707 \times 0.6 = 0.745$, kN,

$R_y = F_2 \times \sin\alpha - F_3 \times \cos\gamma \times \sin\alpha = 3 \times 0.8 - 6 \times 0.707 \times 0.8 = -0.994$, kN,

$R_z = -F_1 + F_3 \times \sin\gamma = -4 + 6 \times 0.707 = -0.2424$, kN.

Main force vector $\vec{R}$ magnitude is:

$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{0.745^2 + (-0.994)^2 + (-0.2424)^2} = 1.266$, kN.

Direction cosines of the main vector $\vec{R}$ are:

$\cos(\vec{R}, \vec{\imath}) = \frac{R_x}{R} = \frac{0.745}{1.266} = 0.588$, $\cos(\vec{R}, \vec{\jmath}) = \frac{R_y}{R} = \frac{-0.994}{1.266} = -0.785$

and $\cos(\vec{R}, \vec{k}) = \frac{R_z}{R} = \frac{-0.2424}{1.266} = -0.191$.

The main moment has the form (Fig. 1.4.6):

$\vec{M}_0 = M_x \times \vec{\imath} + M_y \times \vec{\jmath} + M_z \times \vec{k}$.

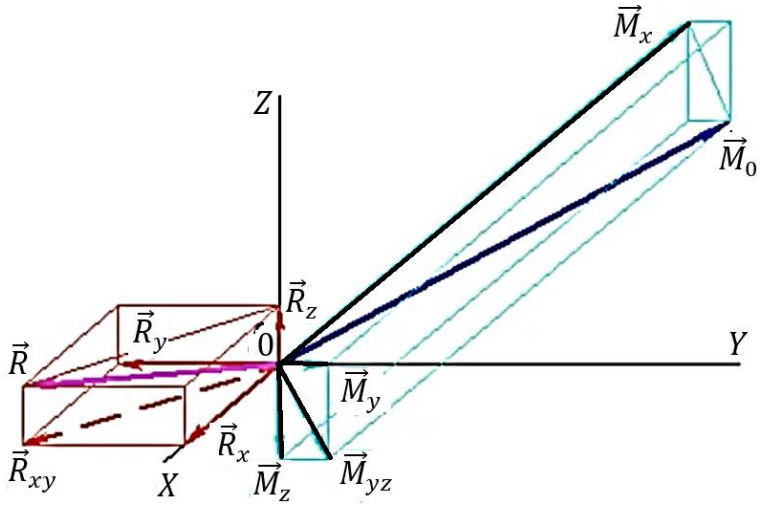


Fig. 1.4.6

The projections of main moment $\vec{M}_0$ on the coordinate axes are:

$M_x = -F_1 \times b - F_2 \times \sin\alpha \times c + F_3 \times \sin\gamma \times b =$

$= -4 \times 4 - 3 \times 0.8 \times 6 + 6 \times 0.707 \times 4 = -13.432$, kN×m,

$M_y = F_1 \times a - F_2 \times \cos\alpha \times c = -4 \times 3 - 3 \times 0.6 \times 6 = 1.2$, kN×m,

$M_z = F_2 \times \sin\alpha \times a + F_3 \times \cos\gamma \times \cos\alpha \times b =$

$= 3 \times 0.8 \times 3 - 6 \times 0.707 \times 0.6 \times 4 = -2.981$, kN×m.

The magnitude of the main moment M_0 is:

$M_0 = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{(-13.432)^2 + 1.2^2 + (-2.981)^2} = 13.811$, kN×m.

Direction cosines of the main moment M_0 are:

$\cos(\vec{M}_0, \vec{\imath}) = \frac{M_x}{M_0} = \frac{-13.432}{13.811} = -0.973$, $\cos(\vec{M}_0, \vec{\jmath}) = \frac{M_y}{M_0} = \frac{1.2}{13.811} = 0.087$

and $\cos(\vec{M}_0, \vec{k}) = \frac{M_z}{M_0} = \frac{-2.981}{13.811} = -0.216$.

Exercise 2. The assigned system, consisting of three forces: $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, bring to the center at the point 0, the data indicated in the Table 1.4.2 and on the diagram (Fig. 1.4.7).

Table 1.4.2

F_1, kN	F_2, kN	F_3, kN	a, m	b, m	c, m
3	5	2	3	4	3

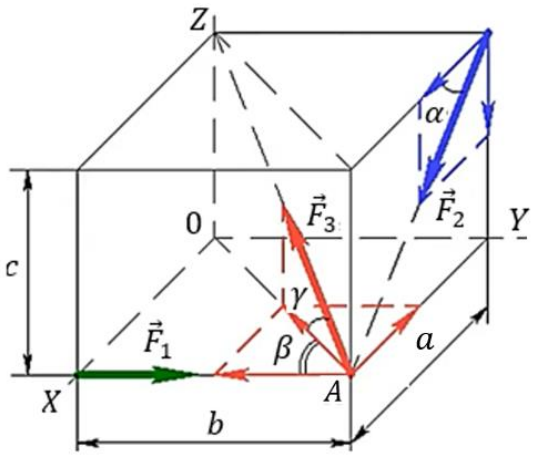


Fig. 1.4.7

Values of trigonometric functions of the angles of projections of the main vector of forces $\vec{R}$ on the axes: $\cos\alpha = 0.707$, $\sin\alpha = 0.707$, $\cos\beta = 0.8$, $\sin\beta = 0.6$, $\cos\gamma = 0.857$ and $\sin\gamma = 0.514$.

The projections of the main vector of forces $\vec{R}$ on the coordinate axes are:

$R_x = F_2 \times \cos\alpha - F_3 \times \cos\gamma \times \sin\beta = 5 \times 0.707 - 2 \times 0.857 \times 0.6 = 2.507$, kN,

$R_y = F_1 - F_3 \times \cos\gamma \times \cos\beta = 3 - 2 \times 0.857 \times 0.8 = 1.629$, kN,

$R_z = -F_2 \times \sin\alpha + F_3 \times \sin\gamma = -5 \times 0.707 + 2 \times 0.514 = -2.507$, kN.

Main force vector $\vec{R}$ magnitude is:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{2.507^2 + 1.629^2 + (-2.507)^2} = 3.902, \text{ kN}.$$

Direction cosines of the main vector $\vec{R}$ are:

$\cos(\vec{R},\vec{\imath}) = \frac{R_x}{R} = \frac{2.507}{3.902} = 0.642$, $\cos(\vec{R},\vec{\jmath}) = \frac{R_y}{R} = \frac{1.929}{3.902} = 0.417$

and $\cos(\vec{R},\vec{k}) = \frac{R_z}{R} = \frac{-2.507}{3.902} = -0.642$.

The projections of the main moment $\vec{M}_0$ of forces on the coordinate axis are:

$M_x = -F_2 \times \sin\alpha \times b + F_3 \times \sin\gamma \times b =$

$= -5 \times 0.707 \times 4 + 2 \times 0.514 \times 4 = -10.028$, kN×m,

$M_y = F_2 \times \cos\alpha \times c - F_3 \times \sin\gamma \times a = 5 \times 0.707 \times 3 - 2 \times 0.514 \times 3 =$

$= 7.521$, kN×m,

$M_z = F_1 \times a - F_2 \times \cos\alpha \times b = 3 \times 3 - 5 \times 0.707 \times 4 = -5.14$, kN×m.

The magnitude of the main moment M_0 is:

$M_0 = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{(-10.028)^2 + 7.521^2 + (-5.14)^2} = 13.548$, kN×m.

Direction cosines of the main moment M_0 are:

$\cos(\vec{M}_0, \vec{\imath}) = \frac{M_x}{M_0} = \frac{-10.028}{13.548} = -0.740$, $\cos(\vec{M}_0, \vec{\jmath}) = \frac{M_y}{M_0} = \frac{7.521}{13.548} = 0.555$

and $\cos(\vec{M}_0, \vec{k}) = \frac{M_z}{M_0} = \frac{-5.14}{13.548} = -0.379$.

Exercise 3. The assigned system, consisting of three forces: $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, bring to center at the point 0, the data indicated in the Table 1.4.3 and on the diagram (Fig. 1.4.8).

Table 1.4.3

F_1, kN	F_2, kN	F_3, kN	a, m	b, m	c, m
5	5	$5 \times \sqrt{2}$	6	10	8

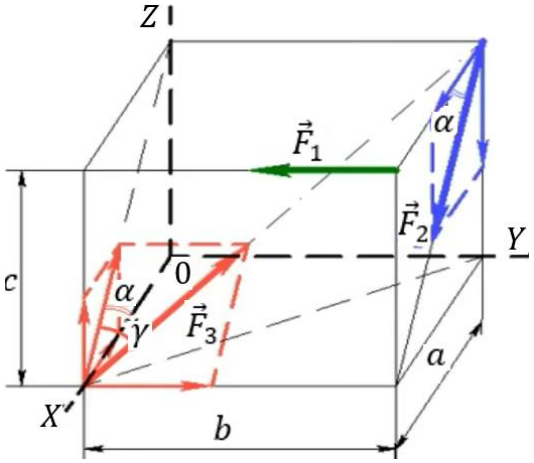


Fig. 1.4.8

Values of trigonometric functions of the angles of projections of the main vector of forces $\vec{R}$ on the axis are: $\cos\alpha = 0.6$, $\sin\alpha = 0.8$, $\cos\gamma = 0.707$ and $\sin\gamma = 0.707$.

Projections of the main vector of forces $\vec{R}$ on the coordinate axes:

$R_x = F_2 \times \cos\alpha - F_3 \times \cos\gamma \times \cos\alpha = 5 \times 0.6 - 5 \times \sqrt{2} \times 0.707 \times 0.6 = 0$,

$R_y = -F_1 - F_3 \times \sin\gamma = -5 + 5 \times \sqrt{2} \times 0.707 = 0$,

$R_z = -F_2 \times \sin\alpha + F_3 \times \cos\gamma \times \sin\alpha = -5 \times 0.8 + 5 \times \sqrt{2} \times 0.707 \times 0.8 = 0$.

Main force vector module $\vec{R}$ is: $R = \sqrt{R_x^2 + R_y^2 + R_z^2} = 0$.

Projections of the main moment of forces $\vec{M}_0$ on the coordinate axes are:

$M_x = F_1 \times c - F_2 \times \sin\alpha \times b = 5 \times 8 - 5 \times 0.8 \times 10 = 0$,

$M_y = F_2 \times \cos\alpha \times c - F_3 \times \cos\alpha \times \sin\gamma \times a =$

$= 5 \times 0.6 \times 8 - -5 \times \sqrt{2} \times 0.707 \times 0.8 \times 6 = 0$,

$M_z = -F_1 \times a - F_2 \times \cos\alpha \times b + F_3 \times \sin\gamma \times a = -5 \times 6 - 5 \times 0.6 \times 10 +$

$+5 \times \sqrt{2} \times 0.707 \times 6 = -30$, kN×m.

The magnitude of the main moment is: $M_0 = -30$ kN×m.

Questions and comments:

1.4.1. Conversion of the spatial force system to a principal vector and force pair.
1.4.2. Reduction of the spatial force system to a force pair if the main vector is zero.
1.4.3. Reducing the spatial force system to a resultant if the moment is zero.
1.4.4. Reduction of the forces system lying on the plane.
1.4.5. The determining of dynamic screw (the torsion).
1.4.6. Reducing the spatial force system to a dynamic screw.

1.5. The action of a pair of forces and the transformation of a pair of forces to the moment

The pair of forces (Fig. 1.5.1) is the system of two equal in magnitude, parallel and directed in opposite directions forces acting on an absolutely rigid body: $F_1 = -F_2$.

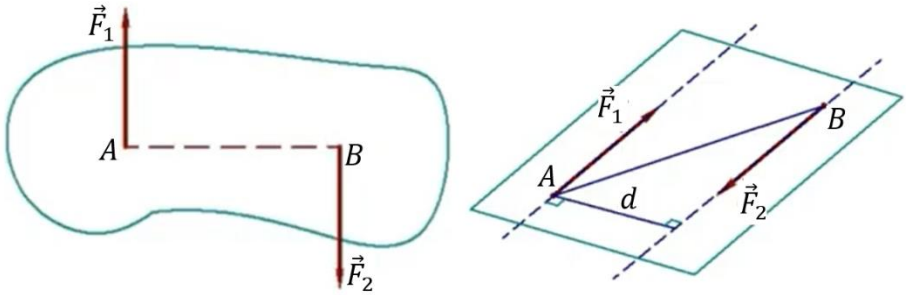


Fig. 1.5.1

A pair of forces is characterized by a plane of action, then is the plane in which the forces of this pair are located and applied. A pair of forces has no resultant.

The lines of action of the forces in the pair are parallel, and the magnitudes of the forces in the pair are equal: $F_1 = F_2 = F$. The arm of a pair of forces is the shortest distance d between the lines of action of the forces of this pair.

The algebraic moment is the product of the force F by the arm d of forces pair (the shortest distance between the lines of action of forces in a pair, Fig. 1.5.2):
$M(\vec{F}_1, \vec{F}_2) = \pm F \times d$.

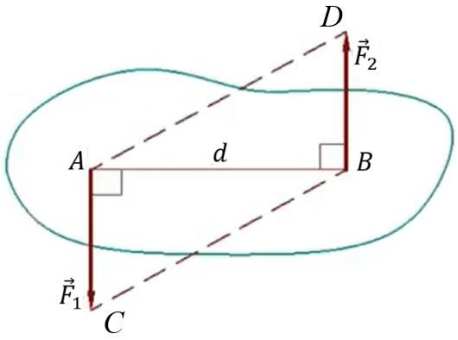


Fig. 1.5.2

It is adopted that the algebraic moment $M(\vec{F}_1, \vec{F}_2)$ is considered positive (from the observer's side) if a pair of forces tends to rotate the body counterclockwise, and negative if a pair of forces tends to rotate the body clockwise.

The algebraic moment of a pair of forces $M(\vec{F}_1, \vec{F}_2)$ magnitude is equal to the area of the parallelogram $ACBD$ (Fig. 1.5.3).

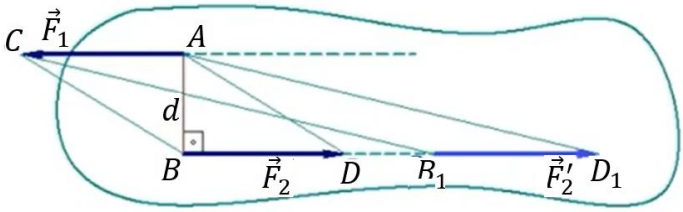


Fig. 1.5.3

Theorem on the equivalence of two pairs of forces located in the same plane. Two pairs of forces are denominated equivalent if their action on a solid surface is the same, other things being equal.

A pair of forces $(\vec{F}_1, \vec{F}_2)$, acting on a rigid body can be replaced with another one pair of forces $(\vec{F}_1, \vec{F}_2')$, located in the same plane of action (Fig. 1.5.3) and having the same algebraic moment: $M(\vec{F}_1, \vec{F}_2) = F \times d = M(\vec{F}_1, \vec{F}_2')$.

Conclusions. A pair of forces $(\vec{F}_1, \vec{F}_2)$ can be rotated as a rigid figure and transferred in any way in the plane of its action. For the pair of forces one can simultaneously change the arm d and the forces: $\vec{F}_1$ and $\vec{F}_2$, keeping wherein the algebraic moment $M(\vec{F}_1, \vec{F}_2)$ and the plane of action.

Theorem on the transfer of a pair of forces into a parallel plane. A pair of forces $(\vec{F}_1, \vec{F}_2)$, acting in the plane, is given P_1 (Fig. 1.5.4). If add two more of the force system equivalent to zero in the parallel plane P_2: $(\vec{F}_1', \vec{F}_1'') \sim 0$ and $(\vec{F}_2', \vec{F}_2'') \sim 0$, because: $\vec{F}_1' = -\vec{F}_1''$, $\vec{F}_2' = -\vec{F}_2''$, then: $(\vec{F}_1', \vec{F}_1'', \vec{F}_2', \vec{F}_2'', \vec{F}_1, \vec{F}_2) = (\vec{F}_1, \vec{F}_2)$.

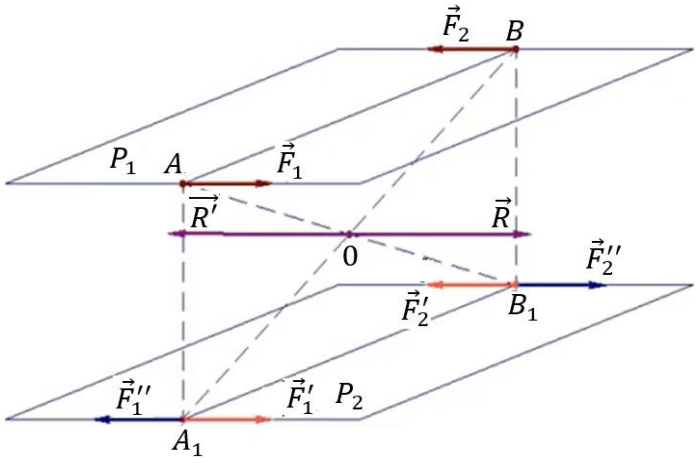


Fig. 1.5.4

One can replace the forces: $\vec{F}_1$ and $\vec{F}_2''$ by their resultant: $\vec{R} = \vec{F}_1 + \vec{F}_2''$.

Likewise, the forces: $\vec{F}_2$ and $\vec{F}_1''$ one can replace by the resultant: $\overrightarrow{R'} = \vec{F}_2 + \vec{F}_1''$.

But these resultant forces: $\vec{R}$ and $\overrightarrow{R'}$ are equal in magnitude and opposite in direction: $\vec{R} = -\overrightarrow{R'}$, that is: $\left(\vec{R}, \overrightarrow{R'}\right) \sim 0$.

After discarding the indicated pair of forces: $\left(\vec{R}, \overrightarrow{R'}\right) \sim 0$ one can obtain a system of two forces $\left(\vec{F}_1', \vec{F}_2'\right)$, which is with pair of forces and is obtained from the original system by the equivalent transformations, therefore, the force systems are equivalent: $\left(\vec{F}_1', \vec{F}_2'\right) \sim \left(\vec{F}_1, \vec{F}_2\right)$.

The vector of moment of a pair of forces is the vector, numerically equal to the product of the force by the arm of the pair of forces. The vector of moment (Fig. 1.5.5) is directed perpendicular to the plane of action of the pair of forces so that, looking towards the vector, one can see the desire of the pair of forces to rotate the body counterclockwise. The vector of moment can be applied at any point on the body affected by the pair of forces. The vector of moment acting on a rigid body is a free vector and can be transferred to any point of the body.

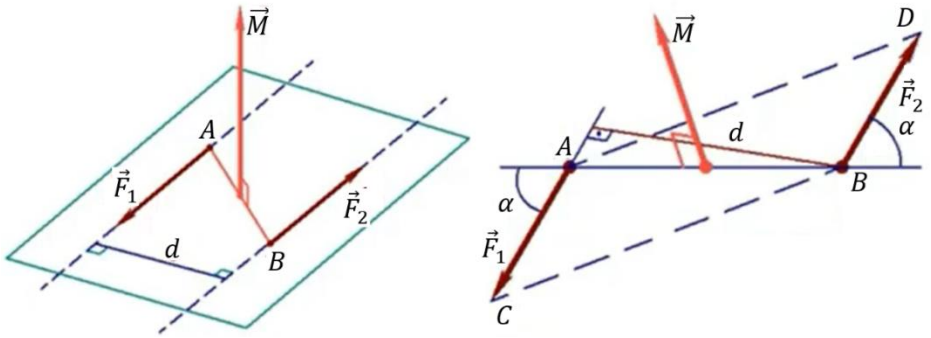


Fig. 1.5.5

The vector of moment of a pair of forces one can find by the product of vectors: of the force ($\vec{F}_1$ or $\vec{F}_2$) and the vector connecting the points of application of the forces ($\overrightarrow{AB}$ or $\overrightarrow{BA}$) as: $\vec{M} = \overrightarrow{AB} \times \vec{F}_2 = \overrightarrow{BA} \times \vec{F}_1$.

The magnitude of the vector moment of pair of forces is equal to:

$\left|\overrightarrow{AB} \times \vec{F}_2\right| = AB \cdot F_2 \cdot \sin\alpha = F_2 \cdot d.$

The magnitude of the vector moment is equal to the algebraic moment:

$M(\vec{F}_1, \vec{F}_2) = F_1 \times d = F_2 \times d.$

Theorem on the sum of moments of a pair of forces.

The sum of vector moments of forces in the pairs relative to any point does not depend on the choice of point and is equal to the vector moment of this pair of forces: $\vec{M}_0(\vec{F}_1) + \vec{M}_0(\vec{F}_2) = \vec{r}_A \times \vec{F}_1 + \vec{r}_B \times \vec{F}_2 = (\vec{r}_B - \vec{r}_A) \times \vec{F}_2$, considering that the forces in the pair are opposite and equal in magnitude: $\vec{F}_1 = -\vec{F}_2$, and vector between the points of the application of forces is equal to the difference between vectors: $\vec{r}_B - \vec{r}_A = \overrightarrow{AB}$ wherein the point 0 of application of the moment $\vec{M}$ is free (Fig. 1.5.6).

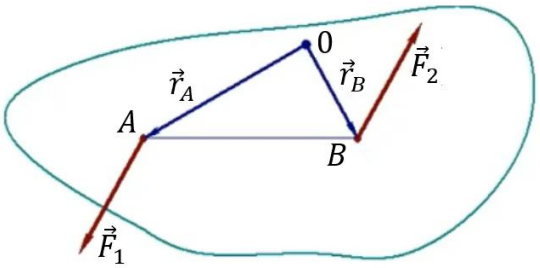


Fig. 1.5.6

Therefore, the sum of vector moments from each of the forces in the pair: $\vec{F}_1$ and $\vec{F}_2$ is equal to the vector product of one of these forces by vector $\overrightarrow{AB}$, connecting the points of the application of the forces: $\vec{M}_0(\vec{F}_1) + \vec{M}_0(\vec{F}_2) = \overrightarrow{AB} \times \vec{F}_2$, which coincides with the vector moment of a pair of forces $\vec{M}$: $\vec{M}_0(\vec{F}_1) + \vec{M}_0(\vec{F}_2) = \vec{M}$, wherein relative to points A and B moments of forces: $\vec{M}_A(\vec{F}_2)$ and $\vec{M}_B(\vec{F}_1)$ are equal: $\vec{M}_A(\vec{F}_2) = \vec{M}_B(\vec{F}_1) = \vec{M}$.

Consequently, the vector of moment of a pair of forces is equal to the vector moment of one of the forces in this pair relative to the point of the application of the other force in the pair.

The addition of force pairs. Two pairs of forces acting on the same body and then lying in intersecting planes, one can replace by one equivalent pair of forces, the vector of moment of which is equal to the sum of vector moments of given pairs of forces.

One can consider two pairs of forces: $(\vec{F}_1, \vec{F}_1')$ and $(\vec{F}_2, \vec{F}_2')$, lying in intersecting planes (Fig. 1.5.7).

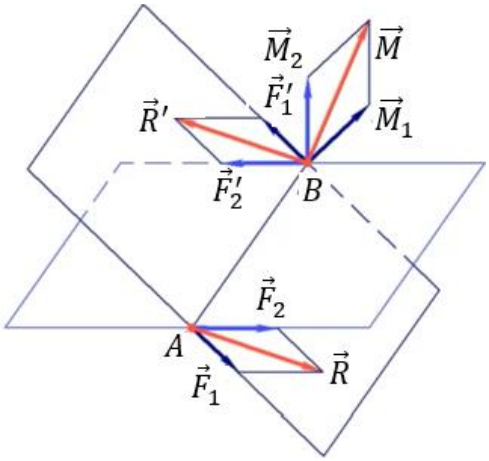


Fig. 1.5.7

One can add up the forces, applied in points A and B, and find two resultant forces: $\vec{R} = \vec{F}_1 + \vec{F}_2$ and $\vec{R}' = \vec{F}_1' + \vec{F}_2'$.

Resultant forces obtained: $\vec{R}$ and $\vec{R}'$ are also the form of pair: $\vec{R} = -\vec{R}'$.

The vector of moment $\vec{M}$ of this pair of resultant forces $(\vec{R}, \vec{R}')$ is equal to: $\vec{M} = \overrightarrow{BA} \times \vec{R} = \overrightarrow{BA} \times (\vec{F}_1 + \vec{F}_2) = \overrightarrow{BA} \times \vec{F}_1 + \overrightarrow{BA} \times \vec{F}_2$.

As a result, one can determine the vector of moment of a pair of resultant forces as the sum of the vector moments of pairs of given forces: $\vec{M}_1 = \overrightarrow{BA} \times \vec{F}_1$, $\vec{M}_2 = \overrightarrow{BA} \times \vec{F}_2$ and $\vec{M} = \vec{M}_1 + \vec{M}_2$.

That is, the vector of moment of an equivalent pair of forces is equal to the sum of the vector moments of the given pairs of forces.

The balance conditions for pairs of forces. For equilibrium of pairs of forces acting on a rigid body, it is necessary and sufficient that the modulus of the vector moment of the equivalent pair of forces be equal to zero: $\vec{M} = 0$, or so that the vector polygon created on the vector moments of the given pairs of forces is closed.

In scalar form, the equilibrium condition has the form: $M = \sqrt{M_x^2 + M_y^2 + M_z^2} = 0$, where M_x, M_y and M_z – the algebraic sums of projections onto the corresponding axes of vector moments of force pairs.

From the above condition of the balance of pairs of forces acting on a solid body, three equations of equilibrium of a system of pairs of forces in space follow in scalar form: $M_x = \sum_{k=1}^{n} M_{kx}$, $M_y = \sum_{k=1}^{n} M_{ky}$ and $M_z = \sum_{k=1}^{n} M_{kz}$.

Thus, for the equilibrium of pairs of forces applied to a solid body it is necessary and sufficient that the sum of the projections of the vector moments of the pairs of forces on each of the three coordinate axes $(0XYZ)$ will be equal to zero.

For equilibrium of pairs of forces acting on a rigid body in one plane it is necessary and sufficient that the sum of the scalar moments of these pairs of forces be equal to zero.

Questions and comments:

1.5.1. Force couple concept.
1.5.2. Algebraic moment of a pair of forces, its magnitude and direction.
1.5.3. Theorem on the equivalence of two pairs of forces located in the same plane.
1.5.4. Theorem on the transfer of a pair of forces into a parallel plane.
1.5.5. Vector moment of a couple of forces.
1.5.6. Theorem on the sum of moments of a pair of forces.
1.5.7. The addition of forces pairs.
1.5.8. Conditions for equilibrium of pairs of forces.

1.6. Equilibrium of a plane system of forces and forms of balance equations

The first (basic) form of the balance equations of an arbitrary plane force system. For the balance of a plane force system acting on a solid body, it is necessary and sufficient that the sum of the projections of these forces on each of the two rectilinear coordinate axes located in the plane of their action be equal to zero, and the sum of the scalar moments of forces relative to any point A, located in the plane of action of forces was also equal to zero, that is: $\sum F_{kx} = 0$, $\sum F_{ky} = 0$ and $\sum M_A(\vec{F}_k) = 0$.

The second form of the equations of the balance of an arbitrary plane force system. For the balance of a flat force system applied to a solid body, it is necessary and sufficient that the sum of the scalar moments of the forces of the system relative to any three points: A, B and C, located in the plane of action of the forces and not lying on the same straight line, is equal to zero, that is: $\sum M_A(\vec{F}_k) = 0$, $\sum M_B(\vec{F}_k) = 0$ and $\sum M_C(\vec{F}_k) = 0$.

The third form of the equations of the balance of an arbitrary flat force system. For the balance of a plane force system applied to a solid body, it is necessary and sufficient that the sums of scalar moments of forces relative to any two points lying in the plane of action of the forces are equal to zero and the sum of the projections of these forces onto any axis of the plane that is not perpendicular to the straight line passing through two moment points was also equal to zero, that is: $\sum M_A(\vec{F}_k) = 0$, $\sum M_B(\vec{F}_k) = 0$ and $\sum F_{kx} = 0$.

Exercise 1. Determine reactions in embedding X_A, Y_A and M_A (Fig. 1.6.1).

Data: $F = 5$ kN, $q = 2$ kN/m, $M = 9$ kN×m, $\alpha = 45^0$, $a = 8$ m, $b = 4$ m and $c = 6$ m.

One can replace the distributed load q with the concentrated force Q and apply it in the middle of the loaded area: $Q = q \times b = 2 \times 4 = 8$, kN.

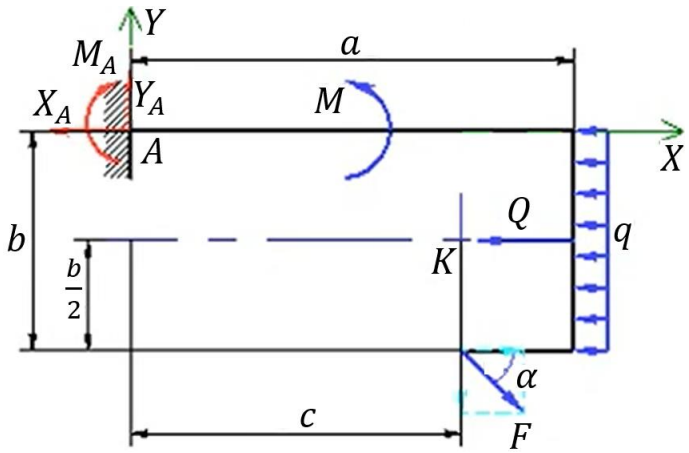


Fig. 1.6.1

One would use the first (basic) form of the balance equations of the arbitrary flat force system:

$\sum F_{kx} = 0, -X_A + F \times \cos\alpha - Q = 0,$

$X_A = F \times \cos\alpha - Q = 5 \times \cos 45^0 - 8 = -4.46,$ kN,

$\sum F_{ky} = 0, Y_A - F \times \sin\alpha = 0$ and $Y_A = F \times \sin\alpha = 5 \times \sin 45^0 = 3.54,$ kN,

$\sum M_A(\vec{F}_k) = 0, -M_A + M - Q \times \frac{b}{2} - F \times \sin\alpha \times c + F \times \cos\alpha \times b = 0,$

$M_A = M - Q \times \frac{b}{2} - F \times \sin\alpha \times c + F \times \cos\alpha \times b,$

$M_A = 9 - 8 \times \frac{4}{2} - 5 \times \sin 45^0 \times 6 + 5 \times \cos 45^0 \times 4 = -14.07,$ kN×m,

$X_A = 4.46$ kN, $Y_A = 3.54$ kN and $M_A = -14.07$ kN×m.

The verification:

$\sum M_K(\vec{F}_k) = -M_A + M + X_A \times \frac{b}{2} - Y_A \times c + F \times \cos\alpha \times \frac{b}{2} =$

$= -(-14.07) + 9 + (-4.46) \times \frac{4}{2} - 3.54 \times 6 + 5 \times \cos 45^0 \times \frac{4}{2} = 0.02 \approx 0.$

Exercise 2. Determine the reactions of the frame supports X_A, Y_A and R_{AB} (Fig. 1.6.2). Data: $F = 3$ kN, $q = 1$ kN/m, $M = 4$ kN×m, $a = 6$ m, $b = 8$ m.

One can replace distributed load with concentrated force:

$Q = q \times a = 1 \times 6 = 6,$ kN.

Geometric dimensions and frame elements: $\sin\alpha = 0.6$, $\cos\alpha = 0.8$, $BD = 10$ m, $AE = 10.5$ m and $HE = 12.5$ m.

Equations of the balance of an arbitrary flat force system (second form):

$\sum M_A(\vec{F}_k) = 0,$

$-M - Q \times \frac{a}{2} + R_B \times \sin\alpha \times (a+b) + F \times \sin\alpha \times \frac{a}{2} - F \times \cos\alpha \times \left(a + \frac{b}{2}\right) = 0,$

$R_B = \frac{1}{\sin\alpha \times (a+b)} \left(M + Q \times \frac{a}{2} - F \times \sin\alpha \times \frac{a}{2} + F \times \cos\alpha \times \left(a + \frac{b}{2}\right) \right) =$

$= \frac{1}{0.6\times(6+8)}\left(4 + 6\times\frac{6}{2} - 3\times 0.6\times\frac{6}{2} + 3\times 0.8\times\left(6+\frac{8}{2}\right)\right) = 4.83$, kN,

$\sum M_B(\vec{F}_k) = 0,\ F\times\frac{BD}{2} + Q\times\left(b+\frac{a}{2}\right) - M - Y_A\times(a+b) = 0,$

$Y_A = \frac{1}{(a+b)}\left(F\times\frac{BD}{2} + Q\times\left(b+\frac{a}{2}\right) - M\right) = \frac{1}{(6+8)}\left(3\times\frac{10}{2} + 6\times\left(8+\frac{6}{2}\right) - 4\right) = 5.5$, kN,

$\sum M_E(\vec{F}_k) = 0,\ -F\times HE - Q\times\frac{a}{2} - M + X_A\times AE = 0,$

$X_A = \frac{1}{AE}\left(F\times HE + Q\times\frac{a}{2} + M\right) = \frac{1}{10.5}\left(3\times 12.5 + 6\times\frac{6}{2} + 4\right) = 5.67$, kN.

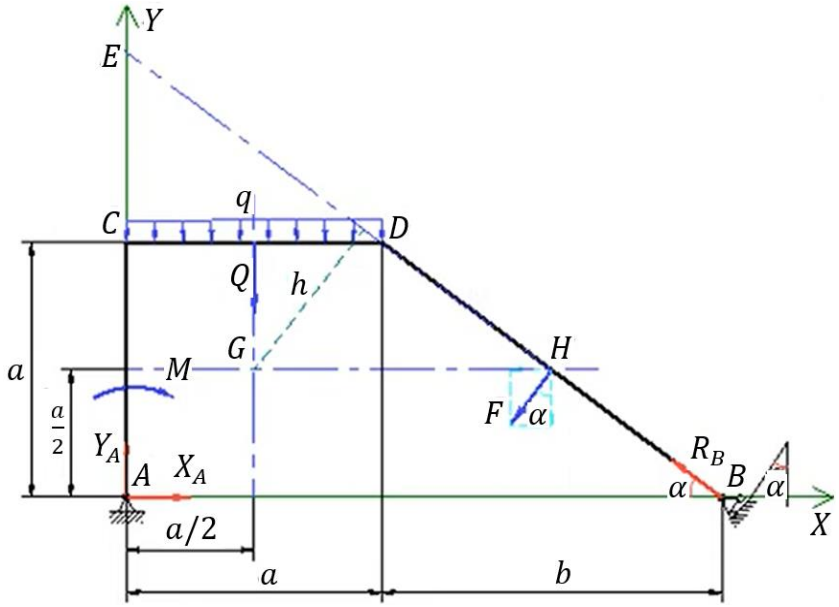


Fig. 1.6.2

The verification:

$\sum M_G(\vec{F}_k) = -M - Y_A\times\frac{a}{2} + X_A\times\frac{a}{2} - F\times\cos\alpha\times\left(\frac{a}{2}+\frac{b}{2}\right) + R_B\times h =$

$= -4 - 5.5\times\frac{6}{2} + 5.67\times\frac{6}{2} - 3\times 0.8\times\left(\frac{6}{2}+\frac{8}{2}\right) + 4.83\times 4.2 = 0,$

where: $h = \left(\frac{a}{2}+\frac{b}{2}\right)\times\sin\alpha = \left(\frac{6}{2}+\frac{8}{2}\right)\times 0.6 = 4.2$, m.

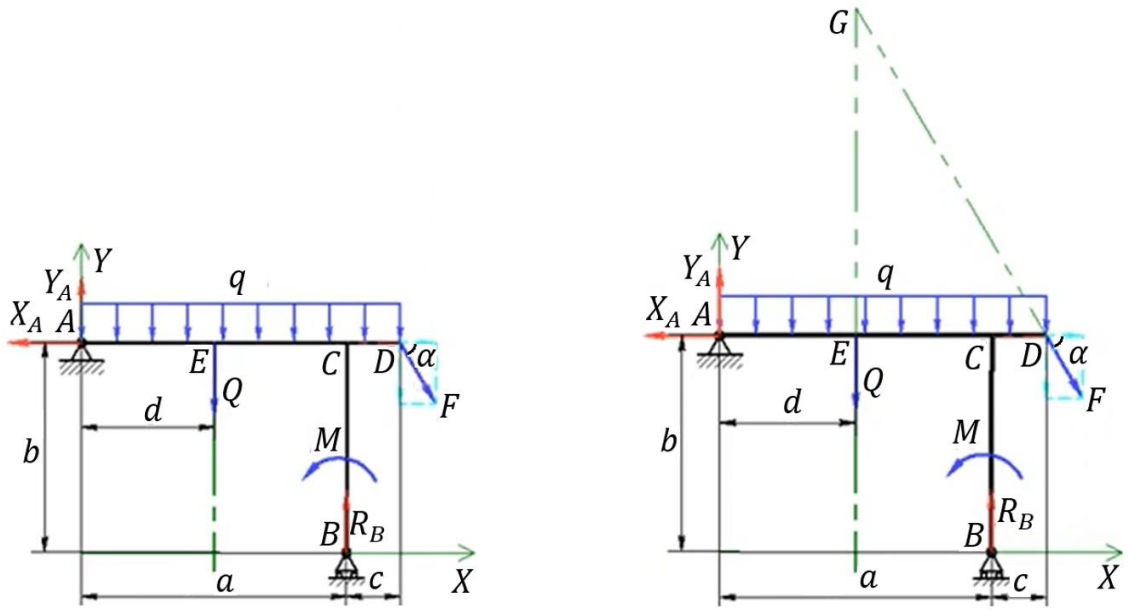
Exercise 3. Determine the reactions of the frame supports X_A, Y_A and R_{AB} (Fig. 1.6.3). Data: $F = 6$ kN, $q = 3$ kN/m, $M = 8$ kN×m, $\alpha = 60^0$, $a = 5$ m, $b = 4$ m, $c = 1$ m.

One can replace the distributed load with concentrated force:

$Q = q\times(a+c) = 3\times(5+1) = 18$, kN.

The geometric dimensions and the frame elements:

$d = \frac{a+c}{2} = \frac{5+1}{2} = 3$ m, $ED = d = 3$ m and $GE = ED\times\tan\alpha = 3\times\tan 60^0 = 5.2$, m.



(a) (b)

Fig. 1.6.3

The third form of the balance equations of the arbitrary flat force system would be used: $\sum F_{kx} = 0$, $-X_A + F \times \cos\alpha = 0$, $X_A = F \times \cos\alpha = 6 \times \cos 60^0 = 3$, kN,

$\sum M_A(\vec{F}_k) = 0$, $R_B \times a + M - Q \times d + F \times \sin\alpha \times (a + c) = 0$,

$R_B = \frac{1}{a}(-M + Q \times d - F \times \sin\alpha \times (a + c)) =$

$= \frac{1}{5}(-8 + 18 \times 3 - 6 \times \sin 60^0 \times (5 + 1)) = 15.44$, kN,

$\sum M_C(\vec{F}_k) = 0$, $-Y_A \times a + M + Q \times (a - d) - F \times \sin\alpha \times c = 0$,

$Y_A = \frac{1}{a}(M + Q \times (a - d) - F \times \sin\alpha \times c) =$

$= \frac{1}{5}(8 + 18 \times (5 - 3) - 6 \times \sin 60^0 \times 1) = 7.76$, kN.

The verification: $\sum M_G(\vec{F}_k) = M - Y_A \times d - X_A \times GE + R_B \times (a - d) =$

$= 8 - 7.76 \times 3 - 3 \times 5.2 + 15.44 \times (5 - 3) = 0.$

Questions and comments:

1.6.1. The first (basic) form of the balance equations of an arbitrary plane system of forces (the equalities to zero of the sum of projections of forces on each of two rectilinear coordinate axes and the sum of scalar moments of forces relative to any point).

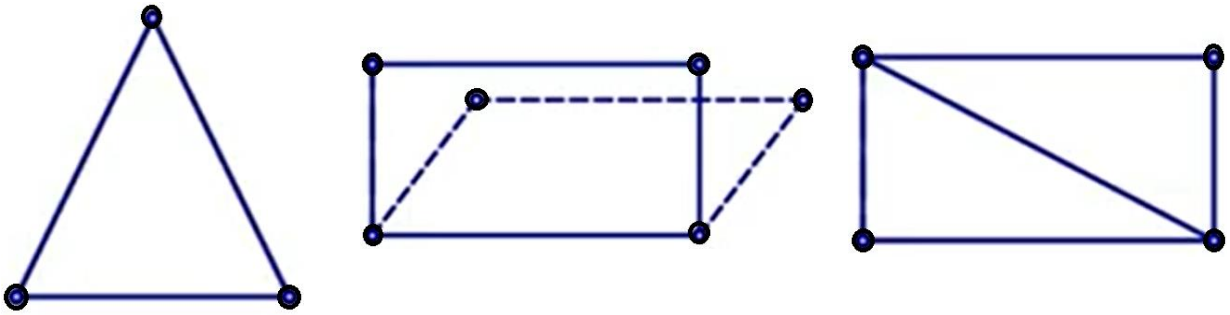
1.6.2. The second form of the balance equations of an arbitrary plane system of forces (the sum of the scalar moments of the system forces relative to any three points is equal to zero).

1.6.3. The third form of the balance equations of an arbitrary plane system of forces (the equality to zero of the sum of scalar moments of forces relative to any two points

lying in the plane of action of the forces, and the sum of the projections of these forces onto any axis of the plane not perpendicular to the line passing through the two moment points)

1.7. Hinged joints in mechanical systems and estimation of degree of freedom

The geometric variability of the system. A mechanical system is geometrically unchangeable (Fig. 1.7.1 a and c), if movements of its points are possible only due to deformation of the links (elements) of the system.



(a) (b) (c)

Fig. 1.7.1

The simplest geometrically unchangeable hinge–rod mechanical system is a triangular truss (Fig. 1.7.1 a).

In the process of kinematic analysis, it is determined whether the system is geometrically invariable. A mechanical system is geometrically variable if changing its shape is possible without deforming its elements.

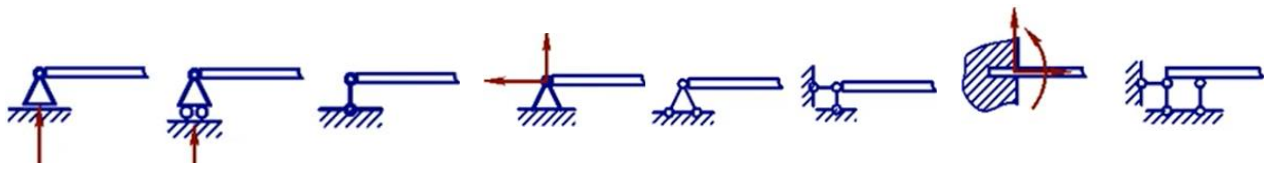
To assess the mobility of a mechanical system, the concept of degrees of freedom is used. The number of degrees of freedom is the number of possible independent movements of points of a mechanical system. The number of degrees of freedom is the minimum number of independent generalized coordinates that determine the position of the points of a mechanical system.

The concept of disk. A disk is a geometrically unchangeable part of a mechanical system. A simple or single hinge connects two discs (Fig. 1.7.2 a, b, c and f). A multiple hinge connects more than two discs (Fig. 1.7.2 d and e).

(a) (b) (c) (d) (e) (f)

Fig. 1.7.2

An elementary link (Fig. 1.7.3 a) corresponds to one support rod in the rod interpretation of support links.



(a) (b) (c)

Fig. 1.7.3

The number of degrees of freedom w of a mechanical system one can find using P.L. Chebyshev's formula: $w = 3 \times D - 2 \times T - C_0$, where D – number of disks of the mechanical system, T – number of simple hinges, C_0 – number of elementary support links.

If the number of degrees of freedom is greater than zero: $w > 0$ the mechanical system is geometrically variable. If the number of degrees of freedom is not more than zero: $w \leq 0$ the mechanical system can be geometrically unchangeable. This requirement is necessary, but not sufficient. The diagrams show examples of variable mechanical systems, the number of degrees of freedom, which is not more than zero ($w \leq 0$, Fig. 1.7.4).

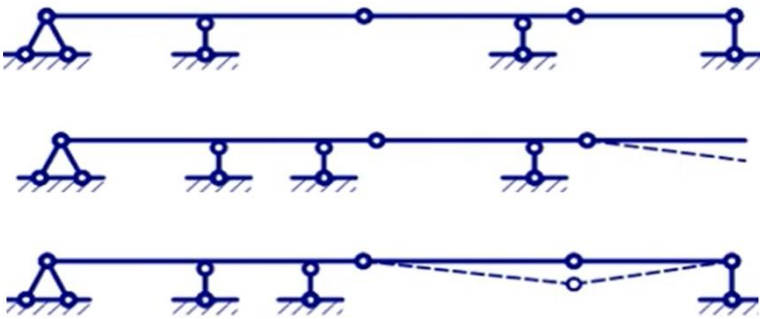


Fig. 1.7.4

The number of degrees of freedom of the truss one can find using the formula: $w = 2 \times Y - S - C_0$, where Y – number of truss nodes, S – number of the truss bars.

An example of finding the number of degrees of freedom of the truss (Fig. 1.7.5): $w = 2 \times 8 - 13 - 3 = 0$.

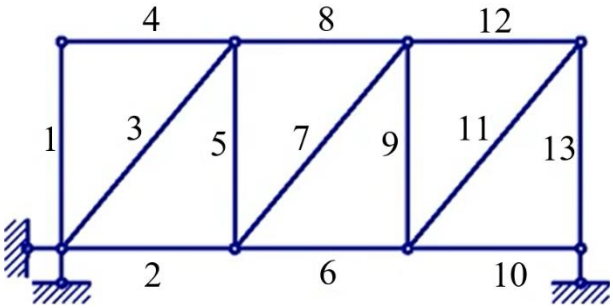


Fig. 1.7.5

Finding the number of degrees of freedom of a truss using P.L. Chebyshev's formula: $w = 3 \times 13 - 2 \times (1 \times 2 + 2 \times 2 + 3 \times 4) - 3 = 0$.

The geometrically unchangeable mechanical system is statically determinable if the forces in it are determined by balance condition. For each disk one can create three independent equations of the balance. For the system one can compose $(3 \times D)$ independent equations of the balance.

In each elementary connection there is one reaction, and the total of the reactions acting in mechanical systems is equal to: $(2 \times T + C_0)$.

Consequently, in statically definable systems the number of simple hinges T between disks and the number of elementary support links C_0 are interdependent with the number of disks: $3 \times D = 2 \times T + C_0$. In statically definable mechanical systems the number of degrees of freedom is zero: $w = 0$.

For a statically definable truss, the number of nodes is related to the number of rods: $2 \times Y = S + C_0$, wherein the number of degrees of freedom of a statically definable truss is zero: $w = 2 \times Y - S - C_0 = 0$.

Questions and comments:

1.7.1. The geometric variability of a mechanical system.

1.7.2. Determination of degrees of freedom (the mobility) of a mechanical system and assessment of its geometric immutability as a subject of kinematic analysis.

1.7.3. The concept of a disk – an element of mechanical system.

1.7.4. Hinges and elementary connections in mechanical systems.

1.7.5. The geometric invariability of a mechanical system and the number of its degrees of freedom.

1.7.6. Finding the degrees of freedom of a truss by P.L. Chebyshev's formula.

1.7.7. Statically definable and statically indefinable mechanical systems.

1.8. Static indefinability of a mechanical system, geometric invariability of the system

For the geometric immutability of the system, the condition: $w \leq 0$ is necessary, but not sufficient. Therefore, the geometric or structural analysis of the system is mandatory, wherein the correct connection of the disks to each other and to the support is checked. The variants of geometrically invariable systems that are formed on the simplest geometrically invariable system – a hinged–rod triangle are the connections:

1. A new node attaching to a disk using two rods (dyads), the axes of which do not lie on the same straight line (Fig. 1.8.1 a).

(a) (b) (c) (d) (e)

Fig. 1.8.1

2. Two disks at the crown with rods whose axes are not parallel and do not intersect at one point (Fig. 1.8.1 b).

3. Two disks with a rod and a hinge that does not lie on the axis of the rod (Fig. 1.8.1 c).

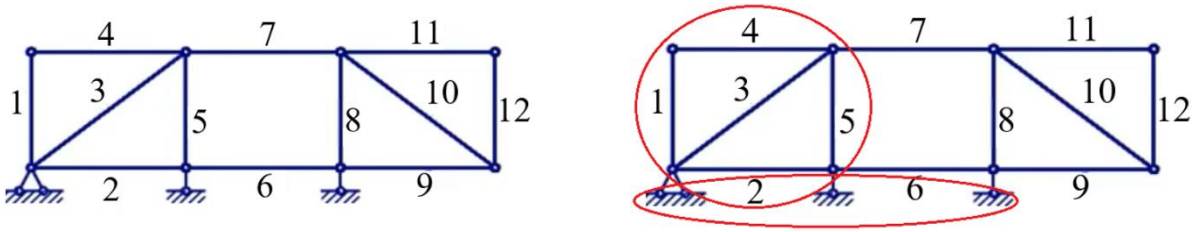
4. Three disks connection by three hinges not lying on the same straight line (Fig. 1.8.1 d).

5. Connection of three disks by six hinges, the intersection points of which do not lie on the same straight line (Fig. 1.8.1 e).

The examples of kinematic analysis of mechanical systems.

1. The system is geometrically immutable and statically determinable (Fig. 1.8.2), the number of degrees of freedom is equal to zero:

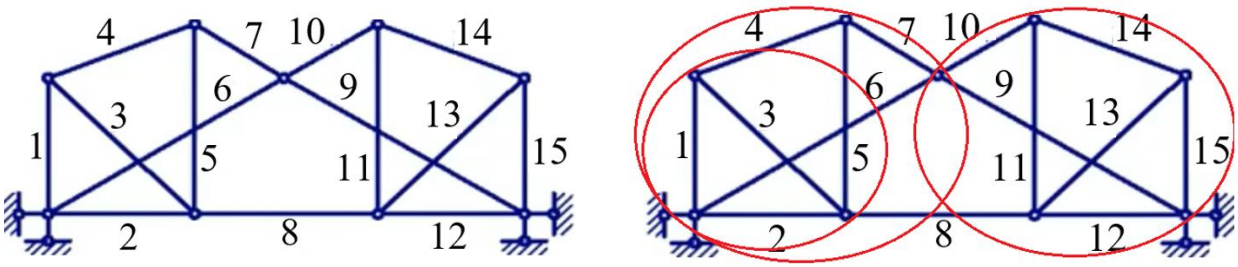
$w = 2 \times Y - S - C_0 = 2 \times 8 - 12 - 4 = 0.$



(a) (b)

Fig. 1.8.2

2. The system is geometrically unchangeable and once statically indefinable (Fig. 1.8.3), the number of degrees of freedom is equal:
$w = 2 \times Y - S - C_0 = 2 \times 9 - 15 - 4 = -1.$



(a) (b)

Fig. 1.8.3

3. The system is geometrically unchangeable and statically indefinable (Fig. 1.8.4 a), and the number of degrees of freedom according to P.L. Chebyshev's formula is equal to: $w = 3 \times D - 2 \times T - C_0 = 3 \times 3 - 2 \times 2 - 5 = 0.$

(a) (b)

Fig. 1.8.4

Questions and comments:

1.8.1. Options for geometrically invariable mechanical systems.
1.8.2. The geometric invariability and static definability.

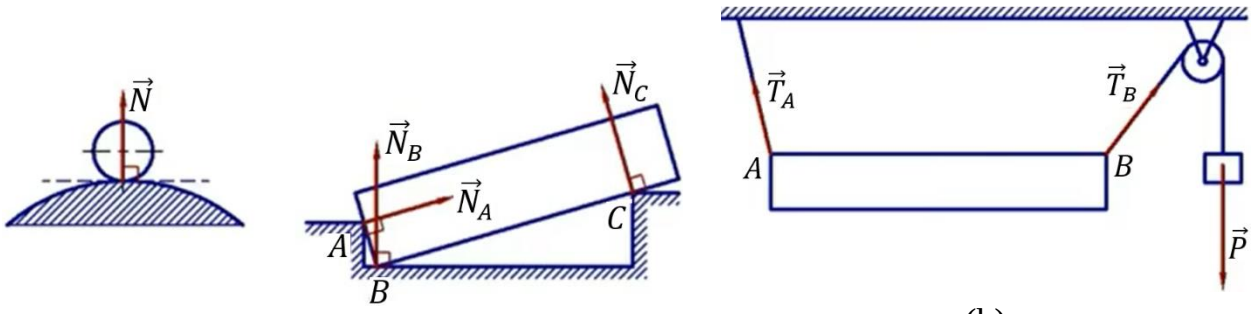
1.9. Types of connections in mechanical systems

Free and non–free solid body. A body whose movement is not subject to any restrictions is free. A free rigid body in space has six degrees of freedom. It can move translationally along the axes: $0X$, $0Y$ and $0Z$ and rotate about each of these axes.

On a plane, a free rigid body has three degrees of freedom – it can move along the axes: $0X$ and $0Y$ and rotate around any point on the plane.

The body that limits freedom of movement of another body is a connection in relation to it. A rigid body whose freedom of movement is limited by bonds is called non–free. The typical types of connections:

1. Perfect smooth surface (Fig. 1.9.1 a). The reaction of an ideal smooth surface is directed along the common normal to the contacting surfaces.

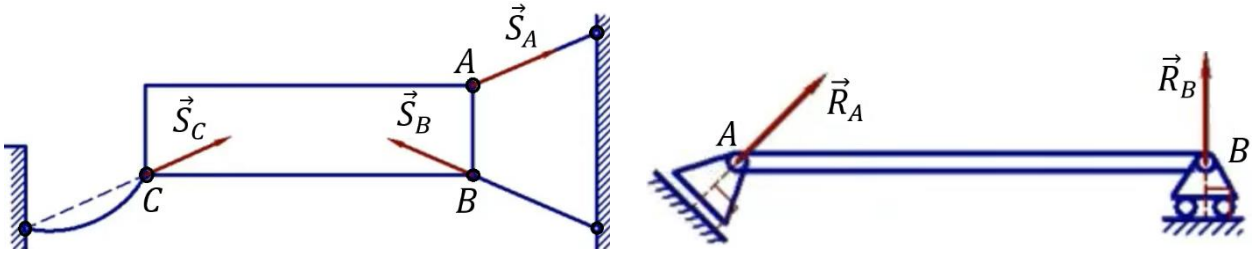


(a) (b)

Fig. 1.9.1

2. Thread (Fig. 1.9.1 b). The reaction of a stretched flexible inextensible thread is directed along the thread to the point of its suspension.

3. The weightless rod (Fig. 1.9.2 a). A weightless rod with hinges at the ends, the weight of which, compared to the perceived load, can be neglected. The reaction of a weightless rod is directed along its axis passing through the hinges at the ends.

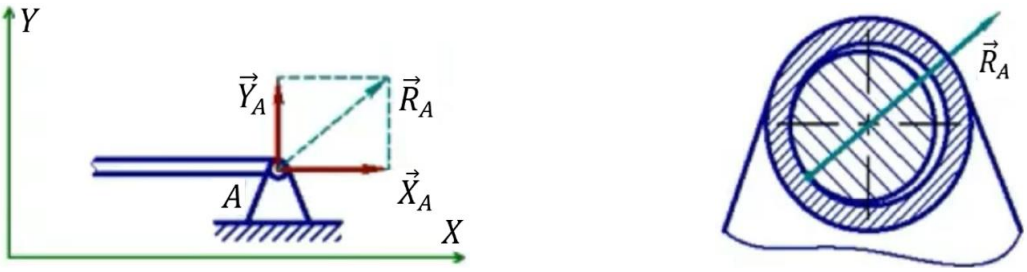


(a) (b)

Fig. 1.9.2

4. The cylindrical articulated support (Fig. 1.9.2 b) prevents the fixed point from moving perpendicular to the supporting surface, so the reaction is also perpendicular to this surface.

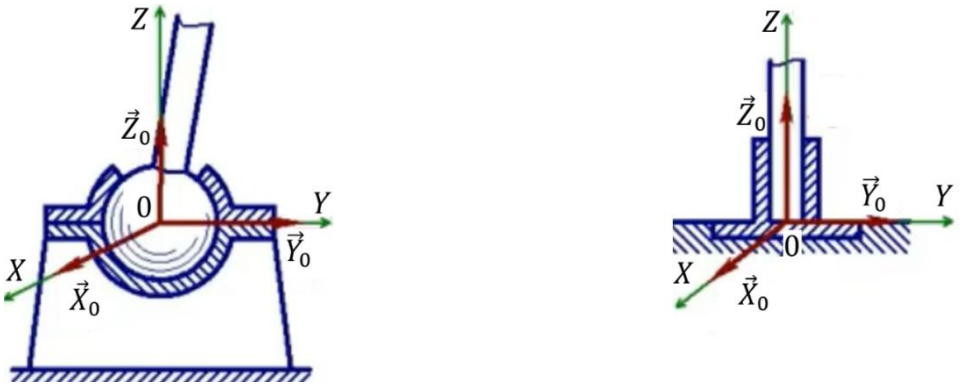
5. Cylindrical articulated – fixed support (Fig. 1.9.3). The two bodies are connected by means of a cylindrical hinge (bearing) and have the ability to rotate one body relative to the other around a common axis, denominated the hinge axis. The reaction of a hinged–fixed support passes through the axis and can have any direction in a plane perpendicular to the hinge axis.



(a) (b)

Fig. 1.9.3

6. The spherical joint and thrust bearing (Fig. 1.9.4) have only one fixed point 0. The reaction of a spherical joint can have different directions in space, but always passes through the fixed point 0.



(a) (b)

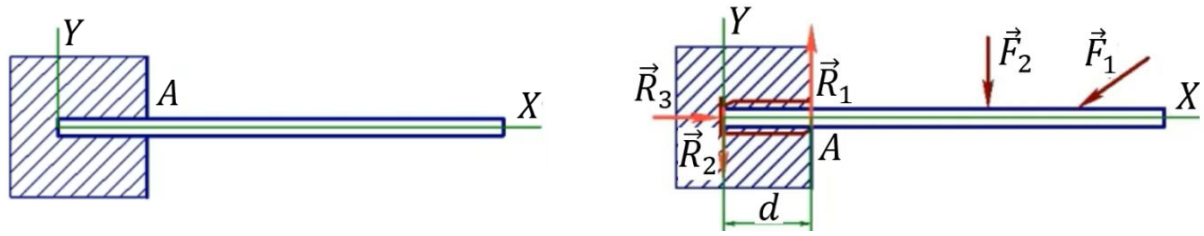
Fig. 1.9.4

Questions and comments:

1.9.1. Free rigid body.
1.9.2. A body that restricts the freedom of movement of another body, which is a connection.
1.9.3. Types of the connections.

1.10. Rigid embedding in the mechanics and its replacement by force vectors

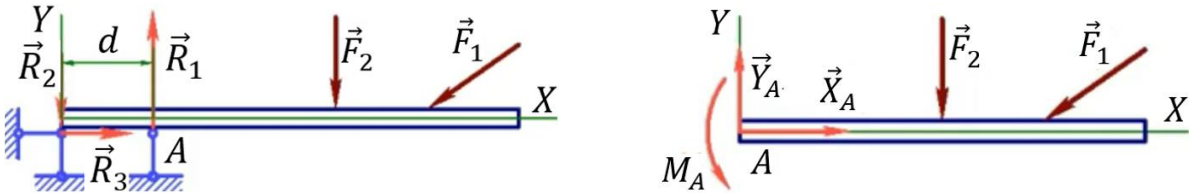
The support at the point A (Fig. 1.10.1) deprives the beam of the possibility of translational movement along the axes: $0X$ and $0Y$, as well as the possibility of rotation in the plane of action of forces.



(a) (b)

Fig. 1.10.1

The rod model of a flat rigid embedment (Fig. 1.10.2) provides the required three connections.



(a) (b)

Fig. 1.10.2

From the three reactions occurring in the model rods, one can form three rigid embedding reactions: $\vec{X}_A = \vec{R}_3$, $\vec{Y}_A = \vec{R}_1 - \vec{R}_2$ and $M_A = R_2 \times d$.

When determining the reactions in a rigid fixation, three forces serve as the unknowns: X_A, Y_A and M_A. They are found from three balance equations for a flat force system.

Questions and comments:

1.10.1. The rod model of flat rigid embedment.

1.10.2. The replacement of rigid embedment model rods with emerging reactions.

1.11. Finding the reaction of supports using balance conditions for elements of mechanical systems

Exercise. The initial conditions. The mechanical system (Fig. 1.11.1 a) consists of the straight rod AC and the rigid angle BDC, connected to each other using the hinge C. External connections imposed on the structure are hinged and fixed supports at points A and B.

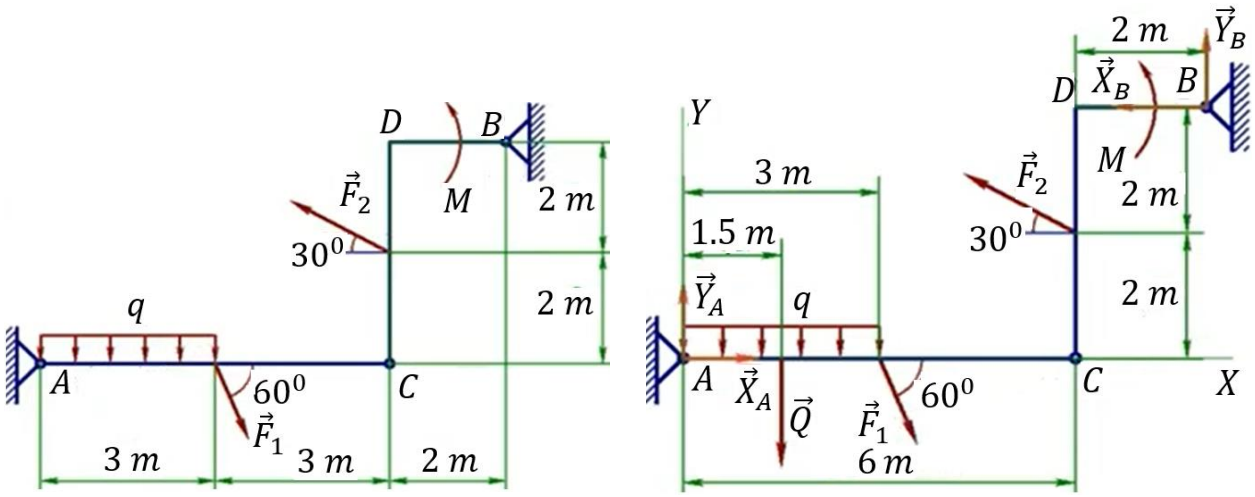


Fig. 1.11.1

The structure is subject to concentrated forces: $F_1 = 40$ kN and $F_2 = 20$ kN, the distributed load of constant intensity: $q = 10$ kN/m and the couple of forces with the moment: $M = 50$ kN×m.

The exercise assigns one to determine the reactions of the support links at points A and B, as well as the reaction in the connecting hinge C.

Equations of the balance. Determination of reactions.

One can replace (Fig. 1.11.1 b) distributed load q with a concentrated force Q and apply it in the middle of the loaded area: $Q = q \times 3 = 10 \times 3 = 30$, kN.

Using the first (basic) form of the balance equations of arbitrary flat force system:

$$\sum F_{kx} = 0,\ X_A - X_B + F_1 \times \cos 60^0 - F_2 \times \cos 30^0 = 0,$$

$$\sum F_{ky} = 0,\ Y_A + Y_B - Q - F_1 \times \sin 60^0 + F_2 \times \sin 30^0 = 0,$$

$$\sum M_A\left(\vec{F}_k\right) = 0,\ -Q \times \frac{3}{2} - F_1 \times \sin 60^0 \times 3 + F_2 \times \cos 30^0 \times 2 +$$

$$+F_2 \times \sin 30^0 \times 6 + X_B \times 4 + Y_B \times 8 + M = 0.$$

Equations of the balance for a rod AC (Fig. 1.11.2 a):

$$\sum F_{kx} = 0,\ X_A + X_C + F_1 \times \cos 60^0 = 0,$$

$$\sum F_{ky} = 0,\ Y_A + Y_C - Q - F_1 \times \sin 60^0 = 0,$$

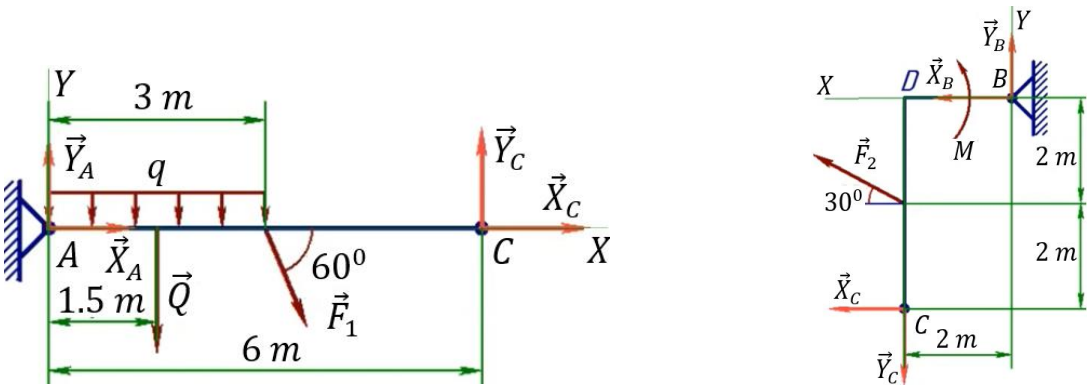
$$\sum M_A\left(\vec{F}_k\right) = 0,\ -Q \times \frac{3}{2} - F_1 \times \sin 60^0 \times 3 + Y_C \times 6 = 0,$$

Equations of the balance for the corner BDC (Fig. 1.11.2 b):

$$\sum F_{kx} = 0,\ X_B + X_C + F_2 \times \cos 30^0 = 0,$$

$$\sum F_{ky} = 0,\ Y_B - Y_C + F_2 \times \sin 30^0 = 0,$$

$$\sum M_C\left(\vec{F}_k\right) = 0,\ F_2 \times \cos 30^0 \times 2 + M + X_B \times 4 + Y_B \times 2 = 0.$$



(a) (b)

Fig. 1.11.2

Solving the system of obtained equations and finding the reactions of the support links at points A and B, as well as the reaction in the connecting hinge C:

$$Y_C = \frac{1}{6} \times \left(Q \times \frac{3}{2} + F_1 \times \sin 60^0 \times 3\right) =$$

$$= \frac{1}{6} \times \left(30 \times \frac{3}{2} + 40 \times \sin 60^0 \times 3\right) = 24.82, \text{ kN},$$

$$Y_A = -Y_C + Q + F_1 \times \sin 60^0 = -24.82 + 30 + 40 \times 0.866 = 39.82, \text{ kN},$$

$$Y_B = Y_C - F_2 \times \sin 30^0 = 24.82 - 20 \times 0.5 = 14.82, \text{ kN},$$

$$X_B = \frac{1}{4} \times (-F_2 \times \cos 30^0 \times 2 - M - Y_B \times 2) =$$

$$= \frac{1}{4} \times (-20 \times 0.866 \times 2 - 50 - 14.82 \times 2) = -28.57, \text{ kN},$$

$$X_C = -X_B - F_2 \times \cos 30^0 = 28.57 - 20 \times 0.866 = 11.25, \text{ kN},$$

$$X_A = -X_C - F_1 \times \cos 60^0 = -11.25 - 40 \times 0.5 = -31.25, \text{ kN}.$$

The found reactions of the support links at points A and B, as well as the reaction at the connecting joint C are shown in the Table 1.11.

Table 1.11

X_A, kN	Y_A, kN	X_B, kN	Y_B, kN	X_C, kN	Y_C, kN
−31.25	39.82	−28.57	14.82	11.25	24.82

The verification of the balance of the corner BDC (Fig. 1.11.2 b):

$$\sum M_B(\vec{F}_k) = M - F_2 \times \sin 30^0 \times 2 - F_2 \times \cos 30^0 \times 2 - X_C \times 4 + Y_C \times 2 =$$

$$= 50 - 20 \times 0.5 \times 2 - 20 \times 0.866 \times 2 - 11.25 \times 4 + 24.82 \times 2 = 0.$$

The verification of the balance of the entire structure (when the origin of coordinates is aligned with the point of application of the force $\vec{F}_2$, Fig. 1.11.2 b):

$$\sum M_{F2}(\vec{F}_k) = -Y_A \times 6 + X_A \times 2 + Q \times 4.5 + F_1 \times \sin 60^0 \times 3 +$$

$$+F_1 \times \cos 60^0 \times 2 + M + X_B \times 2 + Y_B \times 2 =$$

$$= -39.82 \times 6 + 31.25 \times 2 + 30 \times 4.5 + 40 \times 0.866 \times 2 + 40 \times 0.5 \times 2 +$$

$$+50 - 28.57 \times 2 + 14.82 \times 2 = 0.$$

Questions and comments:

1.11.1. Conditions of the balance for an arbitrary plane system of forces.

1.11.2. Finding reactions in support links and in the connecting hinge.

1.11.3. Using the balance equations of a mechanical system to check the found reactions in the support links and in the connecting hinge.

1.12. The balance and degrees of freedom of instantly variable mechanical systems

Signs of instantaneous variability of a mechanical system.

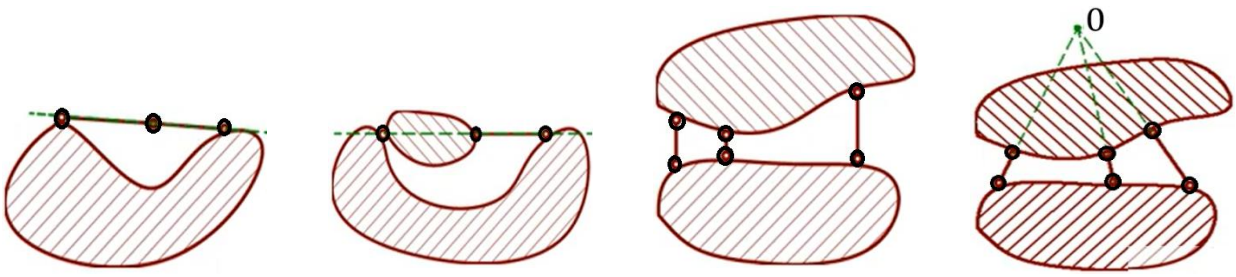
For a given system consisting of three disks (AC, BC and «ground») connected by hinges A, B and C the conditions of the balance have the form (Fig. 1.12.1): $\sum F_{kx} = 0$, $-T \times \cos\alpha + T \times \cos\alpha = 0$ and $\sum F_{ky} = 0$, $2 \times T \times \sin\alpha - F = 0$,

where: $T = \frac{F}{2 \times \sin\alpha}$ at $\alpha \to 0$, $T \to \infty$.

(a) (b)

Fig. 1.12.1

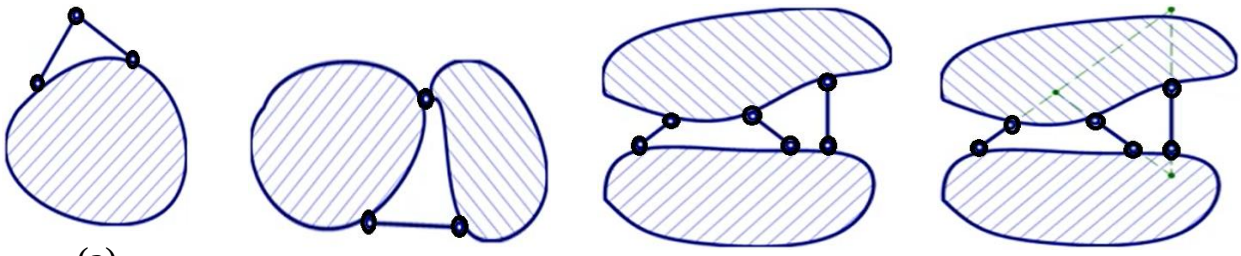
Geometric (the kinematic) sign of the instantaneous variability of the system when connecting three disks is the location of three hinges (three instantaneous centers of rotation) on one straight line (Fig. 1.12.2 a, b, c and d). For two disks, a sign of instantaneous variability of the system is the connection by three parallel rods or three rods whose axes intersect at one point.



(a) (b) (c) (d)

Fig. 1.12.2. Examples of instantaneous dual–disc mechanical systems

The examples of geometrically unchangeable instantaneously mechanical systems of two disks are shown in the diagrams (Fig. 1.12.3 a, b, c and d).



(a) (b) (c) (d)

Fig. 1.12.3

The examples of geometrically unchangeable instantaneously mechanical systems of three disks (Fig. 1.12.4 a and b) and ones changeable instantaneously (Fig. 1.12.4 c and d**)**.

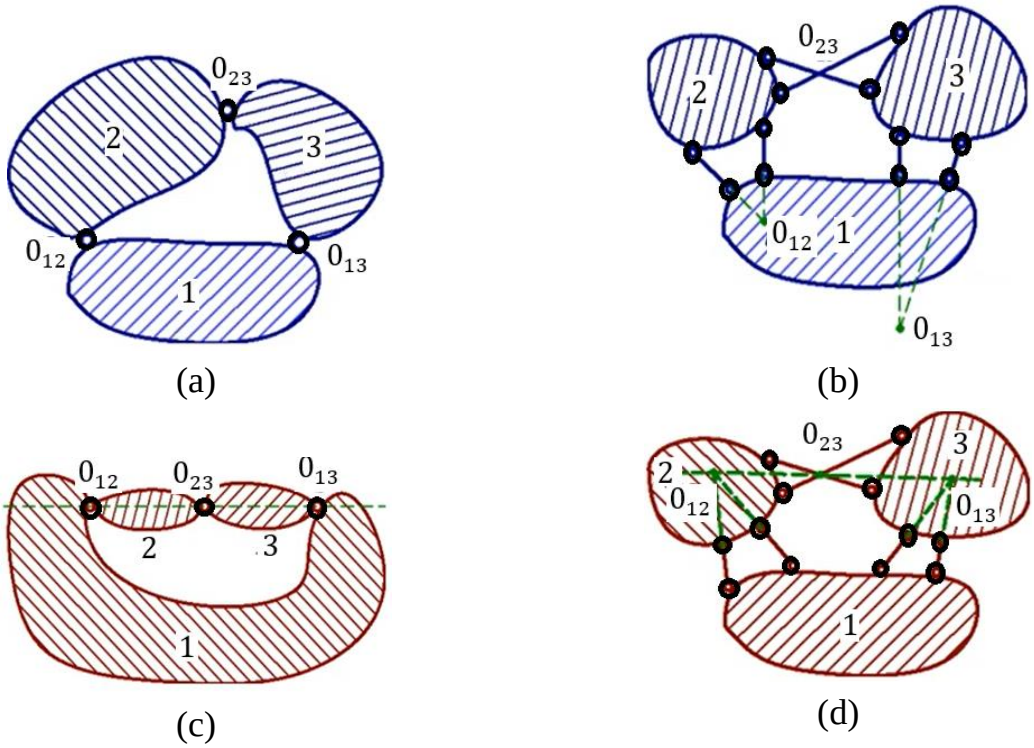


Fig. 1.12.4

Exercise 1. The assess the instantaneous variability of the system (Fig. 1.12.5).

1. According to P.L. Chebyshev's formula, the number of degrees of freedom of a mechanical system is equal to: $w = 3 \times D - 2 \times T - C_0 = 3 \times 1 - 2 \times 0 - 3 = 0$.

2. The system is instantly changeable, because two disks (the frame and the «ground») are connected using a rod and a hinge lying on the axis of the rod.

3. The equation for the sum of moments about the support A:

$\sum M_A(\vec{F}_k) = 0, F \times l - R_B \times 0 = 0.$

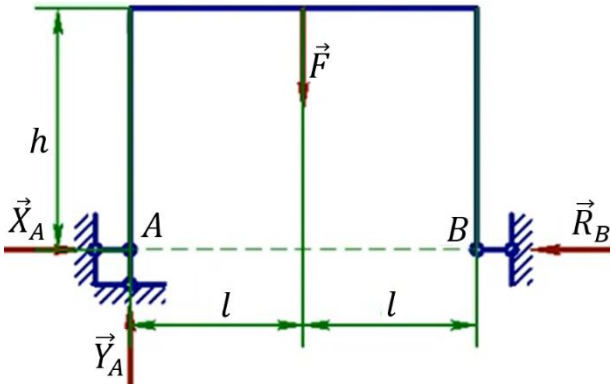
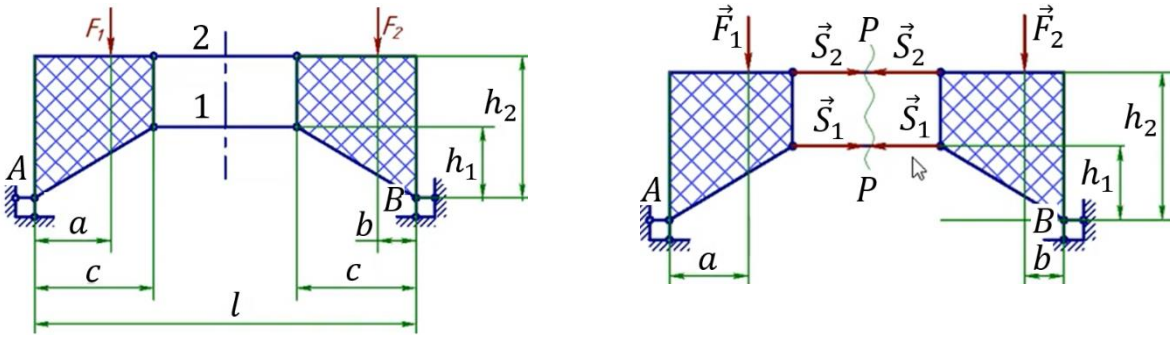


Fig. 1.12.5

The magnitude of the reaction at the point B is equal to: $R_B = \frac{F \times l}{0} = \infty$, the magnitude of the reaction is infinitely large at: $F \neq 0$ (under the assumption of the immutability of the mechanical system). When: $F = 0$ uncertainty arises: $R_B = \frac{0}{0}$.

Exercise 2. Perform a kinematic analysis of the system and evaluate the signs of its geometric instantaneous variability, if: $F_1 = 6$ kN, $F_2 = 8$ kN, $a = 3 \times b$=0.3 m and $h_2 = 2 \times h_1$.

Three disks (two of them are shaded on the diagram in section, and the third is «ground», Fig. 1.12.6) are connected by two pairs of the rods, including parallel rods 1 and 2, as well as rods in connections with the support.



(a) (b)

Fig. 1.12.6

Rods 1 and 2 are parallel, which is a geometric sign of the instantaneous variability of the mechanical system.

According to P.L. Chebyshev's formula, the number of degrees of freedom of a mechanical system is equal to: $w = 3 \times D - 2 \times T - C_0 = 3 \times 4 - 2 \times 4 - 4 = 0$.

The condition of the balance for the left side of the mechanical system is:
$\sum M_A = 0, S_1 \times h_1 + S_2 \times h_2 + F_1 \times a = 0.$

The condition of the balance for the right side of the mechanical system is:

$\sum M_B = 0, S_1 \times h_1 + S_2 \times h_2 + F_2 \times b = 0.$

Due to the instantaneous variability of the mechanical system, the contradiction arises for the conditions of the balance:

$S_1 \times h_1 + S_2 \times h_2 = -F_1 \times a = -6 \times 0.3 = -1.8$, kN,

$S_1 \times h_1 + S_2 \times h_2 = -F_2 \times b = -8 \times 0.1 = -0.8$, kN.

Questions and comments:

1.12.1. The signs of instantaneous variability of a mechanical system.

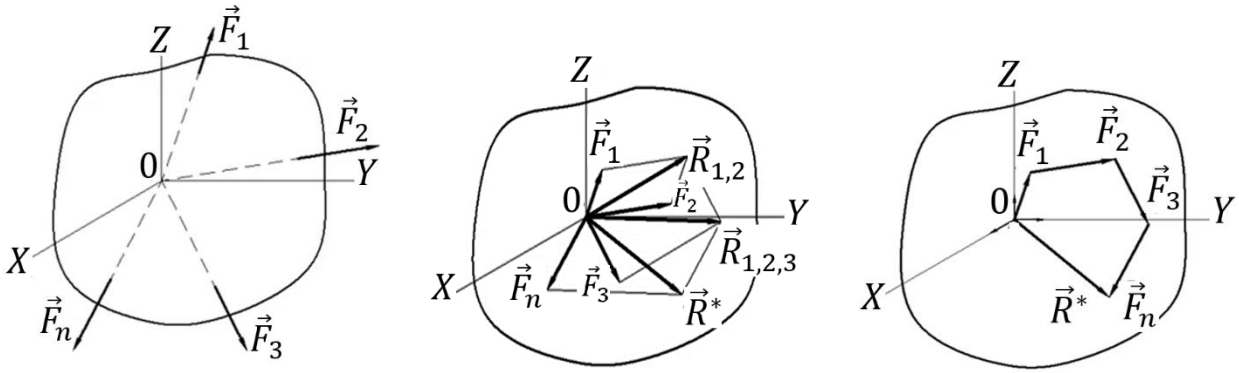
1.12.2. The examples of instantaneous dual–disc mechanical systems.

1.12.3. The examples of instantly unchangeable mechanical systems.

Chapter 2. Static calculations of elements of mechanical systems (frames and trusses), finding reactions and creating force diagrams, equation of three moments, moving load and lines of influence

2.1. The balance of a system of converging forces

Converging forces – lines of action that intersect at one point, are shown in the diagram (Fig. 2.1.1).



(a) (b) (c)

Fig. 2.1.1

The bringing of system of converging forces $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$ to their resultant: $\vec{R}_{1,2} = \vec{F}_1 + \vec{F}_2$, $\vec{R}_{1,2,3} = \vec{R}_{1,2} + \vec{F}_3 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ and $\vec{R}^* = \sum_{k=1}^{n} \vec{F}_k$.

Projections (R_x^*, R_y^* and R_z^*) on the coordinate axes of the resultant $\vec{R}^*$ of the converging forces are equal: $R_x^* = \sum_{k=1}^{n} F_{kx}$, $R_y^* = \sum_{k=1}^{n} F_{ky}$ and $R_z^* = \sum_{k=1}^{n} F_{kz}$.

The magnitude of the resultant of converging forces R^* is determined by its projections (R_x^*, R_y^* and R_z^*) as: $R^* = \sqrt{(\sum_{k=1}^{n} F_{kx})^2 + (\sum_{k=1}^{n} F_{ky})^2 + (\sum_{k=1}^{n} F_{kz})^2}$.

The angles of the resultant R^* with respect to the coordinate axes $0XYZ$ (that is with respect to the unit vectors of the coordinate system: $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$) are characterized by direction cosines: $\cos(\vec{R}^*, \vec{\imath}) = \frac{R_x^*}{R^*}$, $\cos(\vec{R}^*, \vec{\jmath}) = \frac{R_y^*}{R^*}$ and $\cos(\vec{R}^*, \vec{k}) = \frac{R_z^*}{R^*}$.

Condition of the balance of the convergent force system. The scalar notation of the equilibrium condition equation. For the balance of the converging force system it is necessary and sufficient that the magnitude of the resultant forces be equal to zero:

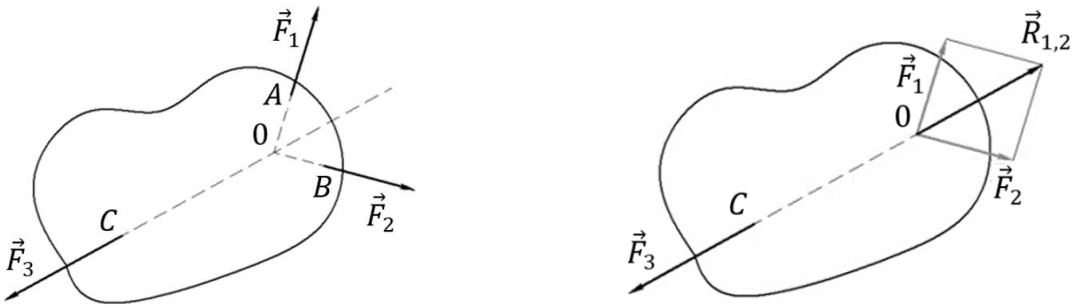
$$R^* = \sqrt{R_x^{*2} + R_y^{*2} + R_z^{*2}} = \sqrt{(\sum_{k=1}^{n} F_{kx})^2 + (\sum_{k=1}^{n} F_{ky})^2 + (\sum_{k=1}^{n} F_{kz})^2} = 0.$$

For the balance of the spatial system of converging forces in scalar form, it is necessary and sufficient that the totals of the projections of forces on each of the three axes of the rectangular Cartesian coordinate system be equal to zero:

$\sum_{k=1}^{n} F_{kx} = 0$, $\sum_{k=1}^{n} F_{ky} = 0$ and $\sum_{k=1}^{n} F_{kz} = 0$.

For the balance of a flat system of converging forces in the analytical form it is necessary and sufficient that the totals of the projections of forces on each of the two axes of the Cartesian coordinate system be equal to zero: $\sum_{k=1}^{n} F_{kx} = 0$ and $\sum_{k=1}^{n} F_{ky} = 0$.

Three Forces Theorem. If a system of three non–parallel forces lying in the same plane is in the balance, then the lines of action of these forces intersect at one point (Fig. 2.1.2).



(a) (b)

Fig. 2.1.2

Let the system of three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be in the balance.

The lines of action of the forces $\vec{F}_2$ and $\vec{F}_3$ intersect at the point 0.

Let's move these of the forces along the lines of their action to the point 0, add them according to the parallelogram rule and replace by their resultant: $\vec{R}_{1,2} = \vec{F}_1 + \vec{F}_2$.

The equilibrium of a body under the action of two forces $\vec{F}_3$ and $\vec{R}_{1,2}$ is possible only if: $\vec{F}_3 = -\vec{R}_{1,2}$. Thus, the line of action of force $\vec{F}_3$ necessarily passes through the point 0.

Questions and comments:

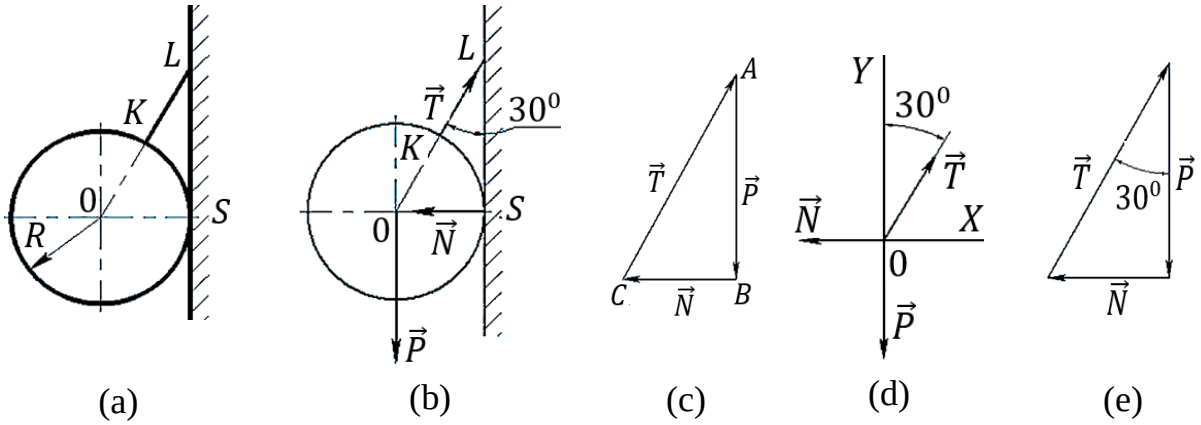
2.1.1. The reducing a system of converging forces to their resultant, the projection of resultant on the coordinate axes, the magnitude of resultant, the cosines of the direction angles.

2.1.2. The balance condition for a convergent system of forces.

2.1.3. Three Forces Theorem.

2.2 Considering the balance of converging forces when analyzing statics issues

Exercise. An ideally smooth ball of arbitrary radius R and weight: $P = 60$ kN is attached to a vertical wall (Fig. 2.2.1 a) at the point L, using a thread, and rests on the smooth surface of the wall at the point S. Wherein the length of the thread is equal to the radius of the ball: $KL = R$. The exercise assigns one to determine the tension force of the thread and the pressure force of the ball on the wall.



(a) (b) (c) (d) (e)

Fig. 2.2.1

Under the influence of three forces, a body is in balance only when the lines of action of the forces lie in the same plane and intersect at one point – in accordance with the Three Forces Theorem. The center of the ball serves as such a point 0. The considering system of forces is convergent. Consequently, in the considered system of forces acting on the ball, the points of application and the directions of all three forces are known.

The force of the ball's weight $\vec{P}$ is applied at its geometric center 0 and is directed vertically downward (Fig. 2.2.1 b). The reaction of the thread $\vec{T}$ is applied at the point K of fastening the thread to the ball and is directed along the thread to the point of suspension L. The reaction $\vec{N}$ from the side of the smooth wall is applied at the point S of contact of the ball with the wall and is directed radially to the center of the ball 0. In a right angled triangle $\Delta 0SL$ the length of the hypotenuse: $0L = 2 \times R$, and the length of the leg: $0S = R$. Thus, the leg is half the size of the hypotenuse, and the angle: $0LS = 30^0$.

Graphical solution. The geometric condition of balance of the converging force system is a closed force polygon (triangle) of these forces (Fig. 2.2.1 c).

To create a force triangle from an arbitrary point A on a scale (for example, 1 mm of vector length corresponds to 1 kN force) is plotted vertically downwards vector of the gravity force $\vec{P}$ of the ball.

The length of the vector is 60 mm, which corresponds to the strength 60 kN.

The end of vector $\vec{P}$ corresponds to a point B. From the beginning and the end of this vector, the straight lines are drawn corresponding to the lines of action of other forces: $\vec{T}$ and $\vec{N}$. They will intersect at the point C, which serves as the third vertex of the closed force triangle.

The end of each previous vector coincides with the beginning of the next one.

The directions of vectors of the required forces are uniquely determined by the need to construct a closed force triangle. The reaction $\vec{N}$ from the side of the smooth wall is directed horizontally to the left. The reaction of the thread $\vec{T}$ is directed along the thread upward to the suspension point, which is at an angle of 30^0 to the vertical. By measuring the lengths of vectors in millimeters with a ruler one can find approximate values of the magnitudes of the reactions sought: $\vec{T} = 69$ kN and $\vec{N} = 35$ kN.

The analytical method of solution (Fig. 2.2.1 d). One needs to move the active ones of the force along their lines of action to the point of «descent» (the center of the suspended ball 0) and align the origin of the plane rectangular coordinate system $0XY$ with this point.

For a converging force system, the analytical condition of the balance is the equality to zero of the totals of projections of the acting forces on the coordinate axes.

Equations of the balance are: $\sum F_{kx} = 0,\ T \times \sin 30^0 - N = 0$ and $\sum F_{ky} = 0,$ $T \times \cos 30^0 - P = 0$. The solving of equations has the form: $T = \frac{P}{\cos 30^0} = 69.28$, kN and $N = P \times \tan 30^0 = 36.64$, kN.

Graphic–analytical method of solution (Fig. 2.2.1 e). The condition of balance of the converging force system is used (as and for the graphical method), that is the force triangle must be closed. The polygon (triangle) has a leg and an acute angle 30^0. One can find the force values using a graphical–analytical method: $T = 69.28$ kN, $N = 36.64$ kN.

Analysis of the applied methods for solving the statics problem shows three approaches available:

1. The graphical approach is clear and simple but is approximate and depends on the accuracy of measurements and graphic drafts.

2. The analytical solution is appropriately universal and accurate; there are no restrictions on the number of active forces in the system.

3. The graphic–analytical approach is accurate and visual, but in practice it can only be used for the systems with a small number of forces: three (with a force triangle) or four (with a force parallelogram).

Questions and comments:

2.2.1. The use of the balance conditions of three forces lying in the same plane to find the tension of the thread and the pressure on a smooth wall when suspending a ball.

2.2.2. The graphical solution method and a converged force polygon to determine the tension of the thread and the pressure of the ball on the wall.

2.2.3. Application of a graphical method for solving and transferring of forces along the lines of their action to the «vanishing point» (the center of a suspended ball).

2.2.4. Using the graphic–analytical method for finding the forces and the reactions and the conditions for the closure of the force triangle.

2.3. Using the three–force theorem in solving statics problems

Three forces theorem. If a rigid body is in balance under the action of three non–parallel forces, then all three of the forces lie in the same plane and their lines of action intersect at one point.

Exercise. A two–body system is specified (Fig. 2.3.1). Weightless three–hinge arch loaded with horizontal force: $F = 10$ kN. One needs to determine the reactions in supports: A and B.

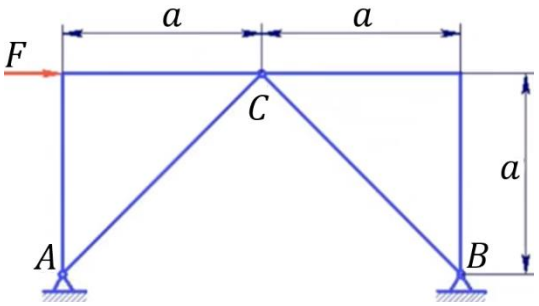
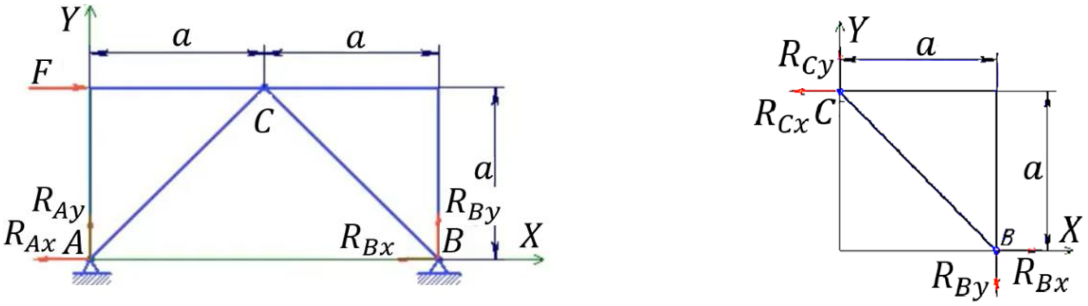


Fig. 2.3.1

The analytical method of solving using the conditions of balance.

Equations of the balance for the entire arch as a whole (Fig. 2.3.2 a), for example, relative to the support A: $\sum F_{kx} = 0$, $\sum F_{ky} = 0$ and $\sum M_A(\vec{F}_k) = 0$.



(a) (b)

Fig. 2.3.2

One equation of the balance for half an arch (Fig. 2.3.2 b), for example regarding a hinge C, is: $\sum M_C(\vec{F}_k) = 0$.

Graphic–analytical method for solving the problem (Fig. 2.3.3). A converged force triangle is drafted graphically.

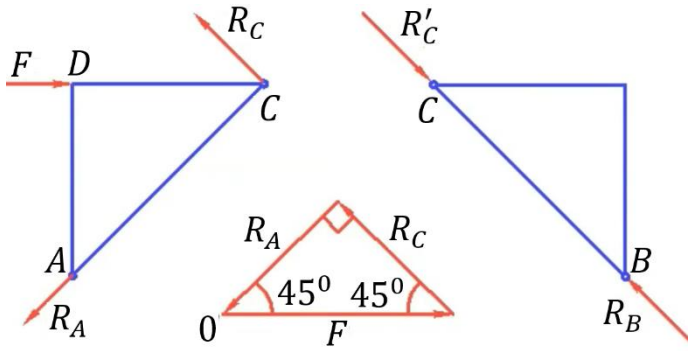


Fig. 2.3.3

When using the graphic–analytical method, there is no need to project forces onto coordinate axes, which would be necessary to compile the balance equations when using the analytical method for determining support reactions.

When finding the support reactions, the following axioms and theorem are used.

The axiom «of two forces»: $R_B = R_C$.

The axiom «of the equality of reaction and action forces»: $R'_C = R_C$.

According to the Three Forces Theorem, the line of action of the force R_A, which close the force triangle, acts along the edge AC of the arch. The magnitude of the reactions of supports A and B one can determine by closed force right triangle as:

$R_A = R_B = F \times \cos 45^0 = F \times \sin 45^0 = 10 \times \frac{\sqrt{2}}{2}$, kN.

Questions and comments:

2.3.1. Three Forces Theorem.

2.3.2. Analytical and graphic–analytical methods for solving the equations of forces balance on a plane.

2.3.3. The axiom about two forces.

2.3.4. The axiom on the equality of reaction and action forces.

2.4. Calculation of flat trusses using the section method

The trusses are the structures consisting of straight rods connected at the ends using hinge joints.

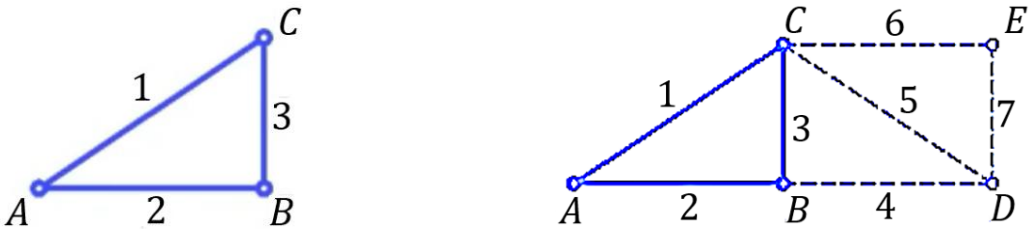
The assumptions made when calculating forces in truss elements:

(a) the external forces are applied only at the truss nodes;

(b) when calculating the truss, one can neglect the mass of the rods and the weight, distributed over the rod;

(c) the truss nodes are ideal hinges: there are no gaps in the hinges, one can neglect friction in the hinges.

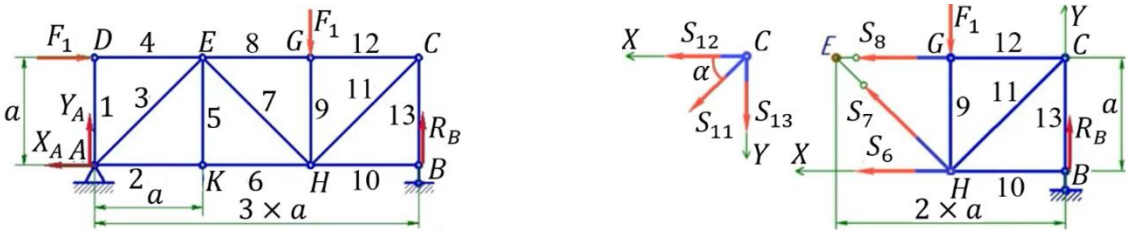
The principles of the truss creating. The simplest truss is triangular, consisting of three rods connected by three hinges (Fig. 2.4.1 a). Each new hinge joint is connected by two new rods (Fig. 2.4.1 b). The relationship between the number of rods s and the number of nodes u of the truss is: $s - 3 = 2 \times (u - 3)$.



(a) (b)

Fig. 2.4.1

Calculation methods of flat trusses. The calculation of a truss consists in determining the forces in its rods (Fig. 2.4.2). The forces in the rods are determined by the method of nodes cutting or the method of sections (the Ritter method, Fig. 2.4.2 b).



(a) (b)

Fig. 2.4.2

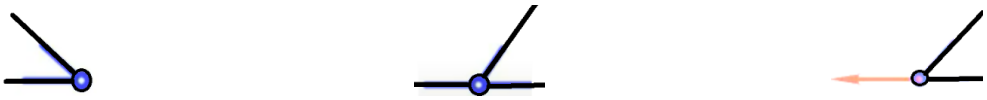
Zero rods of trusses – the forces in which are equal to zero.

Zero rods meet the conditions:

(1) if in an unloaded node, two rods that do not lie on the same line converge, then both rods are zero (Fig. 2.4.3 a);

(2) if three rods converge in an unloaded node, two of which lie on the same straight line, then the third rod, not lying on this straight line, is zero (Fig. 2.4.3 b);

(3) if two rods that do not lie on the same straight line converge in a loaded node, and the external load is directed along one of the rods, then the second rod is zero (Fig. 2.4.3 c).



(a) (b) (c)

Fig. 2.4.3

Exercise 1. Find all zero bars for a given truss (Fig. 2.4.4).

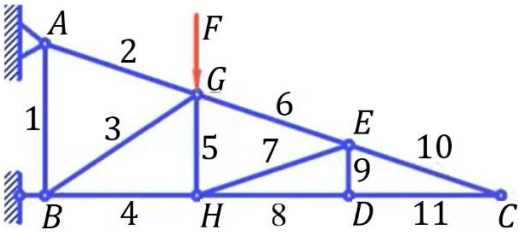
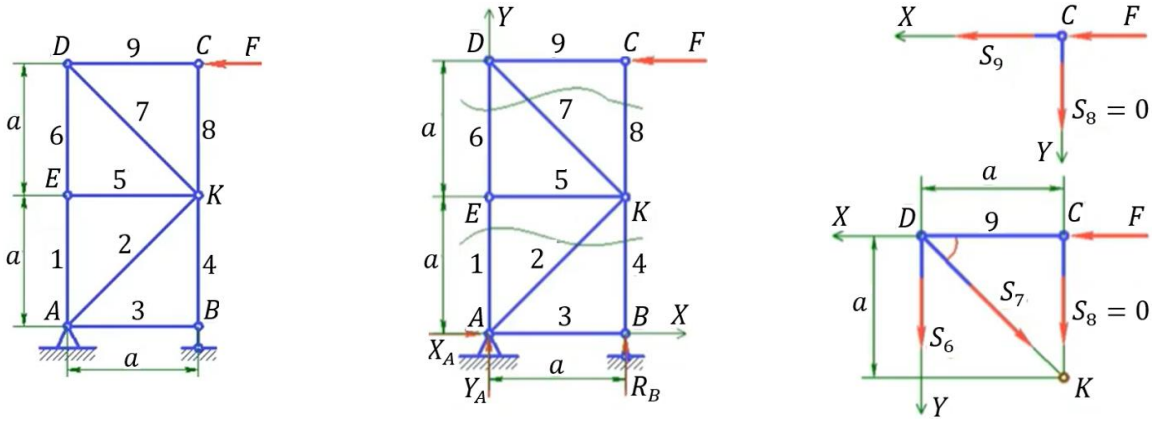


Fig. 2.4.4

Zero rods of truss, the forces in which are equal to zero: 4, 5, 6, 7, 8, 9, 10 and 11.

Exercise 2. Determine the forces in the truss bars for a given external force: F =5 kN (Fig. 2.4.5 a).



(a) (b) (c)

Fig. 2.4.5

The determination of support reactions. Equations of the balance of an integral truss relative to the supports (Fig. 2.4.5 b), for example, relative to the support A, are:

$\sum F_{kx} = 0$, $X_A - F = 0$, $X_A = F = 5$ kN,

$\sum M_B(\vec{F}_k) = 0$, $F \times 2 \times a - Y_A \times a = 0$, $Y_A = 2 \times F = 10$, kN,

$\sum M_A(\vec{F}_k) = 0$, $F \times 2 \times a + R_B \times a = 0$, $R_B = -2 \times F = -10$, kN.

Finding forces in the rods by cutting out the knots. Two equations of the balance for a node C: $\sum F_{kx} = 0$, $F + S_9 = 0$, $S_9 = -F = -5$, kN and $\sum F_{ky} = 0$, $S_8 = 0$.

Equations of the balance for the cut–off part (Fig. 2.4.5 c):

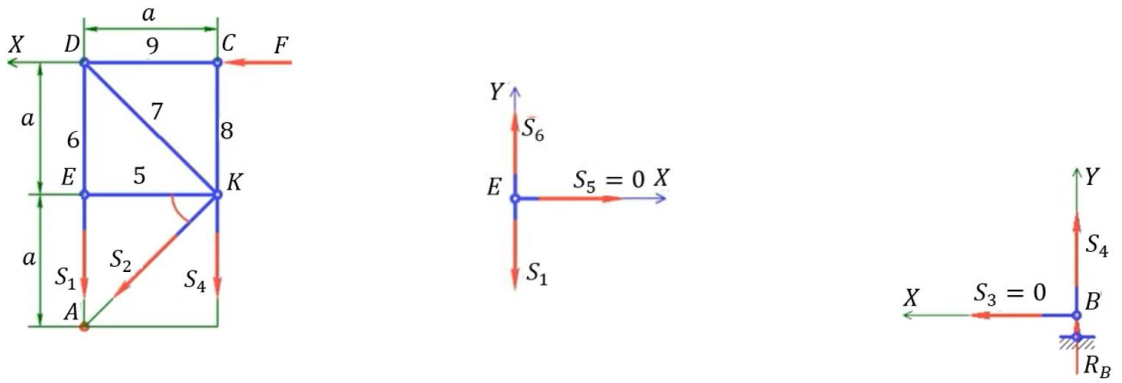
$\sum M_K = 0$, $F \times a + S_6 \times a = 0$, $S_6 = -F = -5$, kN,

$\sum F_{kx} = 0$, $F - S_7 \times \cos 45^0 = 0$, $S_7 = \frac{F}{\cos 45^0} = 5\sqrt{2}$, kN,

$\sum M_D = 0$, $-S_8 \times a = 0$, $S_8 = 0$.

The conditions of balance relative to the node E (Fig. 2.4.6 b):

$S_5 = 0$, $S_6 - S_1 = 0$, $S_6 = S_1 = -5$, kN.



(a) (b) (c)

Fig. 2.4.6

For second cut–off part one defines conditions of the balance (Fig. 2.4.6 a):

$\sum F_{kx} = 0$, $F + S_2 \times \cos 45^0 = 0$, $S_2 = \frac{-F}{\cos 45^0} = -5\sqrt{2}$, kN,

$\sum M_A = 0$, $F \times 2 \times a - S_4 \times a = 0$, $S_4 = 2 \times F = 10$, kN,

$\sum M_K = 0$, $F \times a + S_1 \times a = 0$, $S_1 = -F = -5$, kN.

The conditions of the balance relative to the node B (Fig. 2.4.6 c):

$S_3 = 0$, $S_4 + R_B = 0$, $S_4 = -R_B = -5$, kN.

Questions and comments:

2.4.1. The truss is a rigid structure consisting of straight rods connected at the ends using hinge joints.

2.4.2. The assumptions made when calculating forces in truss elements and the calculation methods used.

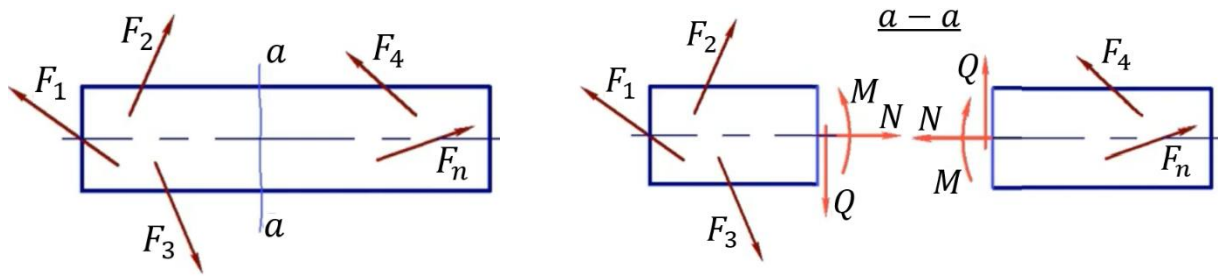
2.4.3. Zero rods of trusses – the forces in which are equal to zero.

2.4.4. Finding support reactions using the balance equations for a complete truss.

2.4.5. Finding forces in the rods by cutting out the nodes.

2.5. Calculation of statically definable mechanical systems using the method of sections

Methods of the internal forces determining. The resultants of normal and tangential mechanical stresses in a plane stress state one can replace (Fig, 2.5.1) by two forces – N and Q and concentrated moment M. The indicated efforts are denominated the internal ones: N – the longitudinal force, Q – the transverse the force and M – the bending moment.



(a) (b)

Fig. 2.5.1

The internal forces can be found by the acting load, the geometric characteristics and the elasticity indicators of the body. Three main methods are used in the structural mechanics to determine forces in statically definable rod systems: the method of sections, the method of replacing the connections and the kinematic method. The first two methods are static, that is are related to the use of equations of the balance.

The section method is the most widely used due to its clarity and the ease of use. The section method is that after the determining of support reactions, the mechanical system (Fig. 2.5.1 a) is cut virtually into two parts and one part is separated by replacing it action with the internal forces (Fig. 2.5.1 b).

Internal forces are determined by the conditions of balance of the mechanical system parts. If one rod falls into the section, then the forces in it are found according to the known rules of the mechanics.

When drawing up the balance equations for internal forces, the rule of signs is used, and the conditions of symmetry and pairing of internal forces in an element that is in balance are also taken into account, in particular, in that element (Fig. 2.5.2), which is located between the separated parts (Fig. 2.5.1 b).

When the elasticity of the material is uniformly distributed over the cross–section, the bending moment M is equal to the sum of the moments relative to the center of the cross–sectional area from all forces acting on the remaining part. The transverse (the shear) force Q is equal to the sum of the projections of the same forces onto the axis directed parallel to the tangential stresses acting in the section. The longitudinal force N is

equal to the sum of the projections of the same forces onto the axis directed perpendicular to the cross–sectional plane.

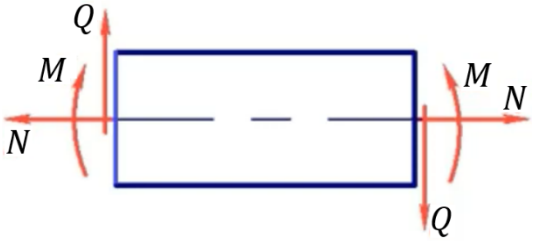


Fig. 2.5.2

The internal forces in frame sections (Fig. 2.5.3 and 2.5.4 a and b) can be found by replacing the links with reactions acting on the part of the replaced links.

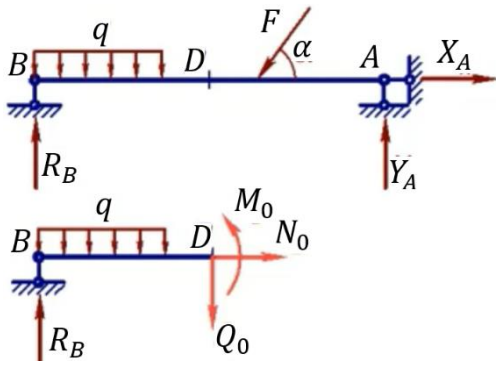


Fig. 2.5.3

For the purpose of systematization, when drawing up and solving the balance equations, the methods of section approach are used: the moment point method, the method of the projections and the method of cutting out the nodes.

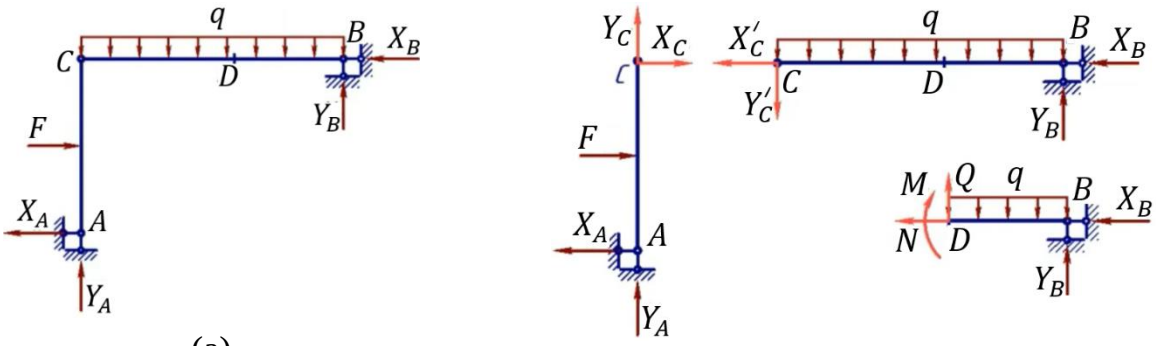
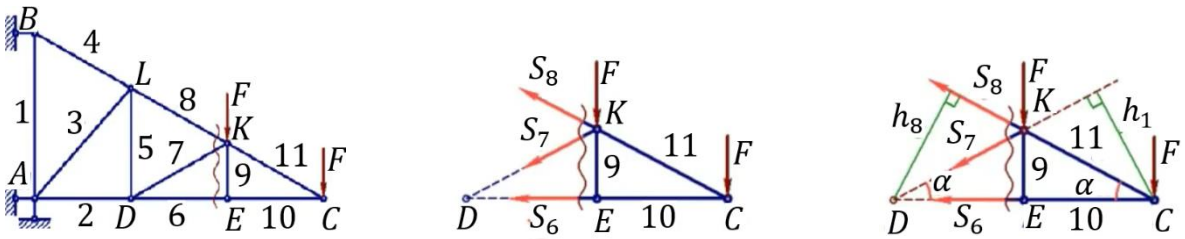


Fig. 2.5.4

The moment point method is applied if the lines of action of all internal forces in the section, except for one of the required forces, intersect at one point. This point is denominated the moment point.

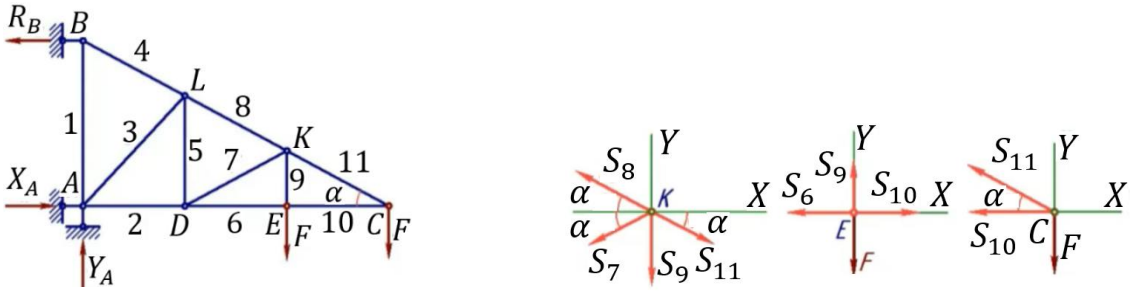
The required force is determined from the equation of the moments of forces total acting on the cut–off part relative to the moment point (Fig. 2.5.5).



(a) (b) (c)

Fig. 2.5.5

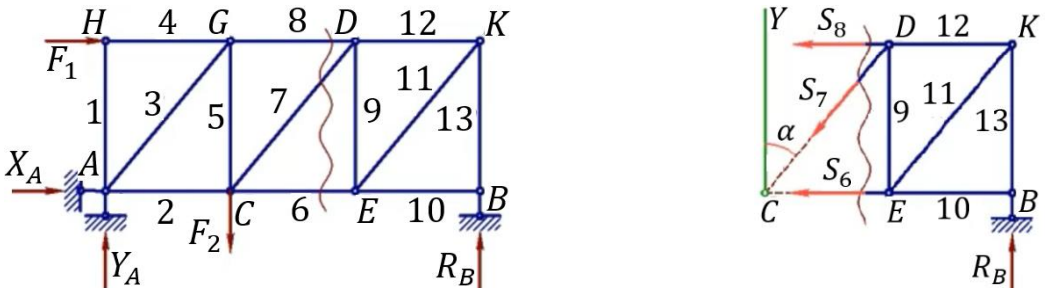
The method of cutting nodes (Fig. 2.5.6) would be used if the forces entering the section form the convergent system of forces and the analytical form of the balance condition one can represent by two equations: $\sum F_{kx} = 0$ and $\sum F_{ky} = 0$. These equations of the balance can be solved if the number of unknowns at a node is no more than two.



(a) (b)

Fig. 2.5.6

The projection method (Fig. 2.5.7) is used in the case when one can select an axis with respect to which all unknown forces in the selected section except the force sought will be perpendicular to this axis. Then the indicated force sought is determined by the equation of the total of projections onto this axis of all forces acting on the cut–off part.



(a) (b)

Fig. 2.5.7

Questions and comments:

2.5.1. The methods for determining of the internal forces in static calculations of mechanical systems.

2.5.2. The replacement of mechanical stresses in a plane stress state with concentrated forces and moment.

2.5.3. Three methods for determining forces in statically definable rod systems: the method of sections, the method of replacing connections and the kinematic method.

2.5.4. The rule of signs and the conditions of symmetry and pairing of internal forces in the elements of mechanical structures that are in the balance.

2.5.5. Methods used in drawing up and solving the balance equations: the moment point method, the projection method and the node cutting method.

2.6. Determination of forces in truss rods using the Ritter method

Exercise 1. Given, the magnitude of the external load: $F_1 = 7$ kN and $F_2 = 11$ kN, dimensions of the truss elements (Fig. 2.6.1): $d = 2$ m and $h = 4$ m. One has to determine the reactions of supports A and B and forces in the rods.

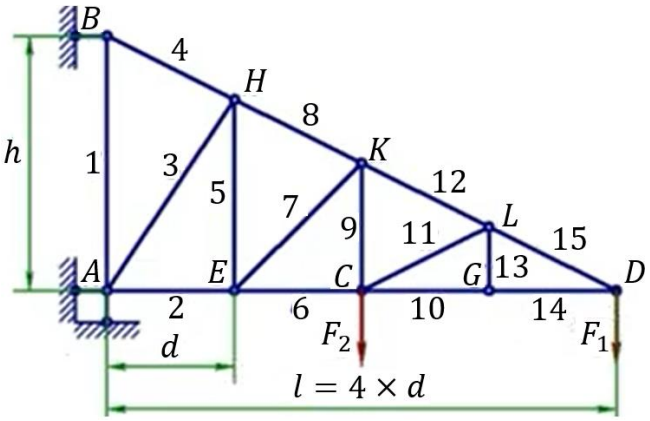


Fig. 2.6.1

Determination of support reactions (Fig. 2.6.2): X_A, Y_A and R_B.

The conditions of the balance of entire truss are:

$\sum F_{kx} = 0$, $X_A - R_B = 0$, $X_A = R_B$,

$\sum F_{ky} = 0$, $Y_A - F_1 - F_2 = 0$, $Y_A = F_1 + F_2 = 7 + 11 = 18$, kN,

$\sum M_A = 0$, $R_B \times h - F_1 \times 4 \times d - F_2 \times 2 \times d = 0$,

$X_A = R_B = \frac{F_1 \times 4 \times d + F_2 \times 2 \times d}{4} = \frac{7 \times 4 \times 2 + 11 \times 2 \times 2}{4} = 25$, kN.

The verification condition of the balance of entire truss is as follows:

$$\sum M_K = 0,\ R_B \times \frac{h}{2} + X_A \times \frac{h}{2} - Y_A \times 2 \times d - F_1 \times 2 \times d =$$

$$= 25 \times \frac{4}{2} + 25 \times \frac{4}{2} - 18 \times 2 \times 2 - 7 \times 2 \times 2 = 0,$$

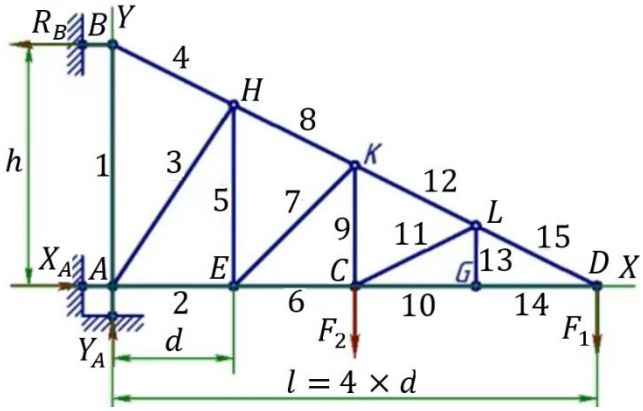


Fig. 2.6.2

Determining of the truss geometric elements (Fig. 2.6.3).

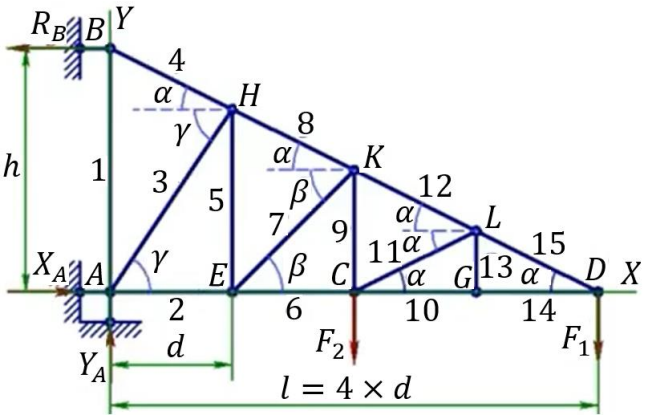


Fig. 2.6.3

The geometric elements of the truss are:

$GL = 1$ m, $EC = CK = 2$ m, $EH = 3$ m,

$\tan\alpha = \frac{h}{l} = \frac{4}{8} = 0.5$, $\alpha = 26.5651^0$, $\cos\alpha = 0.8944$, $\sin\alpha = 0.4472$, $\beta = 45^0$,

$\tan\gamma = \frac{EH}{AE} = \frac{3}{2} = 1.5$, $\gamma = 56.3099^0$, $\cos\gamma = 0.5547$, $\sin\gamma = 0.8320$.

Conditions of the balance of the cut–off part relative to point H (section $a - a$, Fig. 2.6.4), the rod 1 force is determined as:

$$\sum M_H = 0,\ S_1 \times d + R_B \times HM = 0,\ S_1 = -\frac{R_B \times HM}{d} = -\frac{25 \times 1}{2} = -12.5,\ \text{kN}.$$

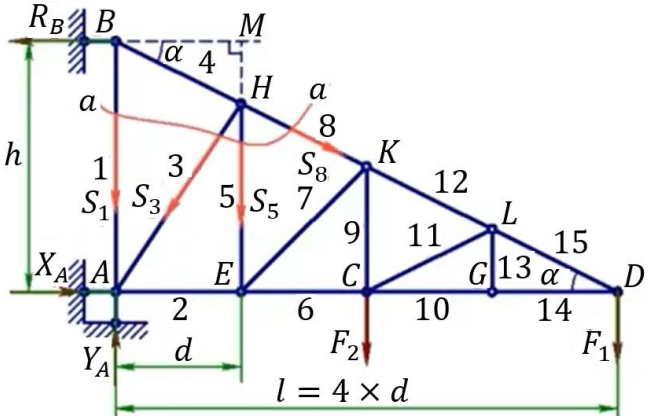


Fig. 2.6.4

The conditions of balance relative to the points: A, D and H (the section $b - b$, Fig. 2.6.5) are considered to find the forces in rods 2, 3 and 4:

$\sum M_H = 0, -S_2 \times EH - F_2 \times d - F_1 \times 3 \times d = 0,$

$S_2 = -\frac{F_2 \times d + F_1 \times 3 \times d}{EH} = -\frac{11 \times 2 + 7 \times 3 \times 2}{3} = -21.33,$ kN,

$\sum M_D = 0, S_3 \times \cos\gamma \times EH + S_3 \times \sin\gamma \times 3 \times d + F_2 \times 2 \times d = 0,$

$S_3 = -\frac{F_2 \times 2 \times d}{\cos\gamma \times EH + \sin\gamma \times 3 \times d} = -\frac{11 \times 2 \times 2}{0.5547 \times 3 + 0.8320 \times 3 \times 2} = -6.61,$ kN,

$\sum M_A = 0, S_4 \times \cos\alpha \times EH + S_4 \times \sin\alpha \times d - F_2 \times 2 \times d - F_1 \times 4 \times d = 0,$

$S_4 = \frac{F_2 \times 2 \times d + F_1 \times 4 \times d}{\cos\alpha \times EH + \sin\alpha \times d} = \frac{11 \times 2 \times 2 + 7 \times 4 \times 2}{0.8944 \times 3 + 0.4472 \times 2} = 27.95,$ kN.

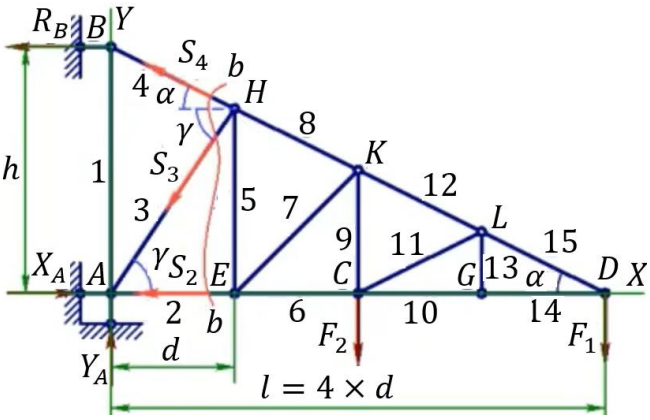


Fig. 2.6.5

The conditions of the balance relative to the points: D and E (the section $c - c$, Fig. 2.6.6) are considered, finding the forces in rods 5 and 8:

$\sum M_D = 0, -S_5 \times 3 \times d + F_2 \times 2 \times d = 0, S_5 = \frac{F_2 \times 2 \times d}{3 \times d} = \frac{11 \times 2}{3} = 7.33,$ kN,

$\sum M_E = 0, S_8 \times \cos\alpha \times CK + S_8 \times \sin\alpha \times d - F_2 \times d - F_1 \times 3 \times d = 0,$

$S_8 = \frac{F_2 \times d + F_1 \times 3 \times d}{\cos\alpha \times CK + \sin\alpha \times d} = \frac{11 \times 2 + 7 \times 3 \times 2}{0.8944 \times 2 + 0.4472 \times 2} = 23.85,$ kN.

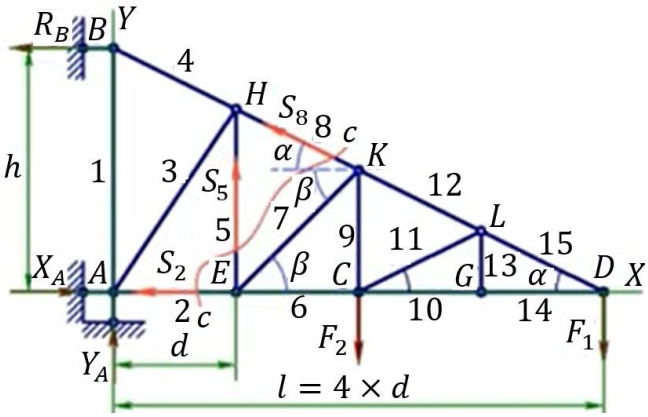


Fig. 2.6.6

To determine the forces in the remaining rods the sections: $d-d$, $e-e$, $f-f$, $g-g$ and $h-h$ are drawn (Fig. 2.6.7). The conditions of the balance are drawn up similarly to the conditions discussed above, and the forces in the rods: 6, 7, 9 – 15 are find similarly with the forces found.

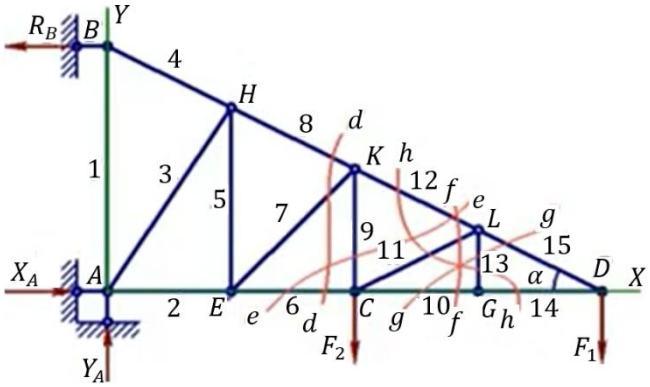


Fig. 2.6.7

Exercise 2. Given, the magnitude of the external load acting on the truss (Fig. 2.6.8 a): $F_1 = 6$ kN, $F_2 = 8$ kN. One needs to determine the reactions of supports A and B and the forces in the rods: S_2, S_5, S_7 and S_8.

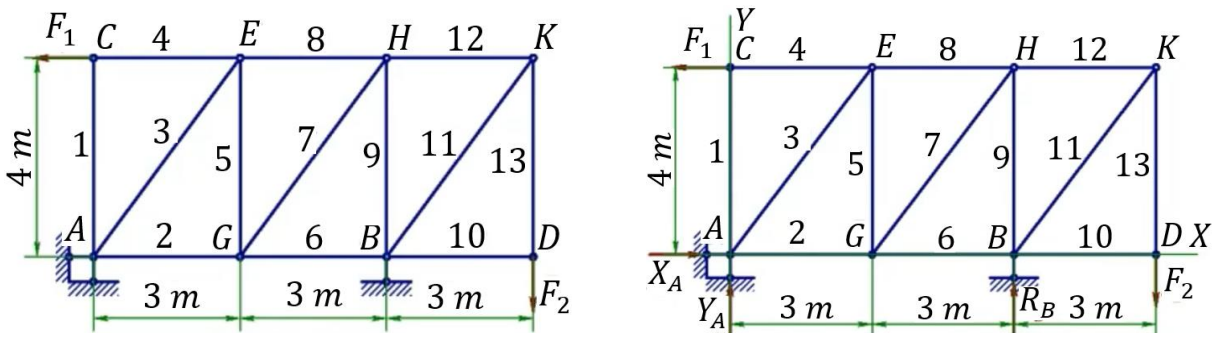


Fig. 2.6.8

The conditions of the balance for the determination of support reactions are:

$\sum F_{kx} = 0, -X_A + F_1 = 0, X_A = F_1 = 6$ kN,

$\sum M_A = 0, R_B \times 6 + F_1 \times 4 - F_2 \times 9 = 0,$

$R_B = \frac{-F_1 \times 4 + F_2 \times 9}{6} = \frac{-6 \times 4 + 10 \times 9}{6} = 11,$ kN,

$\sum M_B = 0, -Y_A \times 6 + F_1 \times 4 - F_2 \times 3 = 0,$

$Y_A = \frac{F_1 \times 4 - F_2 \times 3}{6} = \frac{6 \times 4 - 10 \times 3}{6} = -1,$ kN.

The verification. Finding the total of moments of forces relative to a point K:

$\sum M_K = X_A \times 4 - Y_A \times 9 - R_B \times 3 = 6 \times 4 - (-1) \times 9 - 11 \times 3 = 0.$

Finding forces in rods: 2, 5 (Fig. 2.6.9 a), 8 and 7 (Fig. 2.6.9 b). The section $a - a$. The conditions of the balance of the cut–off part relative to the points: E and G are:

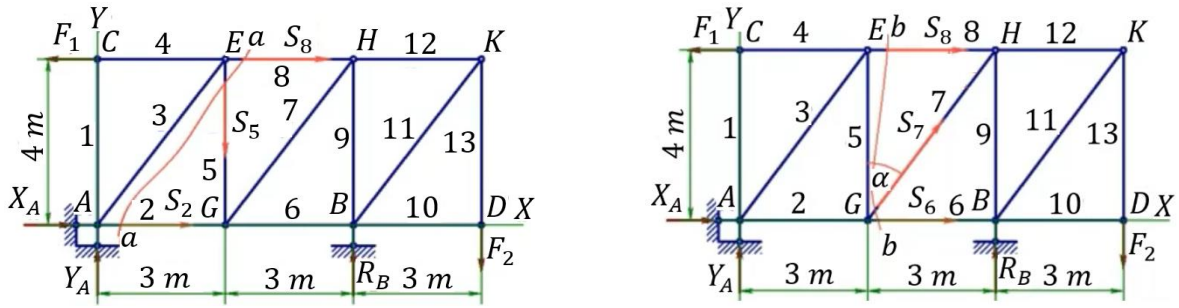
$\sum M_E = 0, X_4 \times 4 - Y_A \times 3 + S_2 \times 4 = 0,$

$S_2 = \frac{-X_4 \times 4 + Y_A \times 3}{4} = \frac{-6 \times 4 - 1 \times 3}{4} = -6.75,$ kN,

$\sum M_G = 0, -S_8 \times 4 + F_1 \times 4 - Y_A \times 3 = 0,$

$S_8 = \frac{F_1 \times 4 - Y_A \times 3}{4} = \frac{6 \times 4 - 1 \times 3}{4} = 5.25,$ kN,

$\sum F_{ky} = 0, -S_5 + Y_A = 0, S_5 = Y_A = -1,$ kN,



(a) (b)

Fig. 2.6.9

The section $b - b$ (Fig. 2.6.9 b). The condition of the balance of the cut–off part according to the projection of forces is: $\sum F_{ky} = 0, S_7 \times \cos\alpha + Y_A = 0,$

$S_7 = -\frac{Y_A}{\cos\alpha} = -\frac{-1}{0.8} = 1.25,$ kN, where: $\cos\alpha = \frac{4}{5} = 0.8.$

Questions and comments:

2.6.1. The determination of support reactions using the equilibrium equations of entire truss.

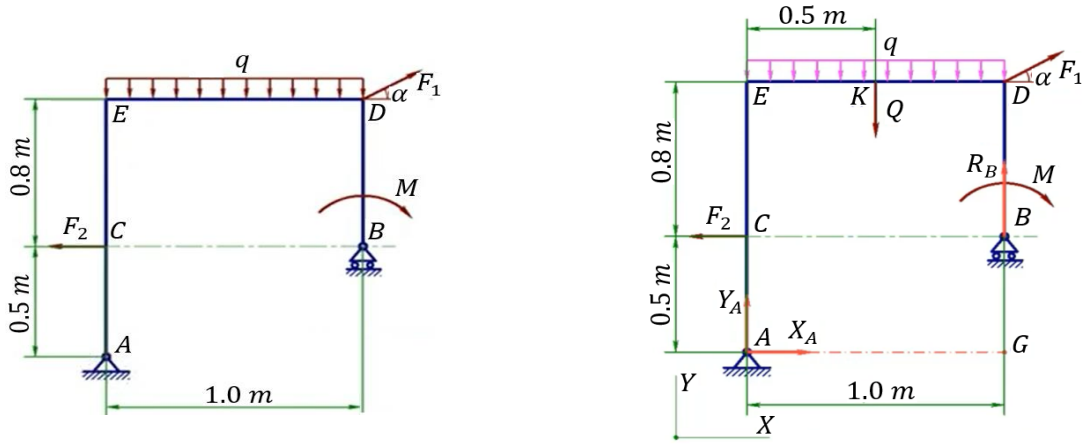
2.6.2. Finding the geometric elements of the truss: the coordinates of nodes, the dimensions of links, the guide angles.

2.6.3. Finding forces in the rods of truss using the section method.

2.7. Finding the reactions of the supports of statically definable frame when it is loaded by a system of forces

Exercise. Given, the magnitude of the external load on a flat statically definable frame (Fig. 2.7.1): $F_1 = 11$ N, $F_2 = 16$ N, $\alpha = 30^0$, $M = 8$ N×m and $q = 6$ N/m. The exercise assigns one to determine the reactions of the supports A and B.

One can replace the distributed load q with the concentrated force Q and apply it in the middle of the loaded area: $Q = q \times 1.0 = 6 \times 1.0 = 6.0$, N.



(a) (b)

Fig. 2.7.1

The use of the equations of balance of the frame:

$\sum F_{kx} = 0$, $X_A + F_1 \times \cos\alpha - F_2 = 0$,

$X_A = -F_1 \times \cos\alpha + F_2 == -11 \times \cos 30^0 + 16 = 6.47$, N,

$\sum M_A(F_k) = 0$,

$F_2 \times 0.5 - Q \times 0.5 - F_1 \times \cos\alpha \times 1.3 + F_1 \times \sin\alpha \times 1.0 - M - R_B \times 1.0 = 0$,

$R_B = -F_2 \times 0.5 + Q \times 0.5 + F_1 \times \cos\alpha \times 1.3 - F_1 \times \sin\alpha \times 1.0 + M =$

$= -16 \times 0.5 + 6 \times 0.5 + 11 \times \cos 30^0 \times 1.3 - 11 \times \sin 30^0 \times 1.0 + 8 =$

$= 9.88$, N,

$\sum M_G(F_k) = 0$, $-Y_A \times 1.0 + F_2 \times 0.5 + Q \times 0.5 - F_1 \times \cos\alpha \times 1.3 - M = 0$,

$Y_A = F_2 \times 0.5 + Q \times 0.5 - F_1 \times \cos\alpha \times 1.3 - M =$

$= 16 \times 0.5 + 6 \times 0.5 - 11 \times \cos 30^0 \times 1.3 - 8 = -9.38$, N.

The verification condition of the balance:

$\sum M_K(\vec{F}_k) = X_A \times 1.3 - Y_A \times 0.5 - F_2 \times 0.8 + F_1 \times \sin 30^0 \times 0.5 - M + R_B \times 0.5 =$

$= 6.47 \times 1.3 - 9.38 \times 0.5 - 16 \times 0.8 + 11 \times 0.5 \times 0.5 - 8 + 9.88 \times 0.5 =$

$-0.009 \approx 0$.

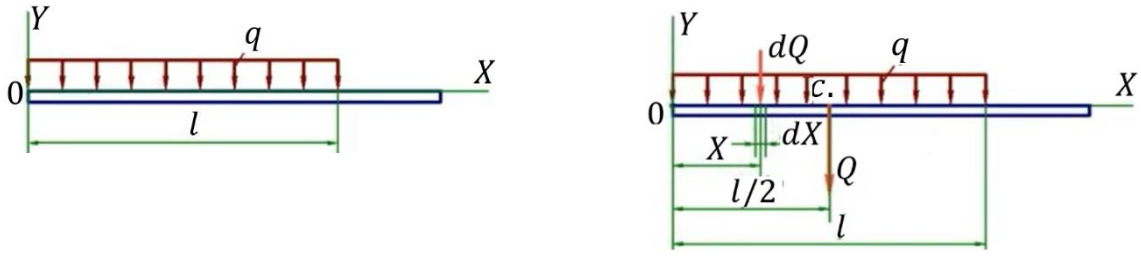
Questions and comments:

2.7.1. The use of balance equations for determination of reactions of supports of statically definable frame under the action of concentrated force, moment and distributed load.

2.7.2. The use of the balance equation to verify the support reactions found.

2.8. Determination of the magnitude and point of application of distributed load resultant

Replacement of a distributed load of constant intensity with a resultant force. The distributed load of constant intensity q acts on a section of length l (Fig. 2.8.1 a). It is required to replace a distributed load of constant intensity with an equivalent concentrated force, that is, to find the resultant Q.



(a) (b)

Fig. 2.8.1

The elementary concentrated force is equal to (Fig. 2.8.1 b): $dQ = q \times dX$.

The resultant Q of the distributed load can be determined by the integrating: $Q = \int_0^l q \times dX = q \times l$.

The resultant Q is equal numerically to the area of the distributed load diagram: $Q = q \times l$ and passes through the center C of this area: $x_c = l/2$ for the even distribution of the load.

The option of load unevenly linear distributing is done on the diagram (Fig. 2.8.2).

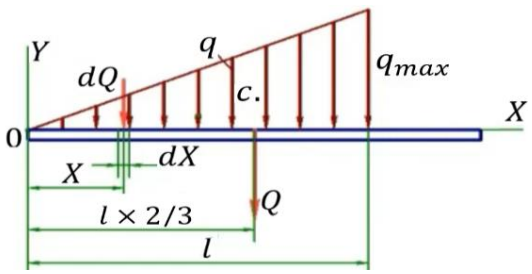


Fig. 2.8.2

The elementary concentrated force is equal to: $dQ = \frac{q_{max}}{l} \times X \times dX$ for unevenly linear distributing of load.

Resultant Q of the distributed load is determined as: $Q = \frac{q_{max} \times l}{2}$.

Position of the center C of area of the distributed load diagram is: $x_c = l \times 2/3$.

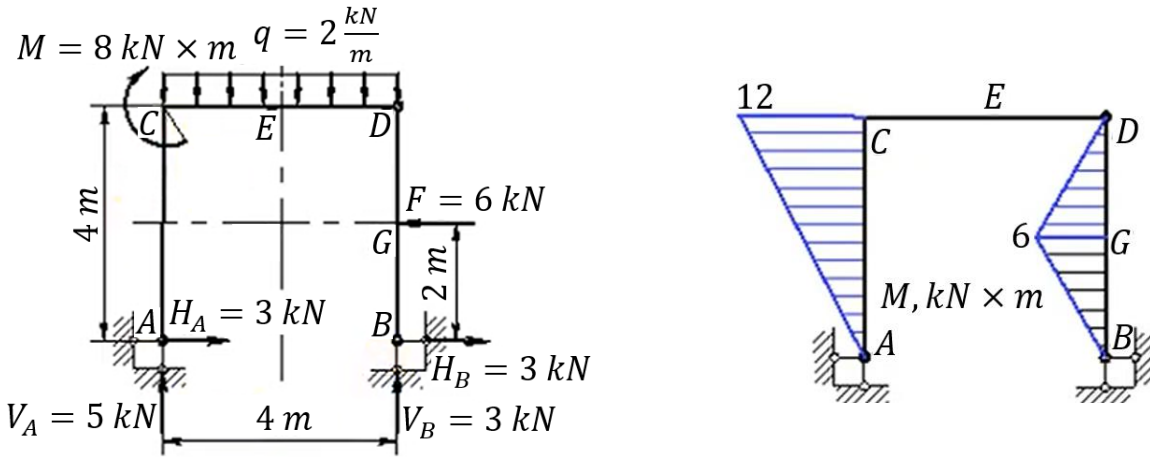
Questions and comments:

2.8.1. The replacing of distributed load with its resultant.

2.8.2. The determining of the center of area of the distributed load diagram.

2.9. Creating diagrams of forces and moments in the calculations of rod structures

The sequence of creating diagrams. It is required for a given statically definable frame (Fig. 2.9.1) to create diagrams of bending moments M, transverse forces Q and longitudinal forces N.



(a) (b)

Fig. 2.9.1

In structural mechanics, the following order of diagrams creating is usual:

(a) determination of support reactions;

(b) creating of the diagram of bending moments M;

(c) creating of the diagram of shear forces Q;

(d) creating of the diagram of longitudinal forces N.

Determination of support reactions. The equation of the balance for the rod BD: $\sum M_D = 0$, $H_B \times 4 - F \times 2 = 0$ and $H_B = \frac{F \times 2}{4} = \frac{6 \times 2}{4} = 3$, kN.

The equation of the balance for entire structure:

$\sum X = 0$, $H_A + H_B - F = 0$ and $H_A = F - H_B = 3$, kN,

$\sum M_A = 0$ and $V_B \times 4 + F \times 2 - q \times 4 \times 2 - m = 0$,

$V_B = \frac{1}{4} \times (-6 \times 2 + 2 \times 4 \times 4 + 8) = 3$, kN,

$\sum Y = 0$, $V_A + V_B - q \times 4 = 0$ and $V_A = -3 + 2 \times 4 = 5$, kN.

The verification of the correctness of the support reactions determining. The balance equation is drawn up relative to point 0 (the intersection of the axes, Fig. 2.9.1 a) to check the reactions found of the supports, because this equation includes all the found reactions of the supports and at the same time it does not include the moments of the given distributed load of assigned intensity q and the concentrated force F:

$\sum M_O = -V_A \times 2 + H_A \times 2 - m + V_B \times 2 + H_B \times 2 =$

$= -5 \times 2 + 3 \times 2 - 8 + 3 \times 2 + 3 \times 2 = 0.$

The general rules of bending moments diagrams determining:

(a) the diagram of bending moments is created on stretched fibers;

(b) the bending moment in the hinge is zero except for the case of the application of concentrated moment in the immediate vicinity of the hinge;

(c) at the point of the application of the concentrated moment on the diagram of bending moments there is a step by the magnitude of this moment;

(d) at the point of the application of concentrated force on the diagram of bending moments there is a kink, that is the angle of inclination of the diagram changes;

(e) in a section with a distributed load of constant intensity the diagram of bending motions has the shape of a parabola convex towards the load;

(a) when determining the diagram of bending moments that is linear on a section of a structure, the moment values at the ends of the section are sufficient, and when determining a curved diagram, another moment value in the middle section can be added.

The determining of the bending moments diagram is shown on the graph (Fig. 2.9.1 b). The diagram of bending moments in the AC section is linear and is created by two points: $M_A = 0$ and $M_C^{AC} = -H_A \times 4 = -3 \times 4 = -12$, kN×m.

On the BD post there are two sections BG and GD, within each of which the magnitude of the bending moment varies linearly: $M_B = M_D = 0$

and $M_G = -H_B \times 2 = -3 \times 2 = -6$, kN×m.

The CD area. Moment equilibrium: M, M_C^{AC} and M_C^{CD} at node C of the frame is considered on the diagram (Fig. 2.9.2): $M_C^{AC} - M - M_C^{CD} = 0$,

then $M_C^{CD} = M_C^{AC} - M = 12 - 8 = 4$, kN×m and $M_D = 0$ in the hinge joint.

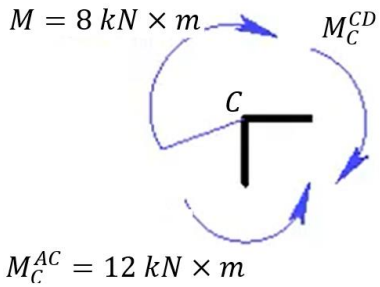
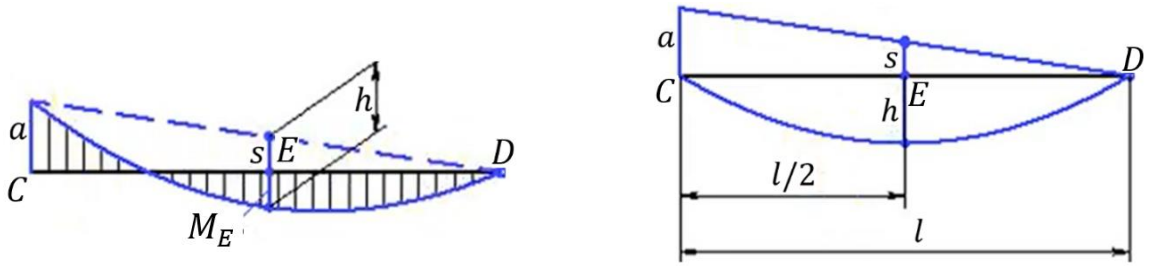


Fig. 2.9.2

The convexity of the parabola in section CD is directed downward (in the direction of action of the distributed load, that is a parabola convex towards the load, Fig. 2.9.3).



(a) (b)

Fig, 2.9.3

Finding the bending moment in the middle section of the element with the distributed load at the point E. The method of stratification of diagrams is used, the curvilinear diagram is presented as the sum of two parts:

(1) one part of the diagram is a triangle with a leg a;

(2) the other part of the diagram is a parabolic segment with a height h.

The bending moment in the middle of the section is equal to the difference:

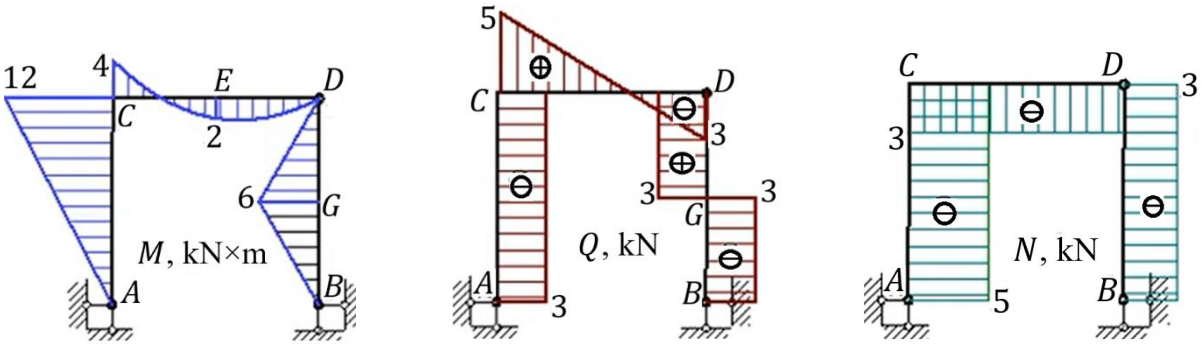
$M_E = h - s$.

The height of the parabolic segment of the diagram of bending moments, which is the magnitude of the moment h for the parabolic part of the diagram for: $x = l/2$, is determined by the dependence: $h = q \times l^2/8$, where q – the distributed load intensity, the section length l is 4 m. In the middle of the E of leg CD the magnitude of the moment s for the triangular part of the diagram is equal to: $s = a/2$.

The resulting value of the bending moment in the middle section of the element with the distributed load at the point E is equal to:

$M_E = q \times l^2/8 - a/2 = 2 \times 4^2/8 - 4/2 = 2$, kN×m.

The resulting diagram of the bending moment M is on the graph (Fig. 2.9.4 a).



(a) (b) (c)

Fig. 2.9.4

Calculation and creating of the diagram of transverse (the shear) forces Q (Fig. 2.9.4 b). The differential interdependence between the bending moment M and shear force has the form: $Q = \frac{dM}{dx}$.

The bending moment M as a function of the longitudinal coordinate x of the cross–section within the section length l is determined as: $M = \frac{q \times l \times x}{2} - \frac{q \times x^2}{2} + \frac{(M_E - M_0) \times x}{l}$, where M_0 and M_E – the values of bending moments at the beginning and at the end of the section. The first two terms serve to determine the bending moment in the presence of a transverse load distributed over the area q.

The shear force is: $Q = \frac{q \times l}{2} - q \times x + \frac{M_E - M_0}{l}$.

Shear force in initial and final sections equals: $Q = \frac{q \times l}{2} + \frac{M_E - M_0}{l}$ at $x = 0$ and $Q = -\frac{q \times l}{2} + \frac{M_E - M_0}{l}$ at $x = l$.

For creating the diagrams of transverse (the shear) forces Q the previously determined diagram of bending moments can be used, in which the values of transverse forces in sections are determined as: $Q_{AC} = \frac{M_C - M_A}{l_{AC}} = \frac{-12-0}{4} = -3$, kN on the AC section,

$Q_C = \frac{q \times l_{CD}}{2} + \frac{M_D - M_C}{l_{CD}} = \frac{2 \times 4}{2} + \frac{0-(-4)}{4} = 4 + 1 = 5$, kN

and $Q_D = -\frac{q \times l_{CD}}{2} + \frac{M_D - M_C}{l_{CD}} = -\frac{2 \times 4}{2} + \frac{0-(-4)}{4} = -4 + 1 = -3$, kN on the CD section,

$Q_{GD} = \frac{M_D - M_G}{l_{GD}} = \frac{0-(-6)}{2} = 3$, kN on the GD section

and $Q_{BG} = \frac{M_G - M_B}{l_{BG}} = \frac{-6-0}{2} = -3$, kN on the BG section.

Determining of the diagram of longitudinal forces N (Fig. 2.9.4 c). To find the longitudinal forces one needs to cut out the frame nodes, to create equations of the balance for the external and internal forces, acting on each node, and to determine the longitudinal forces acting on each node: $N_{AC} = -5$ kN and $N_{CD} = -3$ kN by the conditions of balance of the node C and $N_{BD} = -3$ kN by the balance of node D.

Questions and comments:

2.9.1. The sequence of diagrams determining: the bending moments, the shear forces and the longitudinal forces.

2.9.2. Determination of support reactions using the balance equations of a mechanical system and checking the location of forces.

2.9.3. The general rules of bending moments diagrams determining.

2.9.4. Shapes by sections of the bending moment diagrams depending on the load on mechanical system sections.

2.9.5. Method of stratification when determining the diagrams taking into account the effects of various loads on a section of mechanical system.

2.9.6. The differential interdependence between bending moment and shear force.

2.9.7. Bending moment as a function of longitudinal coordinate of the cross–section.

2.9.8. Using a diagram of the bending moments to determine the diagram of shear (the transverse) forces.

2.9.9. The calculation of longitudinal forces and determining the diagram of longitudinal forces in sections of mechanical system by cutting out it nodes.

2.10. Calculation of forces in multi–span statically definable beams and determining the diagrams of forces and moments

The dependence of static definability on the number of hinges is illustrated by the examples of a hinged cantilever beam (Fig. 2.10.1), including: the two–support statically definable simple beam (Fig. 2.10.1 a), the single statically indefinable two–span beam (Fig. 2.10.1 b), the triple statically indefinable four–span beam (Fig. 2.10.1 c) and the multi–span statically indefinable beam (Fig. 2.10.1 d).

Fig. 2.10.1

The number of hinges of a statically definable hinged cantilever beam can be found using P.L. Chebyshev's formula, showing the number of degrees of freedom:

$w = 3 \times D - 2 \times T - C_0 = 0$, where D – the number of mechanical system disks, T – the number of simple hinges, C_0 – the number of elementary support links.

The number of disks and hinges can be interdependent by the formulas:
$D = T + 1$, $3 \times (T + 1) - 2 \times T - C_0 = 0$ and $T = C_0 - 3$.

Geometric variability of hinged–cantilever beams. The examples of geometrically invariable hinged–cantilever beams are shown in the diagrams (Fig. 2.10.2 a and b).

(a) (b)

Fig. 2.10.2

In order for a statically determined hinged–cantilever beam to be geometrically unchangeable, the following conditions must be met:

1. Each span of the hinged–cantilever beam cannot have more than two hinges.
2. There cannot be two hinges in each span in the adjacent spans.
3. The adjacent spans cannot be without hinges.
4. There cannot be more than one hinge in outer span with an outer hinged support.
5. There must be at least one hinge at the end in the outermost span with a rigid embedment.

The primary and secondary elements of cantilever beams:

1. Basic elements of cantilever beams are those which remain geometrically unchanged when adjacent elements are removed.

2. The secondary elements of cantilever beams due to insufficient number of support connections are not remain geometrically unchanged when the adjacent elements are removed.

Example: the floor diagram. Calculation of the multi–span statically definable beam (Fig. 2.10.3) and the determining of bending moments and transverse (shear) forces diagrams can be performed by the dividing of beam into individual elements using the floor diagram, which is determined taking into account the interaction of elements.

The sequence of floor diagram determining. The hinges are replaced with hinged–fixed supports (with the same number of connections). For example, from left to right: the individual elements are shown above, if the previous one element is geometrically unchangeable, or are shown below – if the previous one changeable.

The system of single–span beams which absorb a given load and pressure from the elements indicated above is the result.

The impact of the main element on the lower secondary one in the floor diagram equals to the reaction of the support of the higher element and has the opposite direction.

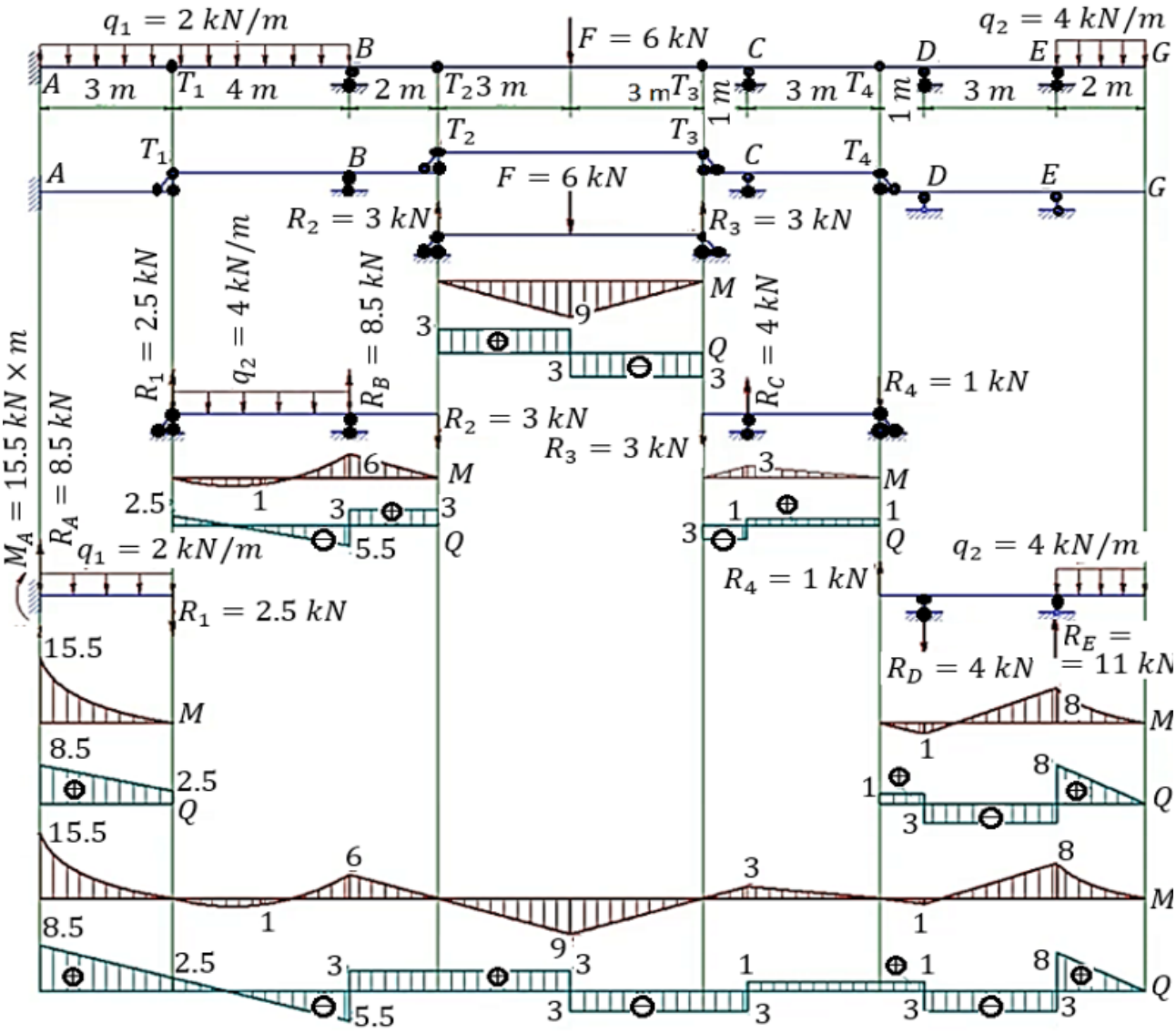


Fig. 2.10.3

Questions and comments:

2.10.1. Static definability of hinged cantilever beam depending on the number of hinges.

2.10.2. Determination of the number of degrees of freedom of a mechanical system by P.L. Chebyshev's formula depending on the number of hinges, disks and support connections.

2.10.3. The geometric variability of hinged cantilever beams.

2.10.4. Conditions of the geometric invariability of a hinged cantilever beam.

2.10.5. Primary and secondary elements of the cantilever beams.

2.10.6. Application of the floor diagram to calculate a multi–span statically determined beam and to make diagrams of bending moments and shear forces.

2.11. Using the forces method in the form of three–moment equation to determine the forces acting in continuous beams

Beams that lie on more than two supports and do not have the intermediate hinges are the continuous beams. The degree of static indefinability n_C of a continuous beam is equal to: $n_C = C_0 - 3$, where C_0 – the number of support connections.

The main system is one that is obtained from a given arrangement of hinges over all intermediate supports.

In this case, the extra (redundant) unknowns are the bending moments M_i in the support sections.

The equivalent system represents a series of simple supported beams loaded with a given forces (concentrated and distributed) and unknown moments.

The action of a given load applies only to the span on which the load is applied.

Its action is transmitted to other spans through supporting bending moments M_i.

An example of a continuous beam resting on more than two supports and not having the intermediate hinges is shown on the diagram (Fig. 2.11.1).

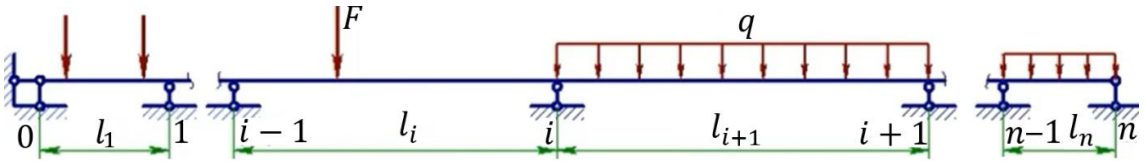


Fig. 2.11.1

The supports are usually numbered from left to right, designating the leftmost support with a number 0. The span number corresponds to the number of its right support.

The use of the forces method in the form of three–moment equation. Physical concept of the three–moment equation means that the angle of mutual turn of the sections adjacent to the cut–out hinge on both sides is equal to zero:

$M_{i-1} \times l_i + 2 \times M_i \times (l_i + l_{i+1}) + M_{i+1} \times l_{i+1} = -6 \times R_{fi}$, where: $R_{fi} = B_{fi} + A_{fi+1}$ – fictitious support reaction on the i–th support (Fig. 2.11.2), equal to the sum of the reactions of single–span beams located to the left and right of the i–th support from the load distributed according to the patterns of bending moments: $B_{fi} = \frac{\omega_i \times a_i}{l_i}$ and $A_{fi+1} = \frac{\omega_{i+1} \times b_{i+1}}{l_{i+1}}$.

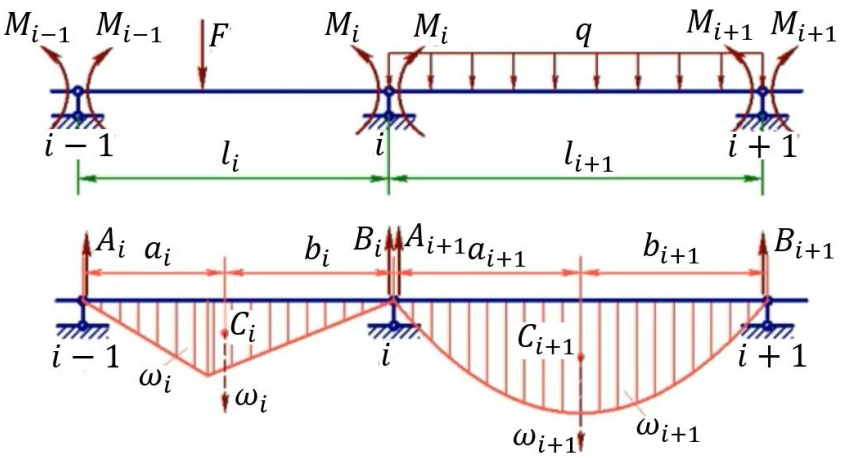


Fig. 2.11.2

The support reactions when loading single–span beams – typical options are shown in the diagrams (Fig. 2.11.3, 2.11.4, 2.11.5 and 2.11.6) and determined by given formulas:

$A_f = \frac{F \times l^2}{6} \times u \times v \times (1+v)$ and $B_f = \frac{F \times l^2}{6} \times u \times v \times (1+u)$ for the single–span beam option shown on the diagram (Fig. 2.11.3);

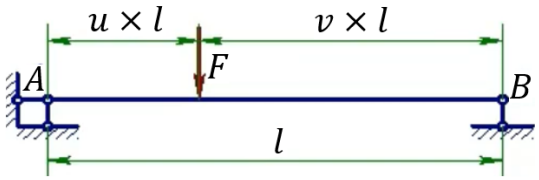


Fig. 2.11.3

$A_f = \frac{F \times a}{6} \times (1-a)$ and $B_f = \frac{F \times a}{6} \times (1-a)$ for the single–span beam option on the diagram (Fig. 2.11.4);

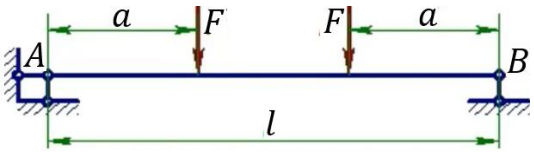


Fig. 2.11.4

$A_f = \frac{q \times l^3}{24}$ and $B_f = \frac{q \times l^3}{24}$ for the option of single–span beam on the diagram (Fig. 2.11.5);

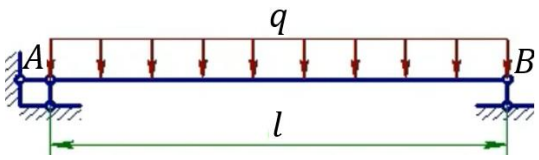


Fig. 2.11.5

$A_f = \frac{9}{384} \times q \times l^3$ and $B_f = \frac{7}{384} \times q \times l^3$ for the option of single–span beam on the diagram (Fig. 2.11.6).

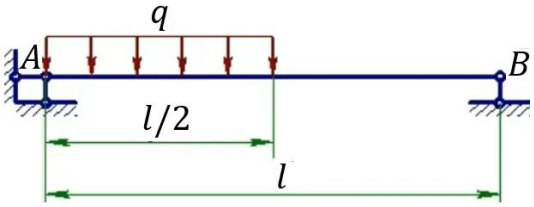


Fig. 2.11.6

An example of the calculation of a continuous beam using the three–moment equation (the forces method). A continuous beam and its basic system are shown on the diagram (Fig. 2.11.7).

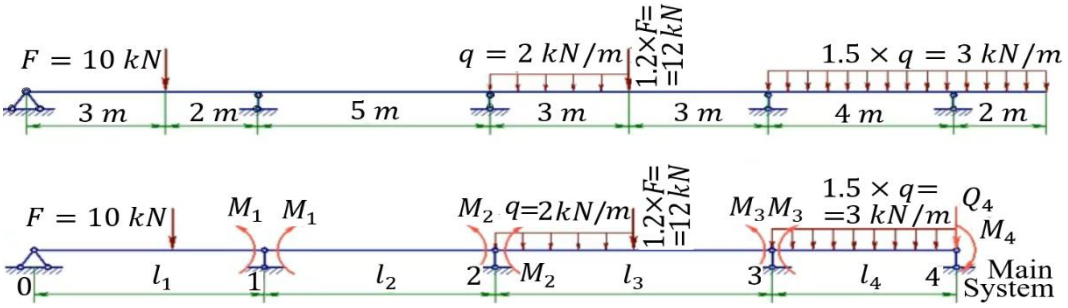


Fig. 2.11.7

To transform the diagram of a continuous beam to its main system, one applies a moment and transverse force on the support instead of the distributed load acting on the console ($1.5 \times q$): $M_4 = -\frac{3\times 2^2}{2} = -6$, kN×m, $Q_4 = 3 \times 2 = 6$, kN.

The three moment equations for each intermediate support have the form:

$M_0 \times l_1 + 2 \times M_1 \times (l_1 + l_2) + M_2 \times l_2 = -6 \times R_{f1}$,

$M_1 \times l_2 + 2 \times M_2 \times (l_2 + l_3) + M_3 \times l_3 = -6 \times R_{f2}$,

$M_2 \times l_3 + 2 \times M_3 \times (l_3 + l_4) + M_4 \times l_4 = -6 \times R_{f3}$.

The bending moments from given span load in the main system are shown on the diagram (Fig. 2.11.8) indicating the dimensions of the spans of a continuous beam.

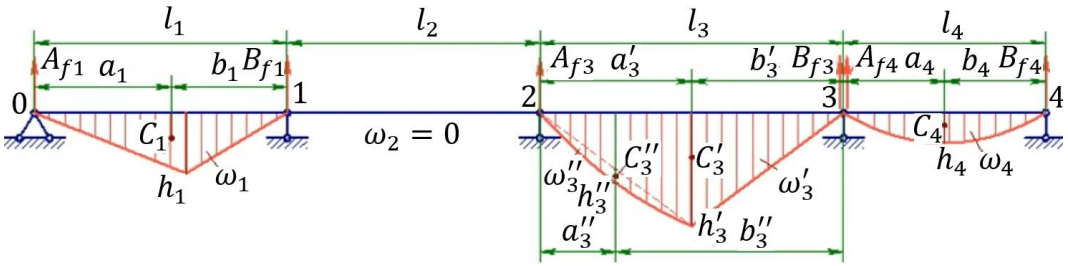


Fig. 2.11.8

Calculation of fictitious (the imaginary) reactions along the spans of a continuous beam. The first span:

$h_1 = 12$ kN×m, $l_1 = 5$ m, $\omega_1 = \frac{1}{2} \times l_1 \times h_1 = \frac{1}{2} \times 5 \times 12 = 30$, kN×m^2,

$a_1 = \frac{1}{3} \times (0 + 3 + 5) = 2.67$, m, $b_1 = \frac{1}{3} \times (0 + 2 + 5) = 2.33$, m,

$A_{f1} = \frac{\omega_1 \times b_1}{l_1} = \frac{30\times 2.33}{5} = 14$, kN×m^2 and $B_{f1} = \frac{\omega_1 \times a_1}{l_1} = \frac{30\times 2.67}{5} = 16$, kN×m^2.

Second span: $\omega_2 = 0$, $A_{f2} = 0$ and $B_{f2} = 0$.

Third span: $h'_3 = 22.5$ kN×m, $l_3 = 6$ m,

$\omega'_3 = \frac{1}{2} \times l_3 \times h'_3 = \frac{1}{2} \times 6 \times 22.5 = 67.5$, kN×m^2,

$a'_3 = b'_3 = 3$ m, $A'_{f3} = B'_{f3} = \frac{\omega'_3 \times b'_3}{l_3} = \frac{67.5\times 3}{6} = 33.75$, kN×m^2,

$h_3'' = 2.25$ kN×m, $l_3'' = \frac{1}{2} \times l_3 = \frac{1}{2} \times 6 = 3$, m,

$\omega_3'' = \frac{2}{3} \times l_3'' \times h_3'' = \frac{2}{3} \times 3 \times 2.25 = 4.5$, kN×m^2,

$a_3'' = 4.5$ m, $b_3'' = 1.5$ m, $A_{f3}'' = \frac{\omega_3'' \times b_3''}{l_3''} = \frac{4.5 \times 4.5}{3} = 3.375$, kN×m^2

and $B_{f3}'' = \frac{\omega_3'' \times a_3''}{l_3''} = \frac{4.5 \times 1.5}{3} = 1.125$, kN×m^2.

The fourth span:

$h_4 = 6$ kN×m, $l_4 = 4$ m, $\omega_4 = \frac{2}{3} \times l_4 \times h_4 = \frac{2}{3} \times 4 \times 6 = 16$, kN×m^2,

$a_4 = b_4 = 2$ m and $A_{f4} = B_{f4} = \frac{\omega_4 \times b_4}{l_4} = \frac{16 \times 2}{4} = 8$, kN×m^2.

Finding of fictitious (the imaginary) external load indicators on the right side of the three–moment equation: $R_{fi} = B_{fi} + A_{fi+1}$,

$R_{f1} = B_{f1} + A_{f2} = 16 + 0 = 16$, kN×m^2,

$R_{f2} = B_{f2} + A_{f3} = 0 + (33.75 + 3.375) = 37.125$, kN×m^2,

$R_{f3} = B_{f3} + A_{f4} = (33.75 + 1.125) + 8 = 42.875$, kN×m^2.

Based on the initial data and when creating the main system, the following supporting points were determined: $M_0 = 0$ and $M_4 = -6$ kN×m.

Finding the remaining supporting moments using three–moment equation (the forces method). The system of the equations for supports 1, 2 and 3 has the form:

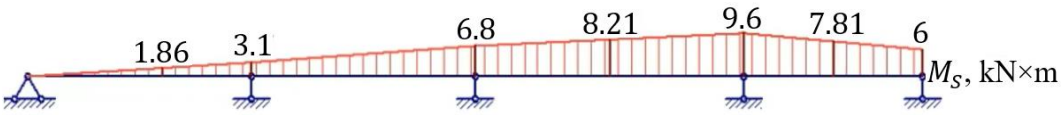
$2 \times M_1 \times 10 + M_2 \times 5 = -6 \times 16$,

$M_1 \times 5 + 2 \times M_2 \times (5 + 6) + M_3 \times 6 = -6 \times 37.125$,

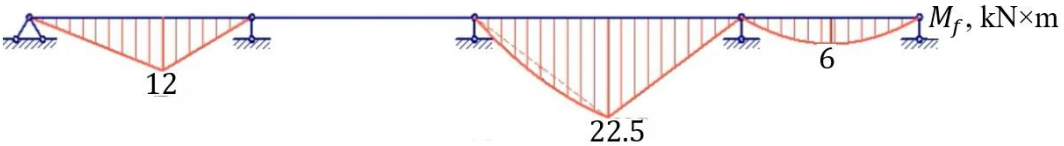
$M_2 \times 6 + 2 \times M_3 \times (6 + 4) - 6 \times 4 = -6 \times 42.875$.

The support moments are found: $M_1 = -3.1$ kN×m, $M_2 = -6.8$ kN×m and $M_3 = -9.6$ kN×m.

Determining the diagram of the support moments and the diagrams of bending moments M and shear forces Q. For determining the diagrams of bending moments one summarizes the diagrams of moments from the support moments M_S (Fig. 2.11.9 a) and from the span load M_f (in the main system, Fig. 2.11.9 b).



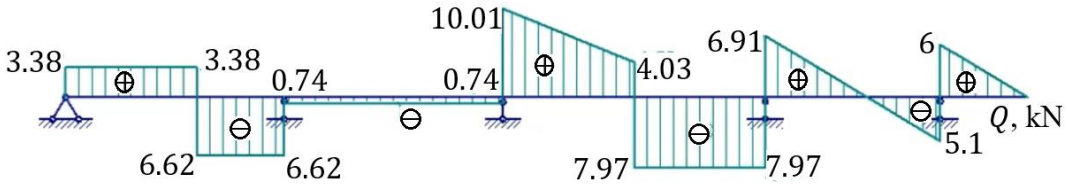
(a)



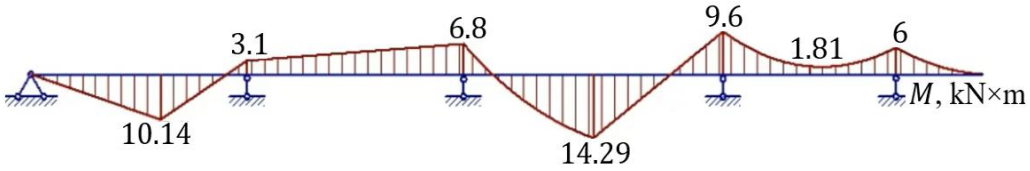
(b)

Fig. 2.11.9

The diagram of transverse (cutting) forces Q (Fig. 2.11.10 a) one can find by differentiating the diagram of bending moments M(Fig. 2.11.10 b).



(a)



(b)

Fig. 2.11.10

Questions and comments:

2.11.1. The concept of a continuous beam, the degree of it static indefinability.

2.11.2. The basic system when calculating a continuous beam.

2.11.3. Using the forces method in the form of three–moment equation. The physical concept of the three–moment balance.

2.11.4. The fictitious support reaction on the support equality to the sum of the reactions of single–span beams located to the left and right of the support from the load distributed according to the pattern of bending moments.

2.11.5. Typical examples of single–span beams used to find the support reactions.

2.11.6. The three–moment equations for each intermediate support.

2.11.7. Calculation of fictitious (the imaginary) reactions along spans of the continuous beam.

2.11.8. Finding of fictitious (the imaginary) external load.

2.11.9. Finding of support moments using the three–moment equation.

2.11.10. Determining the diagrams of bending moments by summing the diagram of moments from the span load and the diagram of the support moments.

2.11.11. Finding the diagram of transverse (the shear) forces by differentiating the diagram of bending moments.

2.12. Impact of a moving load on a beam and creation of a line of influence of a moving load

A line of influence (LI) is a graph of the dependence of a certain factor (the reaction, or the force in any section, or the displacement of point, etc.) on the position of a unit of the force application in a constant direction on a structure (the system of bodies, etc.).

The influence line is not a diagram. Diagram is the graph of any definite dependence of the factor (the internal force, or displacement, etc.) of the stress–strain state of this element (the structure, mechanical system, etc.) on a given stationary load, determined along the axes of the elements of a mechanical system.

When determining a line of influence, the load taken, in particular, is a unit load, which is a unit weight ($P = 1$, when bending a rod as part of a truss or frame – a unit force directed vertically downward or by perpendicular to the axis of this element). The line along which a unit load moves is denominated a load line.

Static or kinematic methods for determining of influence lines are used. The static method is considered below as the example (Fig. 2.12.1).

Determining the lines of influence of the support reactions $LI\ R_A$ and $LI\ R_B$.

It follows: $R_B = 0$ at $x = 0$, the $R_B = 1$ and $R_B = x/l$ at $x = l$ from the equation of balance: $\sum M_A = 0$, $R_B \times l - P \times x = 0$.

At $x = 0$, $R_A = 1$ and at $x = l$: $R_A = 0$ and $R_A = (l - x)/l$ following from the equation of the balance: $\sum M_B = 0$, $-R_A \times l + P \times (l - x) = 0$.

Determining of line of influence on bending moment $LI\ M_K$ of the load position.

The left branch of the influence line $LI\ M_K$. At: $-a_1 \le x \le a$ it follows: $M_K = 0$ at $x = 0$, $M_K = b$ at $x = l$ and $M_K = \frac{a \times b}{l}$ at $x = a$ from the condition of the balance of the cut off right part of the beam: $M_K = R_B \times b = \frac{x}{l} \times b$.

The creation of the right branch of line of influence $LI\ M_K$: $a \le x \le l + b_1$.

From the condition of the balance of the cut off left part of the beam: $M_K = R_A \times a = \frac{l-x}{l} \times a$ it follows: at $x = 0$, $M_K = a$ and at $x = l$, $M_K = 0$.

The determining of lines of influence of the position of unit load on the magnitude of transverse (the shearing) force $LI\ Q_K$ (Fig. 2.12.1).

The determining of the left branch of the line of influence on transverse force $LI\ Q_K$. From the condition of balance of the cut off right part of beam: $Q_K = -R_B = -x/l$ at $-a_1 \le x \le a$ it follows: $Q_K = 0$ at $x = 0$, $Q_K = -1/l$ at $x = l$ and $Q_K = -a/l$ at $x = a$.

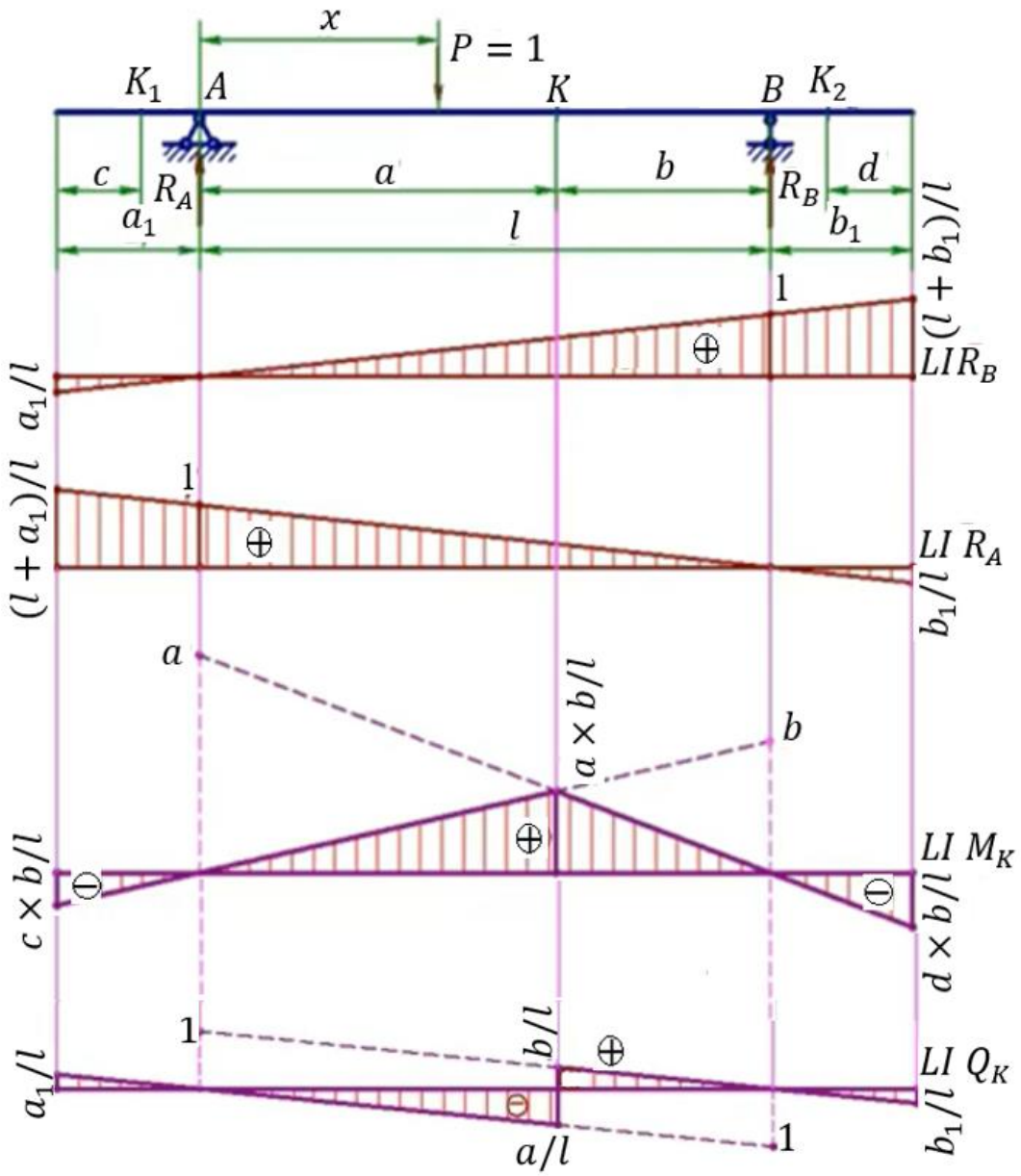


Fig. 2.12.1

The right branch determining of the line of influence on transverse force $LI\ Q_K$. From the condition of the balance of the cut off left part of beam: $Q_K = R_A = (l - x)/l$ at $a \le x \le l + b_1$ it follows: $Q_K = 1$ at $x = 0$, $Q_K = 0$ at $x = l$ and $Q_K = (l - a)/l = b/l$ at $x = a$.

Determination of the internal forces in sections on beam consoles (Fig. 2.12.2).

The internal forces in sections to the left of the console K_1: $M_{K1} = 0$ and $Q_{K1} = 0$ at $x_1 \le 0$, $M_{K1} = -P \times x_1 = -x_1$ at $0 \le x_1 \le c$ and $Q_{K1} = -1$.

Internal forces in sections to the right of the console K_2: $M_{K2} = 0$ and $Q_{K2} = 0$ at $x_2 \le 0$, $M_{K2} = P \times x_2 = x_2$ at $0 \le x_2 \le d$ and $Q_{K2} = 1$.

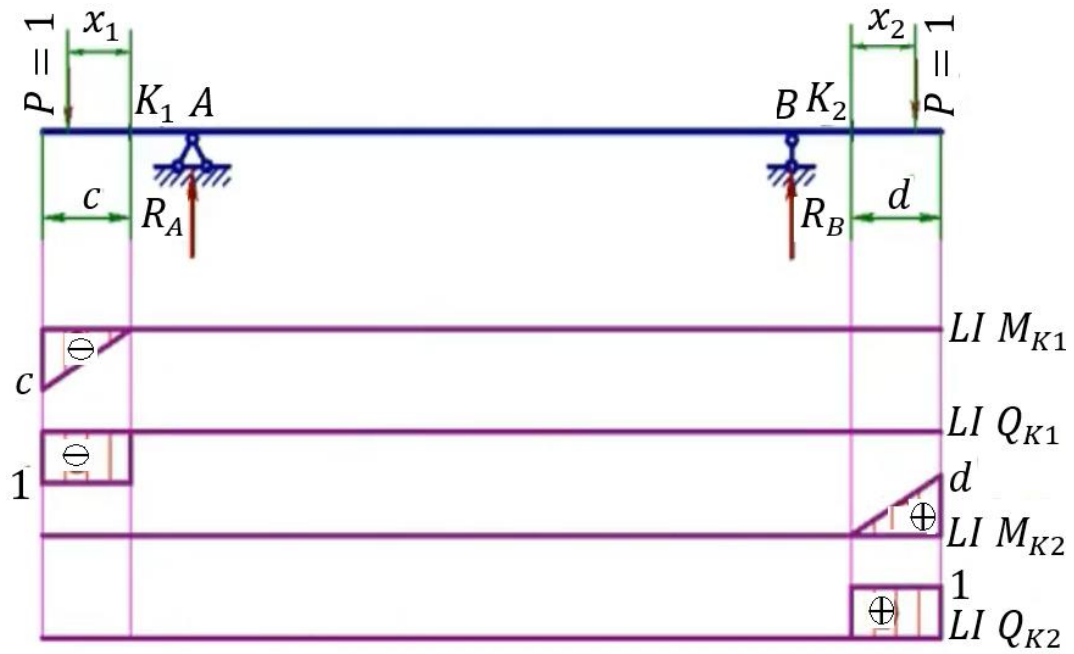


Fig. 2.12.2

Questions and comments:

2.12.1. The line of influence of a force when moving the point of the force application without changing the direction of the force.

2.12.2. The load line along which a unit load moves.

2.12.3. Line of influence of the position of unit load on the magnitude of the transverse (the shearing) force.

2.13. Lines of influence during nodal transmission of moving loads

An example of determining of the line of influence on bending moment $LI\ M_K$ in section K of the main beam ABC with nodal load transfer.

The mechanical system elements: main beam, auxiliary (the longitudinal) beam, cross beam, node, panel. The panel – area between two adjacent nodes.

The data in considered example corresponds to the diagram (Fig. 2.13.1). Longitudinal beams are at the intervals: (0–1) (1–2), … (4–5). The nodes are: 0, 1, 2, 3, 4 and 5.

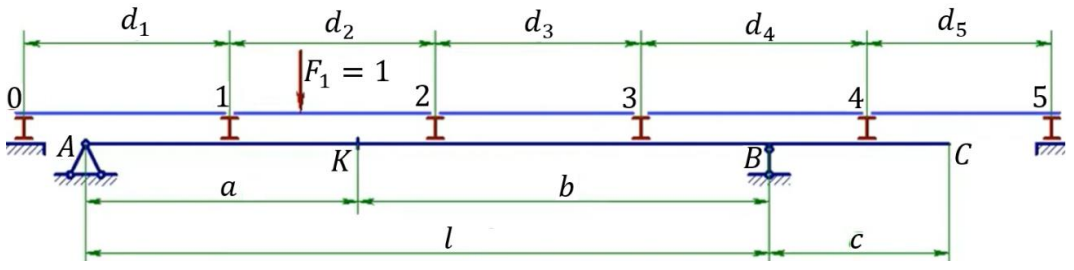


Fig. 2.13.1

The influence line $LI\ M_K$ on bending moment is got without considering the nodal load transfer has the form (Fig. 2.13.2).

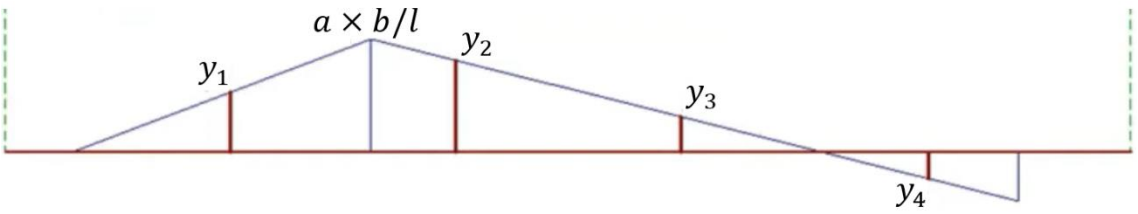


Fig. 2.13.2

Creation of the line of influence LI on bending moment M_K without taking into account the nodal load transfer, which is when moving the unit load along the main beam in the interval (1–2). To determine the line of influence of the bending moment $LI\ M_K$ in section K it is necessary to indicate the ordinate: $a \times b/l$ and to connect the resulting vertex of the line of influence LI with zero ordinates under the left A and right B supports of the main beam (Fig. 2.13.1).

Consider the position of unit load ($F = 1$) applied on one of the secondary beams, for example, on the beam (1–2). The beam calculation (1–2) one can perform independent on others.

The support reactions on the beam (1–2) as a function of the unit force ($F = 1$) are equal: $R_1 = \frac{d_2 - x}{d_2}$ and $R_2 = \frac{x}{d_2}$.

The load forces on the main beam in the nodes 1 and 2 are equal in magnitude to the found reactions R_1 and R_2 and are opposite to them in direction. There are no forces from the remaining beams, because they are not loaded.

The determininf of the line of influence LI on bending moment M_K in section K in when moving a unit load within the panel (secondary beam 1–2). Finding of bending moment M_K in the section K of the main beam when moving a unit load along the secondary beam (1–2). The line of influence LI on the bending moment M_K is used when moving the unit force ($F = 1$) along the main beam directly, loading it with the support reactions R_1 and R_2. The bending moment M_K is equal to:

$M_K = R_1 \times y_1 + R_2 \times y_2 = \frac{d_2 - x}{d_2} \times y_1 + \frac{x}{d_2} \times y_2$, that is: $M_K = y_1$ at $x = 0$ and $M_K = y_2$ at $x = d_2$.

The diagram (Fig. 2.13.3) shows the final decision of the line of influence LI on the bending moment M_K in section K of the main beam ABC with nodal load transfer.

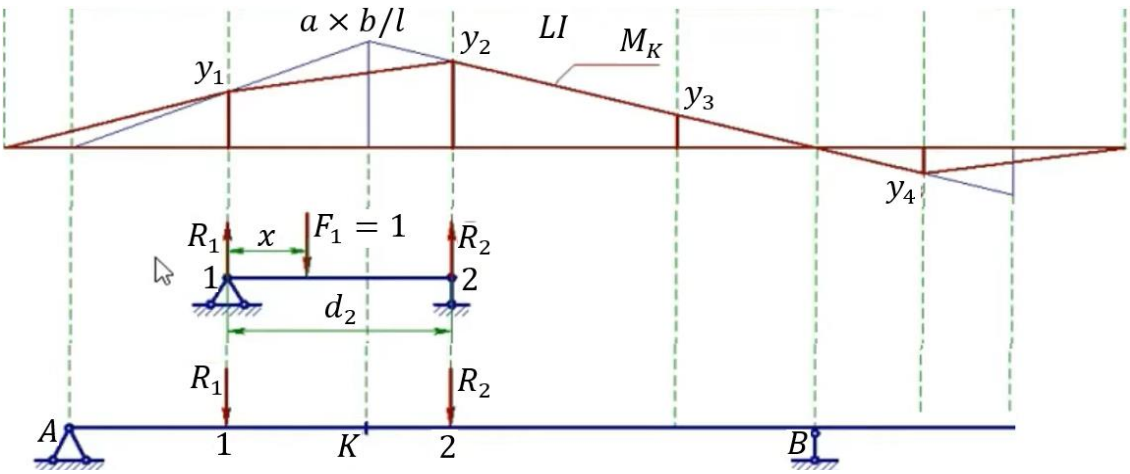


Fig. 2.13.3

Conclusions. When the unit force is positioned on the secondary beam, the line of influence has the shape of a straight–line segment connecting the ordinate vertices (y_1 and y_2) of the line of influence at the nodes. To create the influence line on the main beam one would connect the ordinates at the next nodes (at the beginning and at the end of each panel: y_1, y_2 and others) by the straight segments, having identified the ordinates of the influence line in the nodes. These segments are denoted the transfer lines. The resulting broken line is the final line of influence LI of the bending moment M_K in section K of the main beam.

Questions and comments:

2.13.1. Line of influence of bending moment during nodal load transfer. The considered elements of the mechanical system: main beam, auxiliary (the longitudinal) beam, transverse beam, node, panel.

2.13.2. Creation of the line of influence of the bending moment without taking into account the nodal load transfer, which is when moving a unit load along the main beam.

2.13.3. The creation of the line of influence of the bending moment in section in when moving a unit load within the panel.

2.13.4. The final view of the line of influence of the bending moment in the section of the main beam with nodal load transfer, the transfer lines.

2.14. Lines of influence on forces acting in the truss rods

Example of the determining of line influence *LI* of forces in the rods of second panel of a beam truss. The lower band of the truss can be driven (if serve as a platform for moving cargo). Static method of line influence determining is used, taking into account the nodal load transfer. Data in considered example meets the diagram (Fig. 2.14.1).

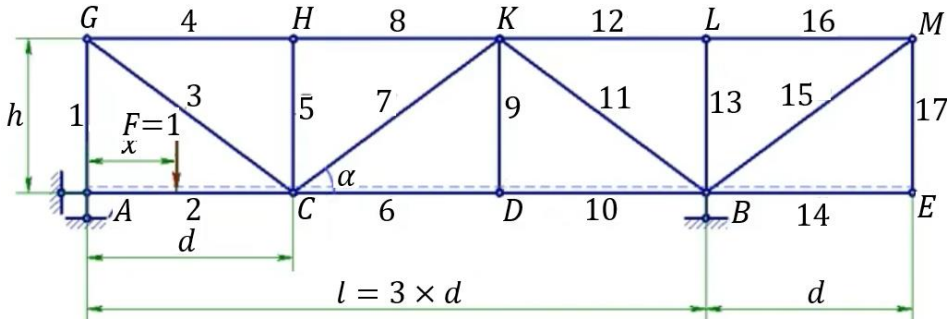


Fig. 2.14.1

The determining of lines of influence of support reactions (Fig. 2.14.2). By the equations, the support reactions are determined as a function of the position (x–coordinate) of a unit load ($F = 1$): $R_A = \frac{l-x}{l}$ and $R_B = \frac{x}{l}$.

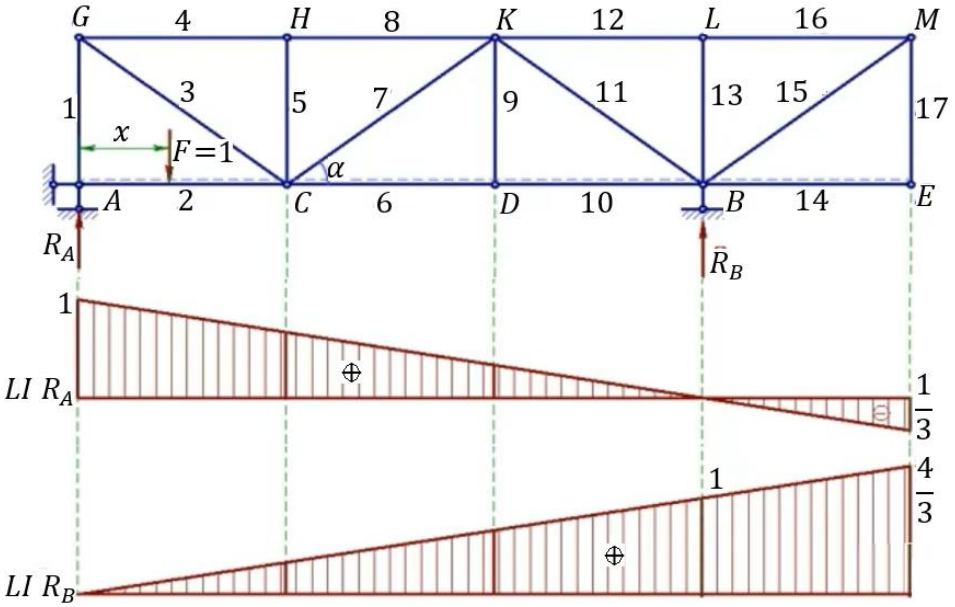


Fig. 2.14.2

In accordance with exercise, for the example, the section is made along the second panel, dividing this second panel along rods 6, 7 and 8. The determining of lines of influence of forces in the truss rods is indicated on the scheme (Fig. 2.14.3).

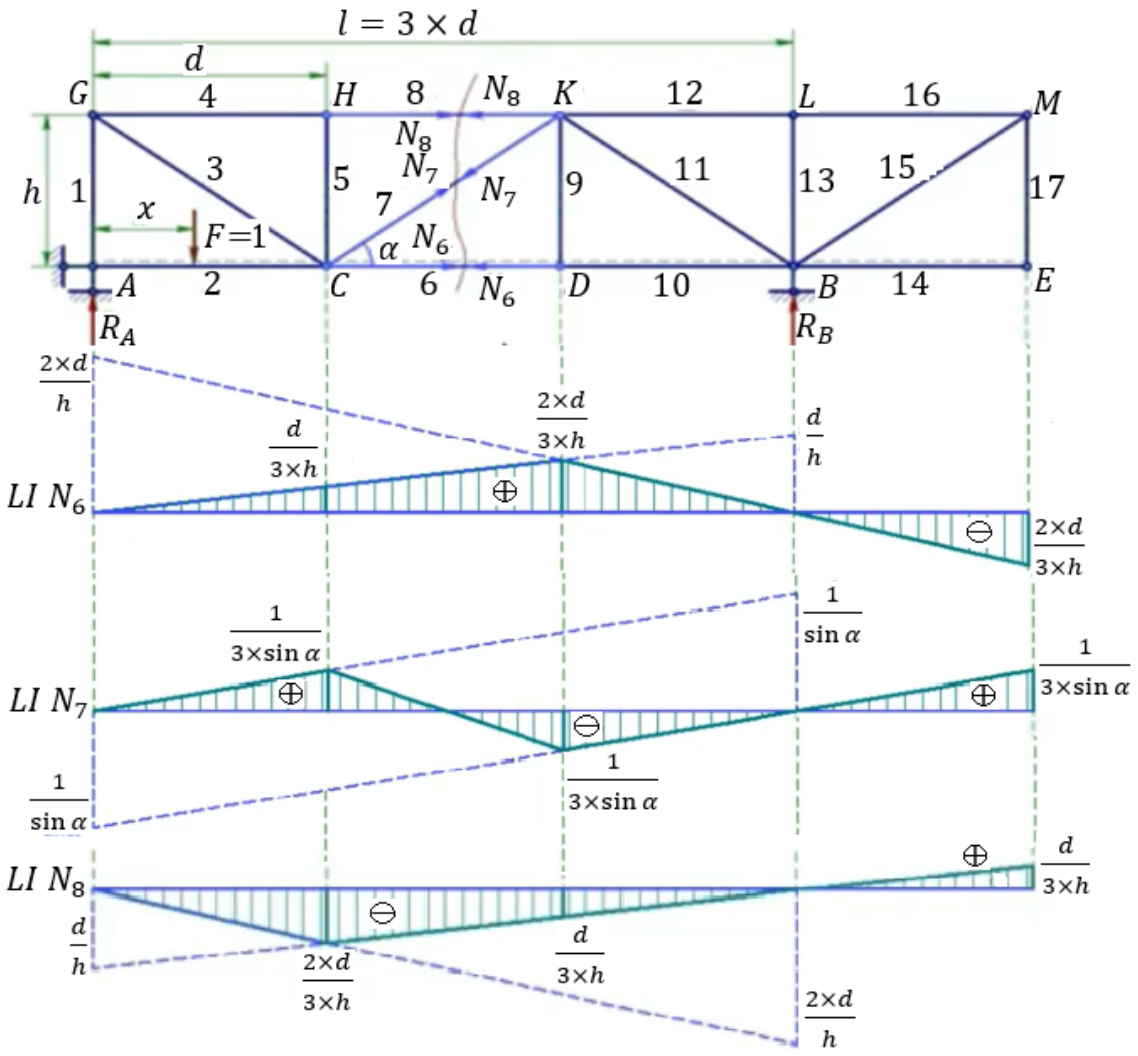


Fig. 2.14.3

The determining of the line of influence $LI\ N_6$ of forces in the sixth rod. The left segment of the line of influence $LI\ N_6$ is determined when a unit load ($F = 1$) is applied on the leg: $0 \le x \le d$. The moment point method is used for the force system acting to the right of the section: $\sum M_K = 0$, $N_6 \times h - R_B \times d = 0$, $N_6 = R_B \times d/h$ or $N_6 = x/(3 \times h)$.

Determining of right segment of the line of influence $LI\ N_6$, when unit load ($F = 1$) is applied to the point: $2 \times d \le x \le 4 \times d$. The moment point method is used for the force system acting to left segment of the section: $\sum M_K = 0$, $N_6 \times h - R_A \times 2 \times d = 0$, $N_6 = R_A \times 2 \times d/h$ or $N_6 = 2 \times (3 \times d - x)/(3 \times h)$. The transfer line connects the co-ordinate vertices at the boundaries of the dissected panel.

Determining the line of influence $LI\ N_7$. The force projection method is used.

Left section: $0 \le x \le d$ of the line of influence $LI\ N_7$. For the right cut–off part, the equation of the balance is drawn up: $\sum Y = 0$, $-N_7 \times \sin\alpha + R_B = 0$, $N_7 = R_B/\sin\alpha$ or $N_7 = x/(3 \times d \times \sin\alpha)$.

Right section: $2 \times d \le x \le 4 \times d$ of the line of influence $LI\ N_7$. For the left cut–off part, the equation of the balance is drawn up: $\sum Y = 0$, $N_7 \times \sin\alpha + R_A = 0$, $N_7 = -R_A / \sin\alpha$ or $N_7 = -(3 \times d - x)/(3 \times d \times \sin\alpha)$.

The transfer line connects the vertices found of the influence line $LI\ N_7$.

The determining of right part of line of influence $LI\ N_8$. Left section: $0 \le x \le d$ of the line of influence $LI\ N_8$. The moment point method is used for the force system acting to the right segment of the section: $\sum M_C = 0$, $N_8 \times h + R_B \times 2 \times d = 0$, $N_8 = -R_B \times 2 \times d/h$ or $N_8 = -2 \times x/(3 \times h)$.

Right section: $2 \times d \le x \le 4 \times d$ of the line of influence $LI\ N_8$.

One uses the moment point method for the force system acting to the left section: $\sum M_C = 0$, $N_8 \times h + R_A \times d = 0$, $N_8 = -R_A \times d/h$ or $N_8 = -(3 \times d - x)/(3 \times h)$.

The transfer line connects the found vertices of the influence line $LI\ N_8$.

The determining of right part of line of influence: N_5 and N_9 and finding the vertices of the influence lines (the forces values) by cutting out nodes (Fig. 2.14.4).

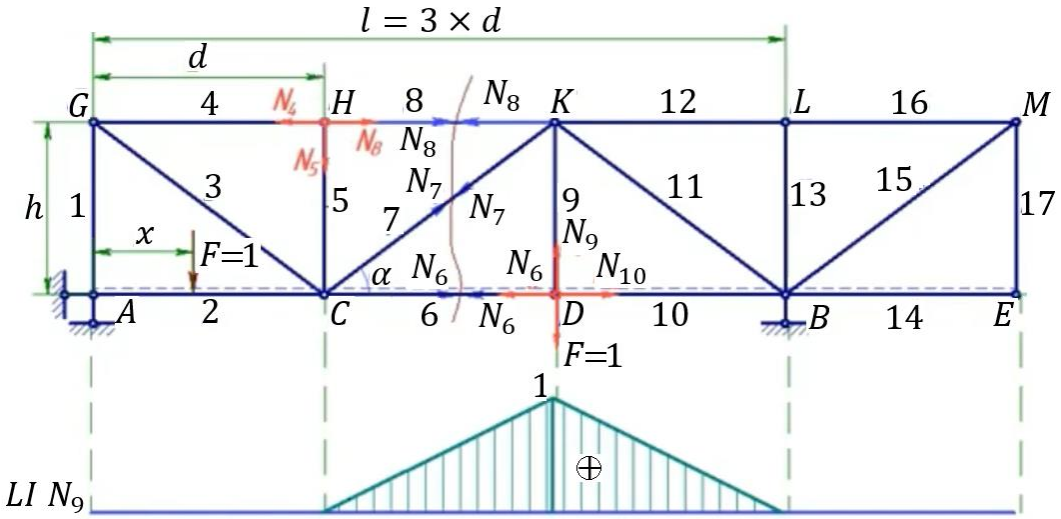


Fig. 2.14.4

The node H is cut out, and force found is: $N_5 = 0$.

Node D is cut out and equations of the balance are drawn up for sections. Coordinate value (force N_9) for sections of the line of influence $LI\ N_9$ is equal to: $N_9 = 0$ in area: $0 \le x \le d$, $N_9 = 1$ at the point: $x = 2 \times d$ and $N_9 = 0$ in the area: $3 \times d \le x \le 4 \times d$.

The transfer line connects the vertices found of the line of influence $LI\ N_9$.

Questions and comments:

2.14.1. The lines of influence on forces in the rods of a beam truss, the lower band of which is the platform for moving the load.

2.14.2. Finding support reactions and determining their lines of influence.

2.14.3. The determining of the lines of influence on the forces in each beam of truss when moving the load.

2.15. Conditions for the balance of a spatial system of forces

The vector forms of the conditions of the balance of a solid body under the influence of the force system $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$ are considered.

For the balance of spatial force system applied to a solid body it is necessary and sufficient that the main vector of the force system $\vec{R}$ be equal to zero, and the main vector of moment $\vec{M}_0$ of the system relative to any center of reduction was also equal to zero, that is: $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n) \sim 0$, that is: $\vec{R} = 0$ and $\vec{M}_0 = 0$.

The scalar form of the conditions of the balance of solid body under the influence of the force system. For the balance of spatial force system $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$, applied to a solid body, it is necessary and sufficient that the magnitude (the modulus) R of the main vector of forces and of the main moment M_0 of these forces relative to any center of the reduction were equal to zero: $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$, that is: $R_x = 0$, $R_y = 0$ and $R_z = 0$, $M_0 = \sqrt{M_x^2 + M_y^2 + M_z^2}$, that is: $M_x = 0$, $M_y = 0$ and $M_z = 0$.

The conditions of the balance of a solid body under the influence of spatial systems of forces $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$. Conditions of the balance under the influence of an arbitrary spatial force system: $\sum F_{nx} = 0$, $\sum F_{ny} = 0$, $\sum F_{nz} = 0$, $M_x = 0$, $M_y = 0$ and $M_z = 0$.

The spatial system of parallel forces (Fig. 2.15.1 a). Conditions of the balance of solid body under the influence of a spatial system of parallel forces have the form: $\sum F_{nz} = 0$, $\sum M_x(\vec{F}_n) = 0$ and $\sum M_y(\vec{F}_n) = 0$.

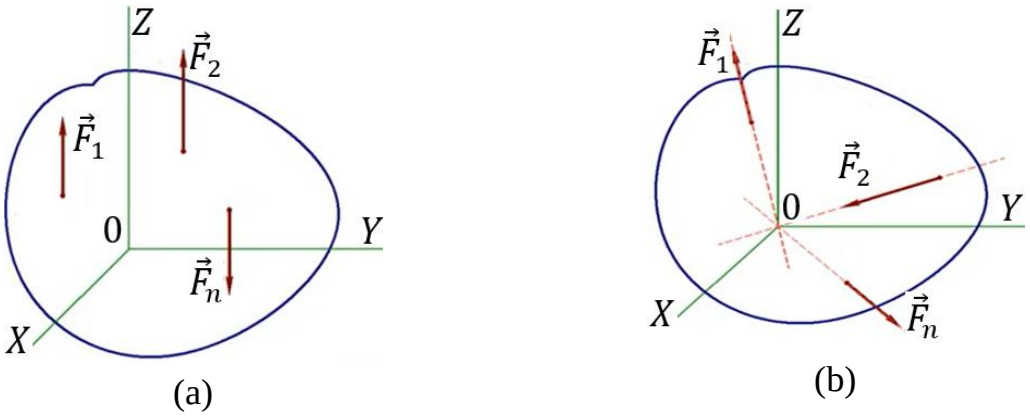
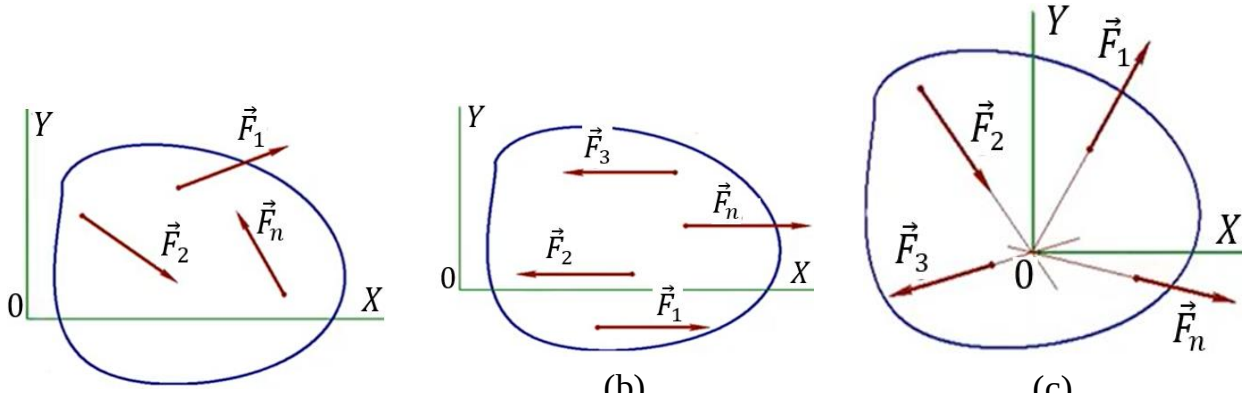


Fig. 2.15.1

The spatial system of converging forces (Fig. 2.15.1 b). Conditions of the balance of a body under the influence of a spatial system of converging forces have the form: $\sum F_{nx} = 0, \sum F_{ny} = 0$ and $\sum F_{nz} = 0$.

Arbitrary flat system of forces (Fig. 2.15.1 a). The necessary and sufficient conditions of balance of a body under the influence of an arbitrary flat force system have the form: $\sum F_{nx} = 0, \sum F_{ny} = 0$ and $\sum M_0(\vec{F}_n) = 0$.



(a) (b) (c)

Fig. 2.15.2

Flat system of parallel forces (Fig. 2.15.2 b). The necessary and sufficient conditions of the balance of solid body under the influence of a plane system of parallel forces have the form: $\sum F_{nx} = 0$ and $\sum M_z(\vec{F}_n) = 0$.

Plane system of converging forces (Fig. 2.15.2 c). The necessary and sufficient conditions of balance of a body under the influence of a flat system of converging forces have the form: $\sum F_{nx} = 0$ and $\sum F_{ny} = 0$.

Questions and comments:

2.15.1. The vector form of the necessary and sufficient conditions of balance of solid body under the action of a spatial system of forces.

2.15.2. Scalar form of the necessary and sufficient conditions of balance of a solid body under the action of a spatial system of forces.

2.15.3. The necessary and sufficient conditions of balance of the body under the influence of a spatial system of parallel forces.

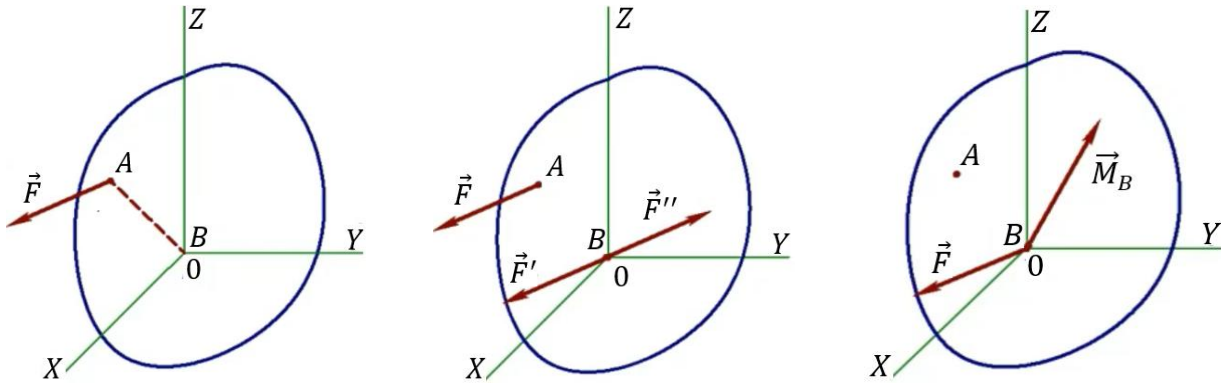
2.15.4. Conditions of the balance of solid body under the influence of an arbitrary flat force system.

2.15.5. Conditions of the balance of solid body under the influence of a spatial system of parallel forces.

2.15.6. The necessary and sufficient conditions of balance of solid body under the influence of a plane system of converging forces.

2.16. Reducing an arbitrary system of forces to its simplest form

Reduction of an arbitrary force system acting on a solid body to a force and a pair of forces. Bringing the force to a given center. At the point A the force $\vec{F}$ operates (Fig. 2.16.1 a). It is required to transfer the force $\vec{F}$ parallel to the point B (Fig. 2.16.1 b).



(a) (b) (c)

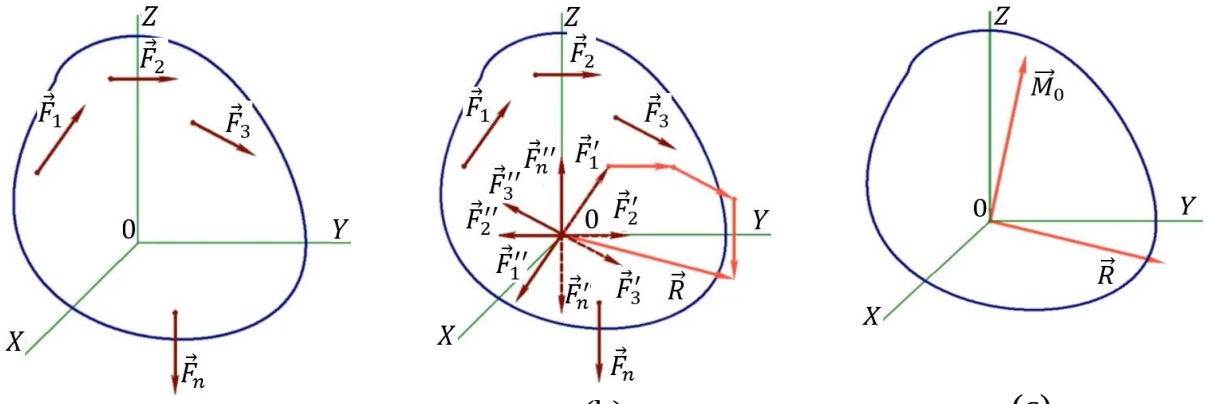
Fig. 2.16.1

At the point B one can apply two oppositely directed forces: $\vec{F}'$ and $\vec{F}''$ equal in magnitude force $\vec{F}$ (Fig. 2.16.1 c): $\vec{F} \sim (\vec{F}, \vec{F}', \vec{F}'') \sim (\vec{F}, \vec{F}'')$ and $\vec{M}_B(\vec{F}, \vec{F}'') = \vec{M}_B(\vec{F})$.

Theorem on parallel force transfer. A force applied to a solid body can be transferred in parallel to any point of the body by adding, wherein, a pair of forces, the moment of which is equal to the vector moment of the transferred force relative to the transfer point.

Reducing an arbitrary system of forces to a force and a pair of forces. The replacing a system of forces with one force and a pair of forces applied to solid body is denominated the bringing the force system to a given center.

Initial system of forces $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$. It is required to bring this system of forces (Fig. 2.16.2 a) to the center 0. To do this, sequentially to each power of the system $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$ one can apply the parallel force transfer theorem.



(a) (b) (c)

Fig. 2.16.2

To do this, one needs to apply at the center 0 the balanced system (Fig. 2.16.2 b), consisting of n oppositely directed forces, the lines of action of which converge at one point (that is converging forces are oppositely directed): $\vec{F}_n' = -\vec{F}_n''$.

The forces applied in a balanced system are equal in magnitudes to the forces in the original system $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$: $\vec{F}_n' = \vec{F}_n$.

The system of converging forces $\vec{F}_n'$ and pairs of forces $(\vec{F}_n, \vec{F}_n'')$ is obtained, jointly equivalent to a given system of forces $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$, applied to a solid body.

The main vector of the force system is their vector sum (the resultant force applied at the center of the reduction): $\vec{R} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = \sum \vec{F}_n$.

The main moment of a given system of forces relative to the center of reduction is the vector sum of the moments of all these forces relative to the specified center:
$\vec{M}_0 = \vec{M}_0(\vec{F}_1) + \vec{M}_0(\vec{F}_2) + \cdots + \vec{M}_0(\vec{F}_n) = \sum \vec{M}_0(\vec{F}_n)$.

Poinsot's theorem. The spatial system of forces $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$, acting on solid body, can lead to a force equal to the main vector of the force system $\vec{R}$, and the pair of forces (Fig. 2.16.2 c), the vector of moment $\vec{M}_0$ relative to the center of reduction 0, which is equal to the main moment of the forces system: $\sum \vec{M}_0(\vec{F}_n)$.

Questions and comments:

2.16.1. The reducing an arbitrary system of forces acting on solid body to a force and a pair of forces.

2.16.2. The theorem on parallel force transfer.

2.16.3. Reducing a system of forces to a point (to the center of the system of forces).

2.16.4. The main vector of the force system is the vector of forces sum.

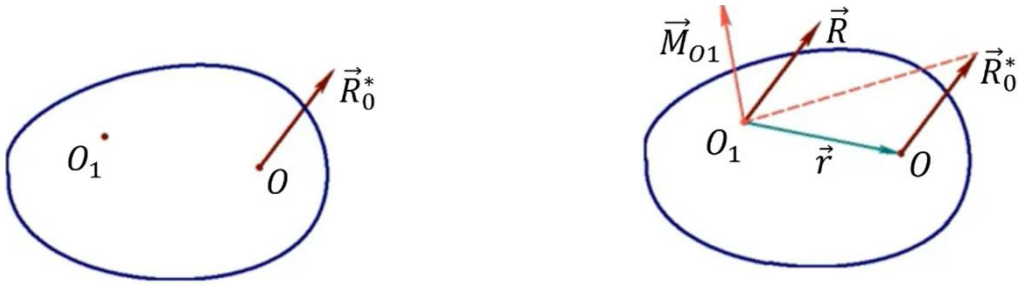
2.16.5. The main moment of force system is the vector sum of the moments of all forces relative to the center of the force system.

2.16.6. Poinsot's theorem – the bringing a spatial system of forces to the main vector of the force system and to the main moment.

2.17. Varignon's theorem on the moment of the resultant force

Pierre Varignon's theorem on the moment of resultant. The moment of the resultant of spatial force system relative to any center is equal to the geometric sum of the moments of all forces relative to this center.

Spatial the system of forces is reduced to the resultant $\vec{R}_0^*$ at the point O (Fig. 2.17.1), which means the main moment of the force is zero: $\vec{M}_O = 0$.



(a) (b)

Fig. 2.17.1

The relationship between the main moments of this system relative to the original center O and the new one O_1 has the form: $\vec{M}_{O1} = \vec{M}_O + \vec{M}_{O1}(\vec{R})$ or $\vec{M}_{O1} = \vec{M}_{O1}(\vec{R})$.

On the other hand, the moment of resultant is equal to sum of the moments of all forces: $\vec{M}_{O1}(\vec{R}) = \sum \vec{M}_{O1}(\vec{F}_K)$ and $\vec{R} = \vec{R}_0^*$, then one can find: $\vec{M}_{O1}(\vec{R}_0^*) = \sum \vec{M}_{O1}(\vec{F}_K)$.

By projecting the resulting vector equality onto coordinate axes, for example, onto the $0X$, axis, one can get the version of Varignon's theorem on the moment of resultant relative to the axis: $\vec{M}_{x1}(\vec{R}_0^*) = \sum \vec{M}_{x1}(\vec{F}_K)$.

The moment of resultant of the force system $\vec{M}_{x1}(\vec{R}_0^*)$ relative to the $0X$ axis is equal to the algebraic sum of the projections of moments of all forces relative to this axis $\vec{M}_{x1}(\vec{F}_K)$.

Version of Varignon's theorem for the flat force system. Since for the flat system the moments of forces on the plane are summed up algebraically, then the Varignon's theorem on the moment of resultant for the flat force system relative to the center O has the form: $M_{O1}(\vec{R}_0^*) = \sum M_{O1}(\vec{F}_K)$.

The moment of the resultant of flat force system relative to any center is equal to the algebraic sum of moments of all forces relative to this center.

The examples of Varignon's theorem use to decompose a force vector into its component vectors are considered below.

Exercise 1. Determine the algebraic moments of the forces $\vec{F}$ and $\vec{P}$ relative to the point A (Fig. 2.17.2).

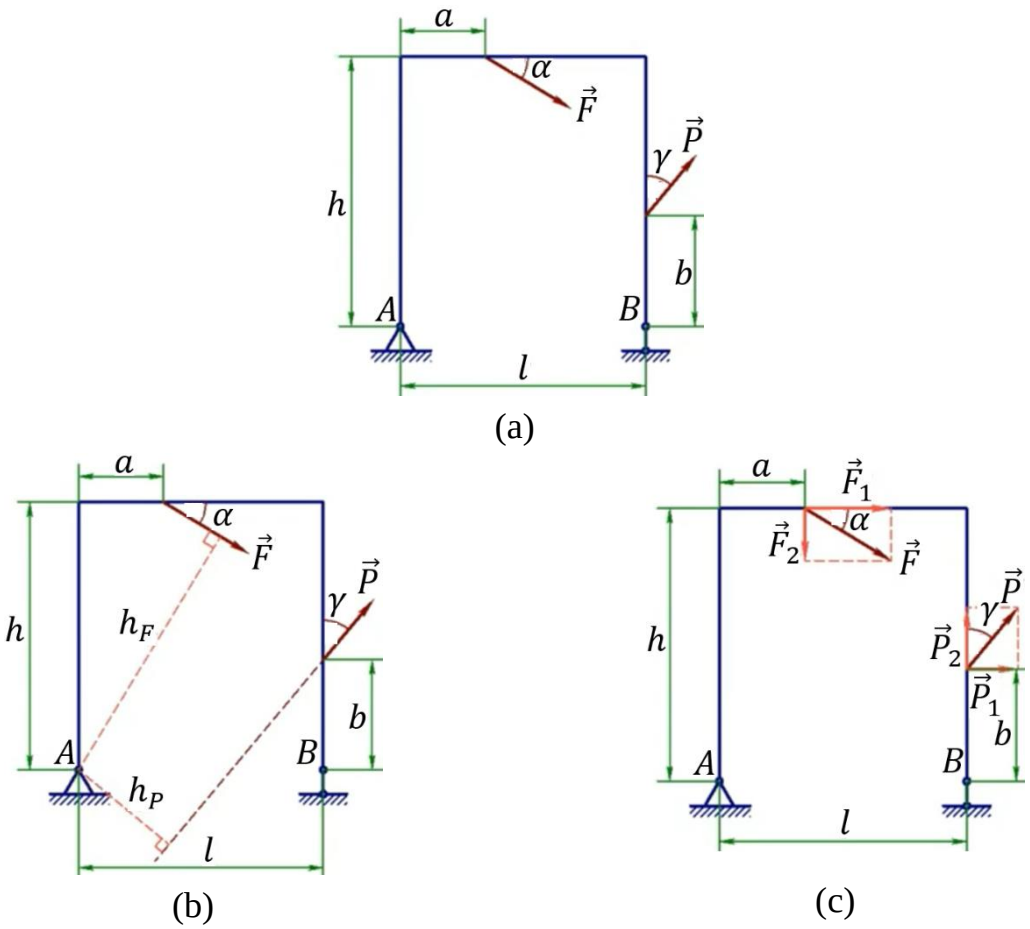


Fig. 2.17.2

The algebraic moments of forces $\vec{F}$ and $\vec{P}$ relative the point A are equal:

$M_A(\vec{F}) = -F \times h_F$, $M_A(\vec{P}) = P \times h_P$,

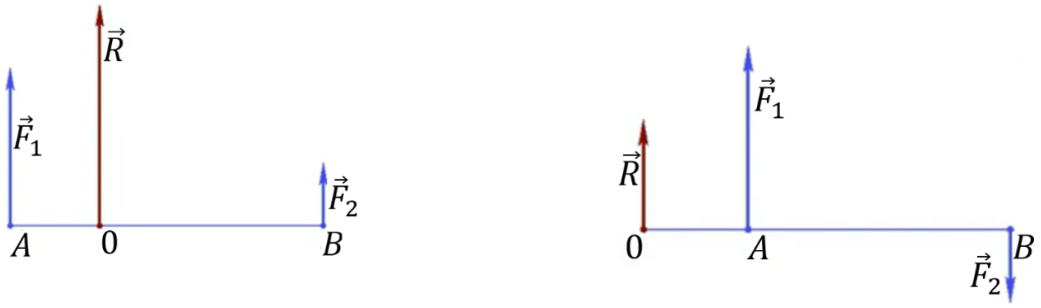
$M_A(\vec{F}) = M_A(\vec{F}_1) + M_A(\vec{F}_2)$, $M_A(\vec{F}) = -F \times \cos\alpha \times h - F \times \sin\alpha \times a$

and $M_A(\vec{P}) = M_A(\vec{P}_1) + M_A(\vec{P}_2)$, $M_A(\vec{P}) = -P \times \sin\gamma \times b + P \times \cos\gamma \times l$.

Exercise 2. Use of Varignon's theorem to add two parallel forces and determine the resultant and the point of its application. Find the resultant $\vec{R}$ of two parallel forces $\vec{F}_1$ and $\vec{F}_2$ (Fig. 2.17.3).

The forces $\vec{F}_1$ and $\vec{F}_2$ form the flat system of forces, which at $\vec{F}_1 \neq -\vec{F}_2$ is reduced to the resultant. To add two parallel forces $\vec{F}_1$ and $\vec{F}_2$ and to determine the resultant $\vec{R}$ and its point of the application A one can use the Varignon's theorem:

$M_O(\vec{R}) = \sum M_O(\vec{F}_K) = 0$, $R = F_1 + F_2$, $F_2 \times 0B - F_1 \times 0A = 0$, that is: $\frac{F_1}{F_2} = \frac{0B}{0A}$.



(a) (b)

Fig. 2.17.3

If the forces $\vec{F}_1$ and $\vec{F}_2$ are oppositely directed (collinear oppositely directed vectors), then one can similarly determine the resultant $\vec{R}$ and its point of the application A: $R = F_1 - F_2$, $-F_2 \times 0B + F_1 \times 0A = 0$ and $\frac{F_1}{F_2} = \frac{0B}{0A}$.

Exercise 3. The use of Varignon's theorem for line determining the resultant action. Find the equation of the line of action of resultant of the force system if the main vector $\vec{R}_1^*$ and the main moment $M_0(\vec{R}_1^*)$ are known relative to the reference center at the origin 0. Let $0_1(x, y)$ – the point of application of the resultant $\vec{R}_1^*$ (Fig. 2.17.4).

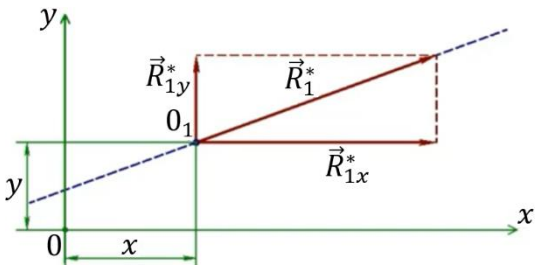


Fig. 2.17.4

The main moment relative to the reduction center at origin 0 is equal to:
$M_0(\vec{R}_1^*) = M_0(\vec{R}_{1x}^*) + M_0(\vec{R}_{1y}^*) = x \times \vec{R}_{1y}^* - y \times \vec{R}_{1x}^*$.

The line equation of the resultant has the form: $y = \frac{\vec{R}_{1y}^*}{\vec{R}_{1x}^*} \times x - \frac{M_0(\vec{R}_1^*)}{\vec{R}_{1x}^*}$, or the form of actions on scalar projections: $y = \frac{R_y}{R_x} \times x - \frac{M_0}{R_x}$.

Questions and comments:

2.17.1. Varignon's theorem on the moment of the resultant.

2.17.2. The connection between the main moments of the system of forces relative to the original and new center.

2.17.3. The variant of Varignon's theorem on the moment of the resultant relative to the axis.

2.17.4. Pierre Varignon's theorem for a plane system of forces.

2.17.5. Application of Varignon's theorem to decompose a force vector into component vectors.

2.17.6. The determination of algebraic moments of forces about a point.

2.17.7. Using Varignon's theorem to add two parallel forces, to determine the resultant and the point of its application.

2.17.8. The use of Varignon's theorem to create the line of action of the resultant.

2.18. The balance of the rod elements of a flat statically definable composite frame

Exercise. Determination of support reactions in a composite frame (Fig. 2.18.1). The initial data: $P = 16$ kN, $F = 7$ kN, $M = 20$ kN×m and $q = 2$ kN/m. The exercise assigns one to find the support reactions in a composite frame: X_A, Y_A, M_A and R_B.

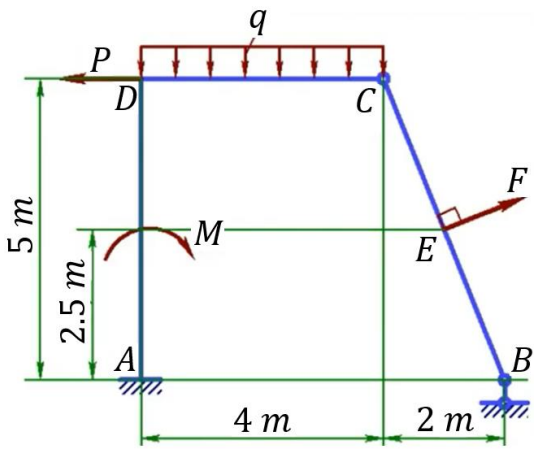


Fig. 2.18.1

Explanations for solving the problem (methods used). To find the support reactions, one can use the first option (1–st form) of the equations of the balance from the set of options: 1–st form: $\sum F_{kx} = 0$, $\sum F_{ky} = 0$ and $\sum M_A(\vec{F}_k) = 0$;

2–nd form, if points A, B and C do not lie on the same line: $\sum M_A(\vec{F}_k) = 0$, $\sum M_B(\vec{F}_k) = 0$ and $\sum M_C(\vec{F}_k) = 0$;

3–rd form, if $0X$ axis is not perpendicular AB: $\sum M_A(\vec{F}_k) = 0$, $\sum M_B(\vec{F}_k) = 0$ and $\sum F_{kx} = 0$.

The rules of projections of forces finding on the axes are shown in the explanatory diagram (Fig. 2.18.2).

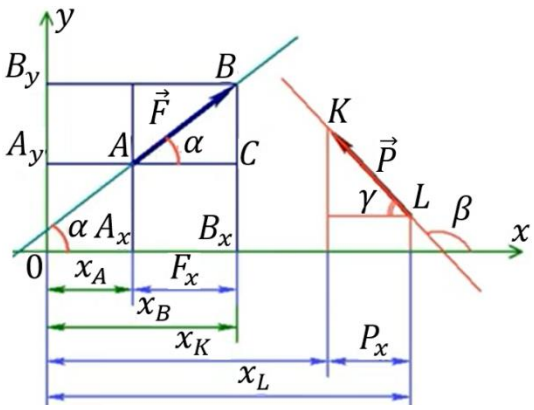


Fig. 2.18.2

Projections F_x and P_x of the forces $\vec{F}$ and $\vec{P}$ are equal: $F_x = x_B - x_A$ and $F_x = F \times \cos\alpha$; $P_x = x_K - x_L$, $P_x = P \times \cos\beta$ and $P_x = -P \times \cos\gamma$.

The rules for finding the algebraic moment of the force relative to a point are shown in the explanatory diagram (Fig. 2.18.3).

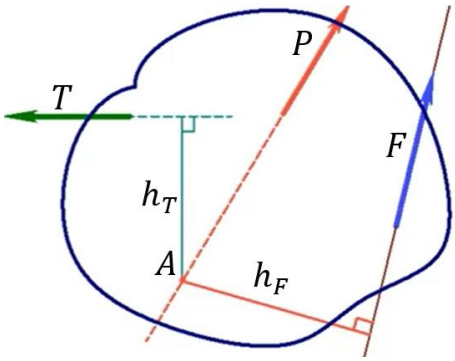


Fig. 2.18.3

Algebraic moments of forces: $\vec{F}$, $\vec{T}$ and $\vec{P}$ about a point A are equal: $\sum M_A(\vec{F}) = F \times h_F$, $\sum M_A(\vec{T}) = T \times h_T$ and $\sum M_A(\vec{P}) = P \times h_P = P \times 0 = 0$.

Using the Varignon's theorem (the moment of the resultant force F is equal to the sum of the moments of the force components, for example, force projections: F' and F'', Fig. 2.18.4) one determines: $M_C(\vec{F}) = -F \times h_C$, where: $h_C = a \times \cos\alpha$ and $M_C(\vec{F}) = -F \times a \times \cos\alpha$, $M_B(\vec{F}) = -F \times (b + c) \times \sin\alpha$ and $M_A(\vec{F}) = -F \times h_A$.

According to Varignon's theorem, the moment of the force F relative to point A is equal to: $M_A(\vec{F}) = M_A(F') + M_A(F'')$, where: $F' = F \times \cos\alpha$ and $F'' = F \times \sin\alpha$.

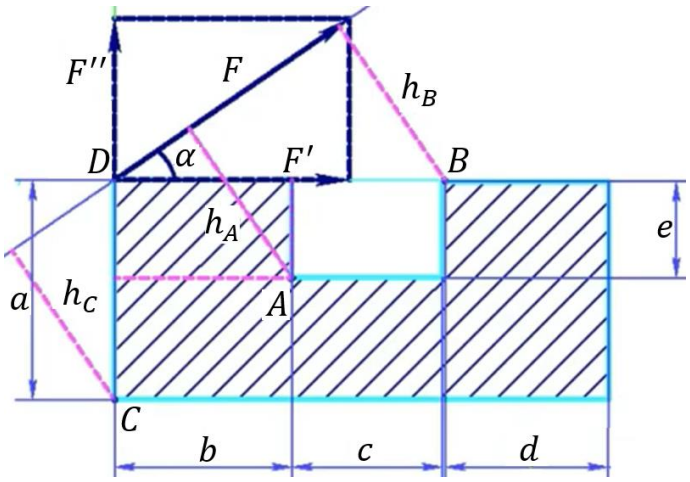
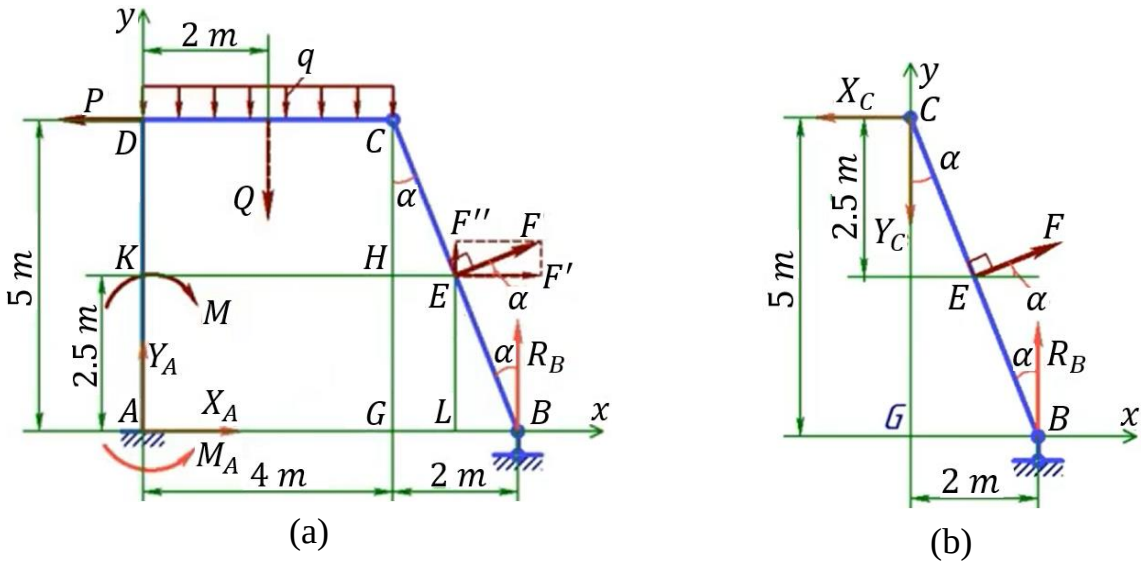


Fig. 2.18.4

Decision of the original issue (Fig. 2.18.5 a and b): the determination of support reactions in a composite frame. Replacing the distributed load q with the concentrated force Q and determining of the composite frame geometric elements:

$Q = q \times 4 = 2 \times 4 = 8$, kN, $BC = \sqrt{2^2 + 5^2} = 5.39$, m,

$CE = BE = \frac{BC}{2} = \frac{5.39}{2} = 2.69$, m, $\cos\alpha = \frac{5}{5.39} = 0.928$ and $\sin\alpha = \frac{2}{5.39} = 0.371$.



(a) (b)

Fig. 2.18.5

Determining of the force F projections: $F' = F \times \cos\alpha = 7 \times 0.928 = 6.5$, kN and $F'' = F \times \sin\alpha = 7 \times 0.371 = 2.6$, kN.

Determination of support reactions in a composite frame using the conditions of the balance: $\sum F_{kx} = 0$, $X_A - P + F \times \cos\alpha = 0$

and $X_A = P - F \times \cos\alpha = 16 - 7 \times 0.928 = 9.5$, kN,

$\sum M_C = 0$, $F \times 2.69 + R_B \times 2 = 0$, $R_B = -F \times \frac{2.69}{2} = -9.42$, kN,

$\sum F_{ky} = 0$, $Y_A - Q + F \times \sin\alpha + R_B = 0$,

$Y_A = Q - F \times \sin\alpha - R_B = 8 - 7 \times 0.371 + 9.42 = 14.82$, kN,

$\sum M_A = 0$, $M_A - M + P \times 5 - Q \times 2 - F' \times 2.5 + F'' \times 5 + R_B \times 6 = 0$,

$M_A = M - P \times 5 + Q \times 2 + F' \times 2.5 - F'' \times 5 - R_B \times 6 =$

$= 20 - 16 \times 5 + 8 \times 2 + 6.5 \times 2.5 - 2.6 \times 5 - 9.42 \times 6 = 15.77$, kN×m.

The verification (equality to zero check of the moments of forces in the hinge C):

$\sum M_C = 0$, $M_A - M + X_A \times 5 - Y_A \times 4 + Q \times 2 + F \times 2.69 + R_B \times 2 =$

$= 15.77 - 20 + 9.5 \times 5 - 14.82 \times 4 + 8 \times 2 + 7 \times 2.69 + (-9.42) \times 2 =.$

$= -0.02 \approx 0.$

Questions and comments:

2.18.1. Using balance conditions to find the support reactions of a composite frame.

2.18.2. Finding projections on the axis of external forces acting on the frame.

2.18.3. Using Varignon's theorem to determine the moment of a resultant force as the sum of the moments of its component forces.

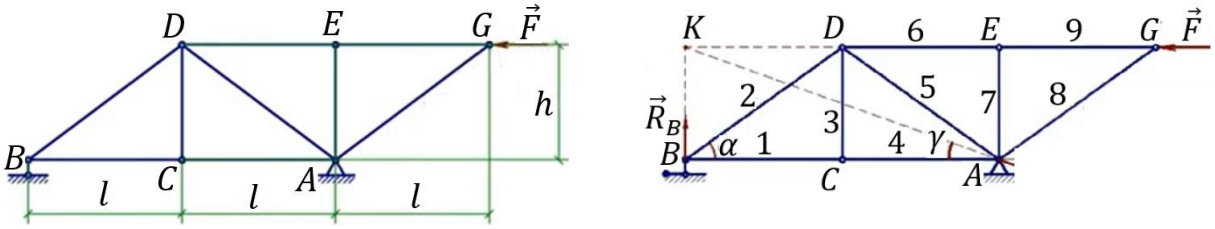
2.18.4. Replacing a distributed load acting on a frame with a concentrated force.

2.18.5. Determination of support reactions in a composite frame using its balance conditions.

2.18.6. Checking the found support reactions using the equations of forces balance in the hinge.

2.19. Calculation of forces in truss rods by cutting out the truss nodes

Exercise. The calculation of flat truss (Fig. 2.19.1). Given: $F = 6$ kN, l=4 m and h=3 m. The exercise assigns one to find the support reactions and forces in rods: $\vec{R}_A$, $\vec{R}_B$ and $\vec{S}_k$, $k - 1, \ldots, 9$.



(a) (b)

Fig. 2.19.1

Defining of the truss geometric elements. From the triangle ΔABK one can determine: $\tan\gamma = \frac{BK}{AB} = \frac{3}{8} = 0.375$, $\gamma = 20.556°$, $\sin\gamma = 0.351$ and $\cos\gamma = 0.936$.

From the triangle ΔBCD found: $\tan\alpha = \frac{CD}{BC} = \frac{3}{4} = 0.75$, $\alpha = 36.870°$, $\sin\alpha = 0.6$ and $\cos\alpha = 0.8$.

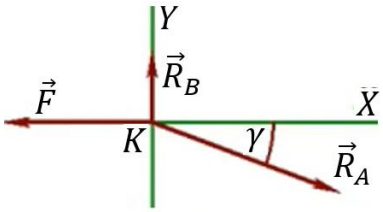
To find the support reactions and forces in the truss rods, the three–force theorem is used. Support reactions are determined using the conditions of the balance, given that the forces intersect at the point K, then the sum of the projections of the forces whose lines of action intersect at this point are equal to zero.

The support reactions at nodes A and B are determined as:

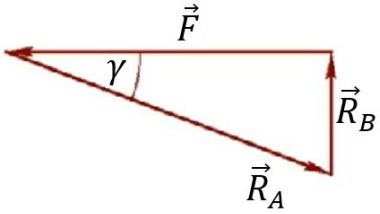
$\sum F_{nx} = 0$, $-F + R_A \times \cos\gamma = 0$, $\sum F_{ny} = 0$, $R_B - R_A \times \sin\gamma = 0$,

$R_A = \frac{F}{\cos\gamma} = \frac{6}{0{,}936} = 6.41$, kN and $R_B = R_A \times \sin\gamma = 6.41 \times 0.351 = 2.25$, kN.

To check that the reaction is correctly located at nodes A and B one can use a force triangle (Fig. 2.19.2 a and b).



(a)



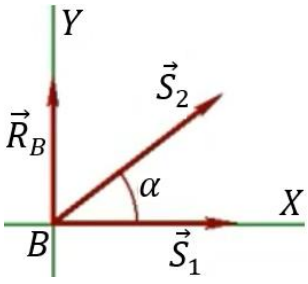
(b)

Fig. 2.19.2

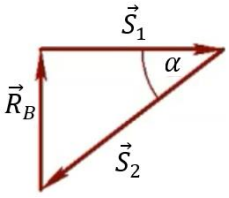
The forces in the truss rods are determined taking into account the fact that the sum of the projections of these forces is equal to zero when crossing their action lines at node B, as: $\sum F_{nx} = 0$, $S_1 + S_2 \times \cos\alpha = 0$, $\sum F_{ny} = 0$, $R_B + S_2 \times \sin\alpha = 0$,

$S_2 = -\frac{R_B}{\sin\alpha} = -\frac{2.25}{0.6} = -3.75$, kN and $S_1 = -S_2 \times \cos\alpha = 3.75 \times 0.8 = 3$, kN.

To check whether the forces in node B are determined correctly, one can use the force triangle (Fig. 2.19.3 a and b).



(a)

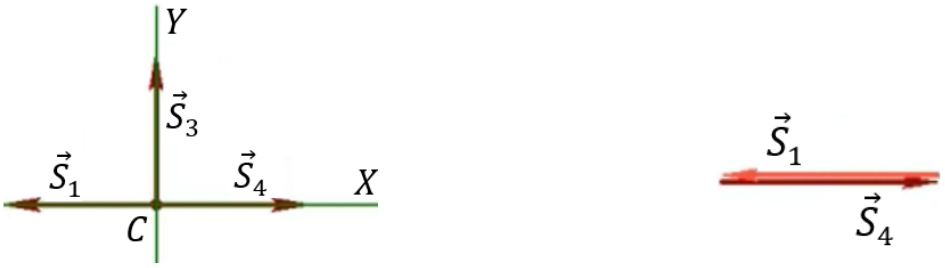


(b)

Fig. 2.19.3

The forces in the truss rods when crossing the lines of their action at node C are determined taking into account the fact that the sum of the projections of these forces is equal to zero, as: $\sum F_{nx} = 0$, $-S_1 + S_4 = 0$, $S_4 = S_1 = 3$, kN and $\sum F_{ny} = 0$, $S_3 = 0$.

To check that the forces in node C are correct one can use a force triangle (Fig. 2.19.4 a and b).



(a) (b)

Fig. 2.19.4

Forces in the truss bars when their action lines intersect in a node D are determined taking into account the fact that the sum of the projections of these forces is equal to zero, as: $\sum F_{nx} = 0$, $-S_2 \times \cos\alpha + S_5 \times \cos\alpha + S_6 = 0$,
$\sum F_{ny} = 0$, $-S_2 \times \sin\alpha - S_3 - S_5 \times \sin\alpha = 0$, $S_3 = 0$, $S_5 = -S_2 = 3.75$, kN
and $S_6 = S_2 \times \cos\alpha - S_5 \times \cos\alpha = -3.75 \times 0.8 - 3.75 \times 0.8 = -6$, kN.

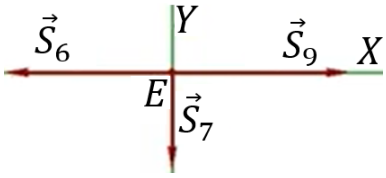
To check that the forces in node D are determined correct, one can use a force triangle (Fig. 2.19.5 a and b).

(a) (b)

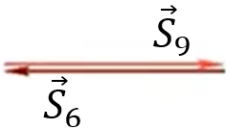
Fig. 2.19.5

The forces in the truss rods when crossing the lines of their action in node E are determined taking into account the fact that the sum of the projections of these forces is equal to zero, as: $\sum F_{nx} = 0$, $-S_6 + S_9 = 0$, $S_9 = S_6 = -6$, kN
and $\sum F_{ny} = 0$, $-S_7 = 0$, $S_7 = 0$.

To check the correct application of the forces in node E one can use a force triangle (Fig. 2.19.6 a and b).



(a)

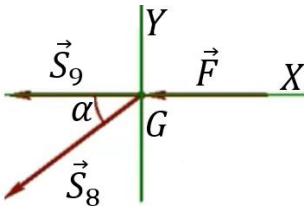


(b)

Fig. 2.19.6

The forces in the truss rods when the lines of their action intersect at node G are determined taking into account the fact that the sum of the projections of these forces is equal to zero, as: $\sum F_{nx} = 0$, $-S_9 - S_8 \times \cos\alpha - F = 0$, $S_9 = -F = -6$, kN
and $\sum F_{ny} = 0$, $-S_8 \times \sin\alpha = 0$, $S_8 = 0$.

To check the correct application of forces in node G one can use a force triangle (Fig. 2.19.7 a and b).



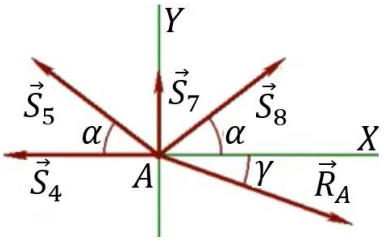
(a)

(b)

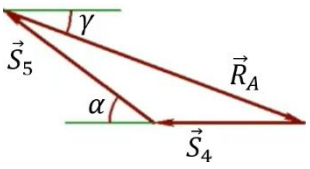
Fig. 2.19.7

The forces in the truss rods when crossing the lines of their action at node A are determined taking into account the fact that the sum of the projections of these forces is equal to zero, as: $\sum F_{nx} = 0$, $-S_4 - S_5 \times \cos\alpha + S_8 \times \cos\alpha + R_A \times \cos\gamma = 0$,
$\sum F_{ny} = 0$, $S_5 \times \sin\alpha + S_7 + S_8 \times \sin\alpha - R_A \times \sin\gamma = 0$,
$S_7 = 0$ and $S_8 = 0$, $S_5 = R_A \times \frac{\sin\gamma}{\sin\alpha} = 6.41 \times \frac{0.351}{0.6} = 3.75$, kN
and $S_4 = -S_5 \times \cos\alpha + R_A \times \cos\gamma = -3.75 \times 0.8 + 6.41 \times 0.936 = 3$, kN.

To check the correct application of forces in the node A one can use a force triangle (Fig. 2.19.8 a and b).



(a)



(b)

Fig. 2.19.8

Calculation results. The reactions in supports: $R_A = 6.41$ kN, and $R_B = 2.25$ kN. Reaction direction angle in thesupport A: $\gamma = 20.556°$. The found forces in the rods ($S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$ and S_9 kN) are indicated in the Table 2.19.

Table 2.19

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
3	–3.75	0	3	3.75	–6	0	0	–6

Questions and comments:

2.19.1. Determining of truss geometric elements.

2.19.2. Finding the support reactions and forces in the truss bars using the three – force theorem.

2.19.3. Checking the correctness of reactions determining in the nodes using a force triangle.

2.19.4. The determination of forces in the nodes of rods, taking into account the condition that the sum of force projections is equal to zero.

Chapter 3. Static indefinability of mechanical systems. Canonical equations of force and displacements methods used to determine reactions under static indefinability

3.1. Statically indefinable systems and the degree of indefinability

Mechanical systems in which the forces cannot be determined using only the equations of the balance, since the number of unknowns (the forces sought and reactions) exceeds the number of equations of the balance, are statically indefinable. The degree of static indefinability of a system corresponds to the number of redundant connections in excess of their minimum number required for the geometric invariability of a mechanical system. The schemes show statically indefinable systems the external (Fig. 3.1.1 a) and internal (Fig. 3.1.1 b).

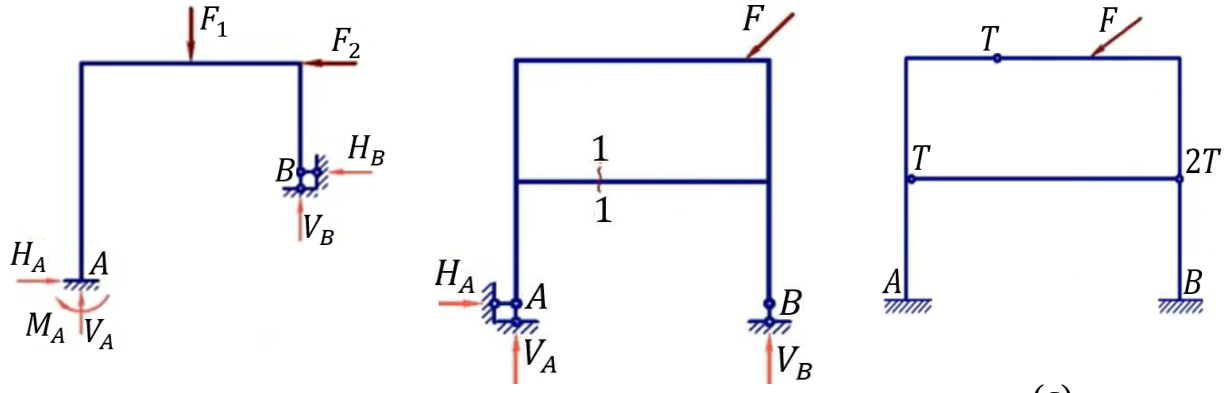


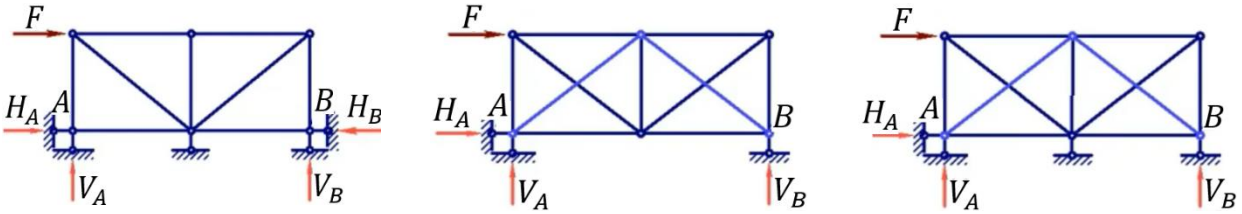
Fig. 3.1.1

The degree of static indefinability n of a mechanical system is equal to: $n = -w$, where w – the number of degrees of freedom of the system.

One can determine the number of degrees of freedom of mechanical system using P.L. Chebyshev's formula (this considered ahead, item 1.7): $w = 3 \times D - 2 \times T - C_0$, where D – the number of disks in the mechanical system (the body on a plane having three degrees of freedom is a disk), T – the number of simple hinges in a mechanical system, C_0 – the number of elementary support links of a mechanical system.

For example, the degrees number of freedom according to P.L. Chebyshev's formula for the mechanical system shown on the diagram (Fig. 3.1.2 a) is equal to: $w = 3 \times 4 - 2 \times 4 - 6 = -2$, and the degree of static indefinability is: $n = -w = 2$.

Static indefinability of a truss (the rod structure with hinges in the nodes, Fig. 3.1.2 b and c) can be determined equal to: $w = 2 \times Y - S - C_0$, where Y – the number of truss nodes, S – the number of truss rods having three degrees of freedom on plane – the disks.



(a) (b) (c)

Fig. 3.1.2

Finding the number of degrees of freedom w and static indefinability n of the truss (Fig. 3.1.2 a) as a mechanical system (using P.L. Chebyshev's formula):

$w = 3 \times D - 2 \times T - C_0 = 3 \times 9 - 2 \times (1 \times 2 + 2 \times 3 + 4 \times 1) - 5 = -2,$

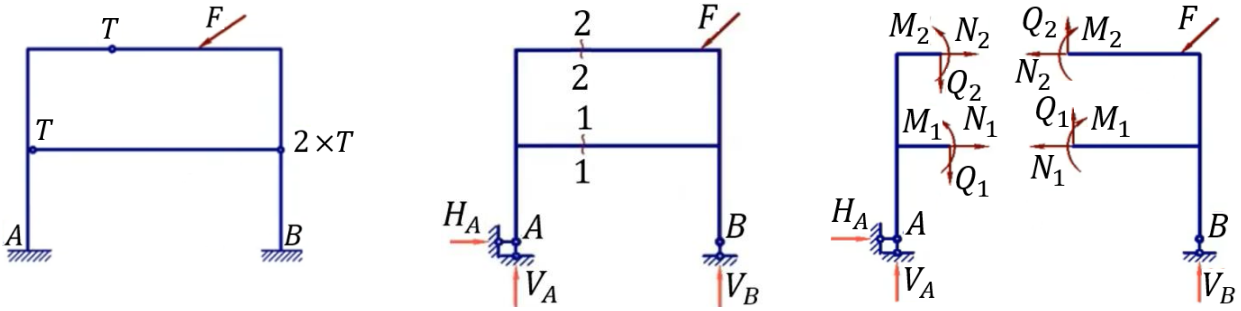
$n = -w = 2.$

Finding the number of degrees of freedom w and the degrees of static indefinability n using the formula for a truss (taking into account the number of truss rods having three degrees of freedom on the plane – the disks, Fig. 3.1.2 a):

$w = 2 \times Y - S - C_0 = 2 \times 6 - 9 - 5 = -2, n = 2.$

For the mechanical system shown on the diagram (Fig. 3.1.3 a) the number of degrees of freedom according to considered formula of P.L. Chebyshev's equal to:

$w = 3 \times 4 - 2 \times 4 - 6 = -2$, and the degree of static indefinability is: $n = -w = 2.$



(a) (b) (c)

Fig. 3.1.3

The closed contour (hinge–less closed mechanical system). An example of internal static indefinability of a frame is considered (Fig. 3.1.3 b and c). The degree of internal static indefinability of a frame with closed loops is related to the number of closed loops K (between the mechanical system and the support) and is defined as: $n = 3 \times K - T_C$, where T_C – the total number of simple hinges (connecting two rods).

The degree of static indefinability defined by the number of closed contours K (which there are three, considering too the contours when closing the structure with supports) is: $n = 3 \times K - T_C = 3 \times 3 - 6 = 3.$

Questions and comments:

3.1.1. Statically indefinable mechanical systems.

3.1.2. External and internal statically indefinable systems.

3.1.3. The degree of static indefinability of a mechanical system.

3.1.4. Determination of the number of degrees of freedom of mechanical system using P.L. Chebyshev's formula.

3.1.5. Static indefinability of a truss.

3.1.6. Finding the number of degrees of freedom and static indefinability of the truss as a mechanical system (using P.L. Chebyshev's formula).

3.1.7. The closed contour. Internal static indefinability of the frame.

3.2. Using the forces method to find forces in statically indefinable mechanical systems, choosing the main system of the forces method

After determining the degree of static indefinability, the main definable system of a statically indefinable mechanical system is selected, for which equations of compatibility of deformations are compiled, which are additional to the equations of the balance when finding support reactions and forces in the elements of the mechanical system.

The main system (the basic one) of the method of forces is such a geometrically immutable and statically definable system, which is obtained from the statically indefinable assigned system by eliminating unnecessary connections and replacing them with the corresponding efforts – the reactions of discarded unnecessary connections.

These efforts are denominated extra (the redundant) unknowns or the main unknowns of the forces method. These force indexes can be designated as a vector (the mathematical set): $X_1, X_2, \dots, X_n$.

Variants of the main system formed on the basis of given system are shown in the diagrams (Fig. 3.2.1 a–f).

The ways to eliminate unnecessary connections when creating main systems:

1. Discarding support connections by completely removing individual supports or replacing them with supports with fewer connections. The reactions acting in direction of the discarded connections are the main unknowns of the forces method.

2. The placement of hinges in intermediate sections of the rod is used in transverse connections corresponding to this bending moment.

3. Cutting the rod eliminates three connections corresponding to internal force factors: the bending moment, transverse (the shear) force and the longitudinal force (M, Q and N, Fig. 3.2.2).

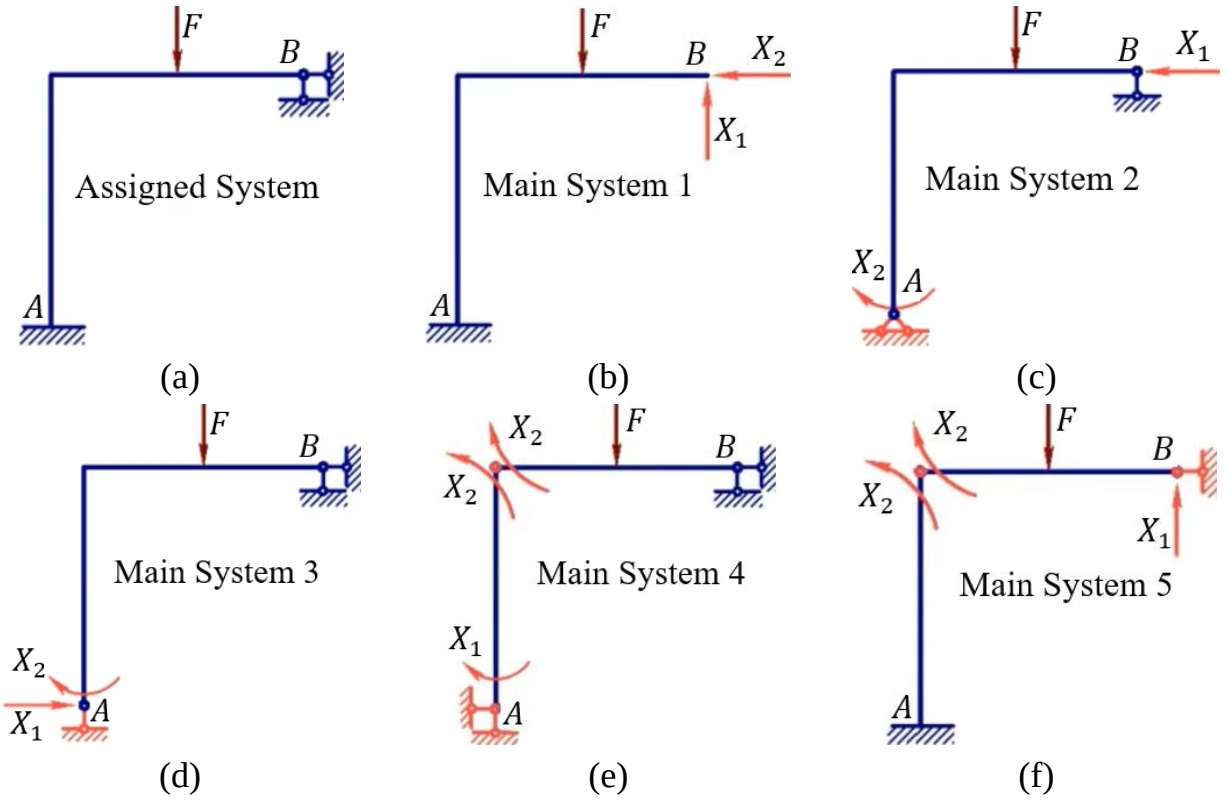


Fig. 3.2.1

In particular cases, cutting a rod at hinge releases two connections, and cutting a straight rod with hinges at the ends releases one connection (Fig. 3.2.2 b).

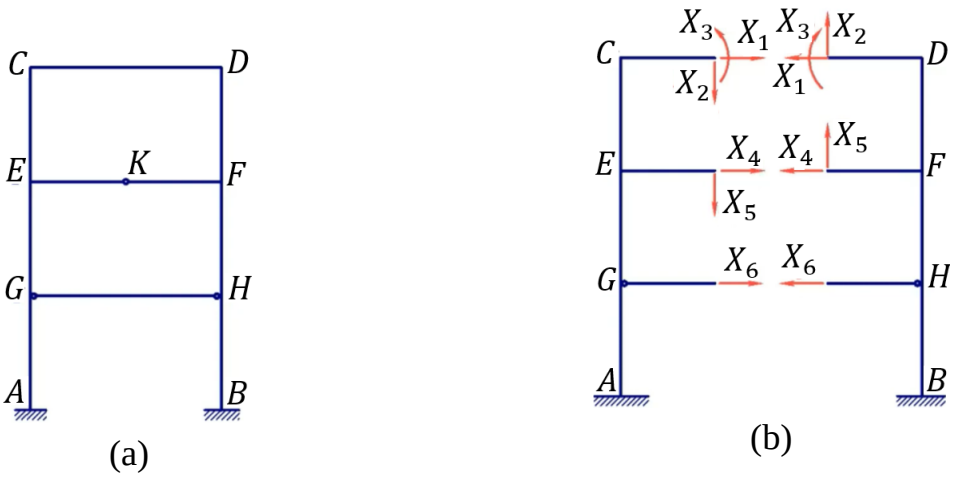
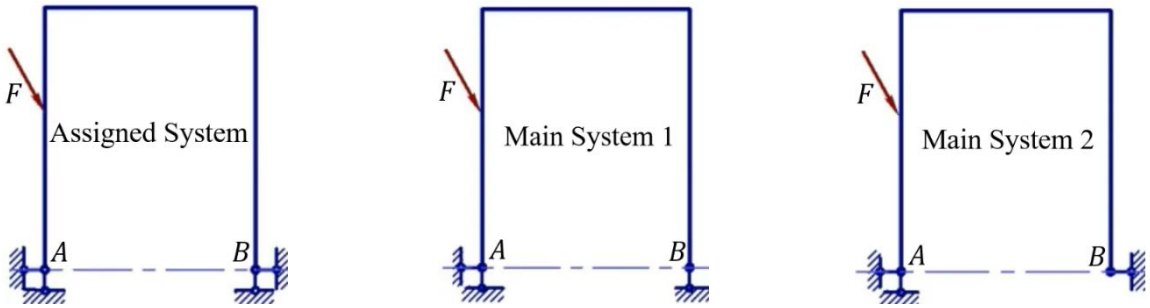


Fig. 3.2.2

The absolutely necessary and conditionally necessary connections of mechanical systems. Absolutely necessary are those connections, which elimination turns a statically indefinable system (Fig. 3.2.3 a) into a geometrically variable one (Fig. 3.2.3 b).



(a) (b) (c)

Fig. 3.2.3

Conditionally necessary are the connections, which when removed, the statically indefinable system remains geometrically unchangeable (Fig. 3.2.3 c).

Questions and comments:

3.2.1. The main system of a statically indefinable mechanical system, for which the equations of compatibility of deformations are compiled.

3.2.2. Extra (the redundant) connections of statically indefinable mechanical system.

3.2.3. The options of basic (the main) system.

3.2.4. The ways to eliminate unnecessary connections when creating main systems.

3.2.5. Conditionally necessary connections in mechanical systems.

3.3. Canonical equations of the method of forces when finding forces in the elements of statically indefinable system

Physical meaning of the canonical equations of the forces method. The canonical equations of the forces method are the equations that complement the balance missing conditions of statically indefinable systems. Canonical equations are a form of compatibility conditions for deformations (the displacements) of a mechanical system. The number of canonical equations is equal to the degree of static indefinability of the mechanical system.

After obtaining the main system by discarding unnecessary connections in assigned statically indefinable mechanical system, it is necessary to ensure the equivalence of the resulting main system and the given statically indefinable system. This equivalence one can define in the form of canonical equations, which are drawing up for the appropriate statically definable main system.

When defining the main system each the canonical equation means that the displacement in the direction of one of extra (the redundant) connections discarded is equal to zero: $\Delta_i = 0$.

The displacement Δ_i in the direction of each of the discarded connections is equal to zero for exactly determined values of reactions (the forces), which are the main unknowns in the canonical equations of the forces method.

Taking into account the concept of independence of the action of forces, the sum of displacements in the direction of the discarded connection from the action of forces in the direction of each of the discarded connections Δ_{ik}, as well as from the action of the external load Δ_{iF} is equal to zero, which in the form of the canonical equation of the forces method has the form: $\Delta_i = \Delta_{i1} + \Delta_{i2} + \cdots + \Delta_{ik} + \cdots + \Delta_{in} + \Delta_{iF} = 0$, where Δ_{ik} – the displacement in the direction of connection i from the action of force from the side of connection k, Δ_{iF} – the displacement in the direction of connection i from the action of an external load F.

Having into account the proportionality of the displacement Δ_{ik} in direction i of each of the connections discarded during the creation of a statically definable main system from a given statically indefinable system, from the action of the force in each direction k of all discarded connections n and of external load F, the term of the canonical equation of the forces method one can draw up as: $\Delta_{ik} = \delta_{ik} \times X_k$, where X_k – the force in the direction of the reaction from the connection k, discarded during the formation of a statically definable main system from a statically indefinable assigned system, δ_{ik} – the «unit displacement», that is displacement in the direction of the connection i from a unit force ($X_k = 1$) corresponded the side of the discarded connection k that is unit displacement δ_{ik} (the displacement in definite direction i under impact of unit force X_k: $k = 11, \dots, n$), the canonical equation of the forces method one can draw up as:

$$\Delta_i = \delta_{i1} \times X_1 + \delta_{i2} \times X_2 + \cdots + \delta_{ik} \times X_k + \cdots + \delta_{in} \times X_n + \Delta_{iF} = 0.$$

The system of canonical equations of the forces method has the form:

$$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \cdots + \delta_{1k} \times X_k + \cdots + \delta_{1n} \times X_n + \Delta_{1F} = 0,$$
$$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \cdots + \delta_{2k} \times X_k + \cdots + \delta_{2n} \times X_n + \Delta_{2F} = 0,$$
$$\dots\dots\dots

Questions and comments:

3.3.1. The physical meaning of canonical equations of the method of forces – the canonical are the missing equations that complement the balance conditions of statically indefinable systems.

3.3.2. The correspondence in the form of canonical equations of the resulting main system and a given statically indefinable system.

3.3.3. The concept of independence (the linear superposition) of the actions of forces when drawing up the canonical equations.

3.3.4. Proportionality the effort in each direction of all discarded connections and the displacement in the direction of each discarded connection when drawing up the main system of the considered mechanical system.

3.4. Coefficients and free terms of canonical equations of the forces method when calculating forces in the elements of statically indefinable system

The statically indefinable frame is assigned, consisting of two connected elements of equal stiffness ($E \times I$), loaded with an external force F (Fig. 3.4.1 a). Statically definable main system is considered (Fig. 3.4.1 b) obtained from a given statically indefinable system.

Fig. 3.4.1

The system of canonical equations of the forces method for a doubly statically indefinable mechanical system has the form:

$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \Delta_{1F} = 0,$

$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \Delta_{2F} = 0.$

The coefficients and free terms of canonical equations of the forces method are the displacements under the action of unit forces.

Unit displacement δ_{ik} in direction of connection i, that is the displacement under action of unit force ($X_k = 1$) considering the rigidity of the corresponding section of the

mechanical system ($E \times I$, where E – the modulus of elasticity of the structural material, I – the moment of inertia of the cross–section of the considered element of the mechanical system) one can found by multiplying the moment diagrams (M_i and M_k) and by using of the Christian Otto Mohr's integral over sections of a mechanical system, as:

$\delta_{ik} = \sum \int \frac{M_i \times M_k}{E \times I} dx.$

There is also a well–known point of view that priority in relation to the C.O. Mohr's integral method in the mechanics of elastic bodies when finding the displacements belongs to another researcher – James Clerk Maxwell.

The load displacements under the influence of an external load are found in a similar way – by multiplying the moment diagrams (M_i and M_F) and applying the Mohr's integral over sections of the mechanical system: $\Delta_{iF} = \sum \int \frac{M_i \times M_F}{E \times I} dx.$

To find the displacements (δ_{ik} and Δ_{iF}) numerically, in particular, the method of multiplying the area of one diagram by the ordinate of the second diagram under the center of the area of the first one can be used, if the diagrams are located in a projective link (the Vereshchagin's method). To perform this calculation, one has to determine the indicated diagrams.

Diagrams of the moments (M_1 and M_2) under unit forces (X_1 and X_2) are shown in the schemes (Fig. 3.4.2 a and b).

(a) (b)

Fig. 3.4.2

Finding of the coefficients and free terms of canonical equations of the forces method, using diagrams (Fig. 3.4.2 a and b, Fig. 3.4.3 a) when determining the terms in the resulting displacement formulas (δ_{ik} and Δ_{iF}) looks like:

$\delta_{11} = \sum \int \frac{M_1 \times M_1}{E \times I} dx = \frac{1}{E \times I} \times \left[\left(\frac{1}{2} \times l \times l \right) \times \frac{2}{3} \times l + (l \times l) \times l \right] = \frac{4 \times l^3}{3 \times E \times I},$

$\delta_{12} = \delta_{21} = \sum \int \frac{M_1 \times M_2}{E \times I} dx = -\frac{1}{E \times I} \times (l \times l) \times \frac{l}{2} = -\frac{l^3}{2 \times E \times I},$

$\delta_{22} = \sum \int \frac{M_2 \times M_2}{E \times I} dx = \frac{1}{E \times I} \times \left(\frac{1}{2} \times l \times l \right) \times \frac{2}{3} \times l = \frac{l^3}{3 \times E \times I},$

$\Delta_{1F} = \sum \int \frac{M_1 \times M_F}{E \times I} dx = \frac{1}{E \times I} \times \left(\frac{1}{2} \times \frac{F \times l}{2} \times \frac{l}{2} \right) \times l = \frac{F \times l^3}{8 \times E \times I},$

$$\Delta_{2F} = \sum \int \frac{M_2 \times M_F}{E \times I} dx = -\frac{1}{E \times I} \times \left(\frac{1}{2} \times \frac{F \times l}{2} \times \frac{l}{2}\right) \times \left(\frac{1}{3} \times \frac{l}{2} + \frac{2}{3} \times l\right) = -\frac{5 \times F \times l^3}{48 \times E \times I}.$$

(a) (b)

Fig. 3.4.3

The verification of the correctness of determination of unit displacements δ_{ik}, that is each of the displacements in the direction of the i–th connection from the k–th unit force X_k. For a line–by–line check one can create and use the total diagram M_S of bending moments (Fig. 3.4.3 b) from both unit forces ($X_1$and X_2), which is the sum of diagrams (M_1 and M_2, Fig. 3.4.2 a and b) from each from unit forces ($X_1$and X_2) separately.

The sum of displacements along the direction of the first of the replaced connections from of unit forces application ($X_1$and X_2) during the formation of the main statically definable mechanical system from a given statically indefinable system is equal to: $\sum \int \frac{M_1 \times M_S}{E \times I} dx = \frac{1}{E \times I} \times \left[\left(\frac{1}{2} \times l \times l\right) \times \frac{2}{3} \times l + \left(\frac{1}{2} \times l \times l\right) \times l\right] = \frac{5 \times l^3}{6 \times E \times I}$.

On the other hand, the sum of displacements is: $\sum \int \frac{M_1 \times M_S}{E \times I} dx = \delta_{11} + \delta_{12}$.

The values of displacements along the direction of the first of the replaced connections are determined above: $\delta_{11} = \frac{4 \times l^3}{3 \times E \times I}$ and $\delta_{12} = -\frac{l^3}{2 \times E \times I}$, therefore, the sum of displacements equals to: $\delta_{11} + \delta_{12} = \frac{4 \times l^3}{3 \times E \times I} - \frac{l^3}{2 \times E \times I} = \frac{5 \times l^3}{6 \times E \times I}$.

Sum of displacements along the direction of the second of the replaced connections of unit forces ($X_1$and X_2) application is equal to:

$\sum \int \frac{M_2 \times M_S}{E \times I} dx = -\frac{1}{E \times I} \times \left(\frac{1}{2} \times l \times l\right) \times \frac{1}{3} \times l = -\frac{l^3}{6 \times E \times I}$.

On the other hand, the sum of displacements is: $\sum \int \frac{M_2 \times M_S}{E \times I} dx = \delta_{21} + \delta_{22}$.

The values defined above are: $\delta_{21} = -\frac{l^3}{2 \times E \times I}$ and $\delta_{22} = \frac{l^3}{3 \times E \times I}$, therefore, the sum of displacements is equal to: $\delta_{21} + \delta_{22} = -\frac{l^3}{2 \times E \times I} + \frac{l^3}{3 \times E \times I} = -\frac{l^3}{6 \times E \times I}$.

Assessing the correctness of finding the total amount of displacements from unit forces ($X_1$and X_2): $\sum \int \frac{M_S \times M_S}{E \times I} dx = 2 \times \frac{1}{E \times I} \times \left(\frac{1}{2} \times l \times l\right) \times \frac{2}{3} \times l = \frac{2 \times l^3}{3 \times E \times I}$.

On the other hand, the sum of displacements is:

$\sum \int \frac{M_S \times M_S}{E \times I} dx = \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22} = \frac{5 \times l^3}{6 \times E \times I} - \frac{l^3}{6 \times E \times I} = \frac{2 \times l^3}{3 \times E \times I}$.

Assessing the correctness of finding the total of displacements under the action of a given load F: $\sum \int \frac{M_S \times M_F}{E \times I} dx = \frac{1}{E \times I} \times \left(\frac{1}{2} \times \frac{F \times l}{2} \times \frac{l}{2}\right) \times \frac{1}{3} \times \frac{l}{2} = \frac{F \times l^3}{48 \times E \times I}$.

On the other hand, the total sum of displacements under the action of a given load F is equal to: $\sum \int \frac{M_S \times M_F}{E \times I} dx = \Delta_{1F} + \Delta_{2F} = \frac{F \times l^3}{8 \times E \times I} - \frac{5 \times F \times l^3}{48 \times E \times I} = \frac{F \times l^3}{48 \times E \times I}$.

Questions and comments:

3.4.1. The main system of a statically indefinable frame.

3.4.2. System of canonical equations of the forces method for a doubly statically indefinable mechanical system.

3.4.3. The determination of the coefficients of the canonical equations of forces method using Mohr's integral: of displacements in the direction of replaced connections under unit forces applied in the direction of each replaced connection and of displacements in the direction of replaced connections under the action of applied load.

3.4.4. Finding the Mohr's integral (of displacements from unit forces and from a specified load) by multiplying diagrams using Vereshchagin's method.

3.4.5. The use of the Mohr's integral to check the correctness of finding displacements under unit forces (coefficients of the canonical equations of the forces method) and under the action of the specified load by determining and using the total diagram of bending moments from all unit forces.

3.5. Decision of the canonical equations of the force method and determining the diagrams of efforts in the statically indefinable mechanical system

The use of canonical equation of the forces method to find reactions of supports of a statically indefinable mechanical system and forces in the elements of the system. The system of canonical equations of the forces method was obtained above (section 3.4) for a doubly statically indefinable frame consisting of two connected elements of equal stiffness ($E \times I$), loaded with an external force F (Fig. 3.5.1 a):

$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \Delta_{1F} = 0$,

$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \Delta_{2F} = 0$, with the coefficients:

$\delta_{11} = \frac{4 \times l^3}{3 \times E \times I}$, $\delta_{12} = \delta_{21} = -\frac{l^3}{2 \times E \times I}$, $\delta_{22} = \frac{l^3}{3 \times E \times I}$, $\Delta_{1F} = \frac{F \times l^3}{8 \times E \times I}$ and $\Delta_{2F} = -\frac{5 \times F \times l^3}{48 \times E \times I}$.

(a) (b)

Fig. 3.5.1

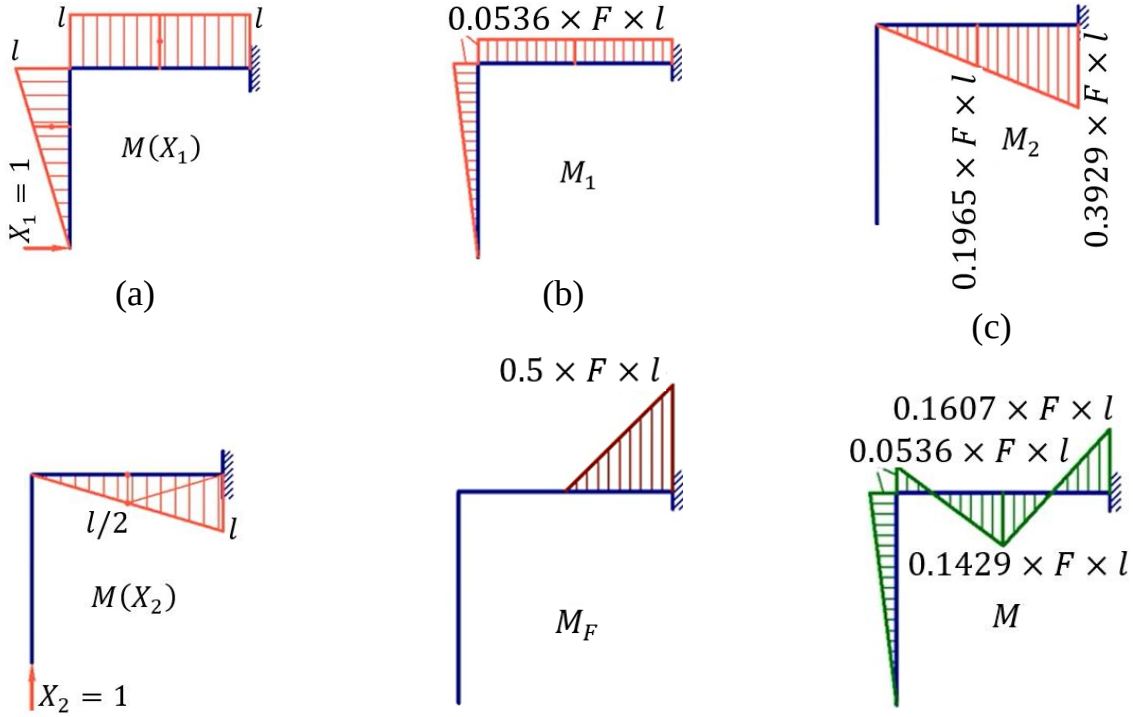
The solution of system of the canonical equations of the forces method has the form:

$\frac{4}{3} \times X_1 - \frac{1}{2} \times X_2 + \frac{1}{8} \times F = 0,$

$-\frac{1}{2} \times X_1 + \frac{1}{3} \times X_2 - \frac{5}{48} \times F = 0.$

The reactions in support B are equal: $X_1 = 0.0536 \times F$ and $X_2 = 0.3929 \times F$.

Diagrams of bending moments from unit forces ($X_1 = 1$ and $X_2 = 1$) are shown on the schemes (Fig. 3.5.2 a and d). Considering them one can create the diagrams of bending moments (Fig. 3.5.2 b and c) for the found values of the reactions in the supports (forces in the replaced connections: X_1 and X_2).



(a) (b) (c)

(d) (e) (f)

Fig. 3.5.2

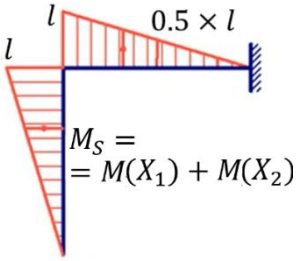
Determining the diagrams of bending moments (Fig. 3.5.2 b and c) for the found reaction values in the supports (X_1 and X_2): $M_1 = M(X_1) \times X_1$ and $M_2 = M(X_2) \times X_2$.

The final diagram of bending moments M (Fig. 3.5.2 f) is the summation of the obtained diagrams (M_1 and M_2, Fig. 3.5.2 b and c) for the found values of reactions in the supports (X_1 and X_2), as well as diagrams of bending moments of external load (M_F, Fig. 3.5.2 e): $M = M_1 + M_2 + M_F$.

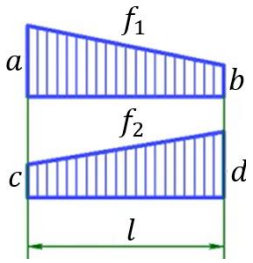
To check the correctness of determining of the final diagram of bending moments M (Fig. 3.5.2 f) one can use found displacements that would be equal to zero in the directions of reactions in the supports (X_1 and X_2), using the determined diagrams (Fig. 3.5.2 a, d and f) and formulas: $\Delta_1 = \sum \int \frac{M_1 \times M}{E \times I} dx = 0$ and $\Delta_2 = \sum \int \frac{M_2 \times M}{E \times I} dx = 0$, and also, the universal verification would be done using the total diagram M_S of bending moments (Fig. 3.5.3 a) both under unit forces (X_1 and X_2), which is the sum of diagrams (Fig. 3.5.2 a and d): $M_S = M(X_1) + M(X_2)$ and $\Delta_S = \sum \int \frac{M_S \times M}{E \times I} dx = 0$.

Initially, the universal verification is performed (one searches the displacement Δ_S), and if the universal verification does not confirm the correctness of the determination of the reactions in the supports, then the displacements are sought separately, which would be equal to zero in directions of the reactions in supports (X_1 and X_2).

To find the integral of a product of functions, the formula of trapezoid can be used (Fig. 3.5.3 b): $\int f_1(x) f_2(x) dx = \frac{l}{6} \times (2 \times a \times c + 2 \times b \times d + a \times d + b \times c)$.



(a)



(b)

Fig. 3.5.3

One can use the formula of trapezoid for the universal test conditions, find displacement Δ_S in the support B: $\Delta_S = \sum \int \frac{M_S \times M}{E \times I} dx = \frac{l}{6} \times \frac{F \times l^2}{E \times I} \times 2 \times 0.0536 \times 1 +$

$+ \frac{l}{2} \times \frac{F \times l^2}{6 \times E \times I} \times (2 \times 0.0536 \times 1 - 2 \times 0.1429 \times 0.5 + 0.0536 \times 0.5 - 1 \times 0.1429) +$

$+ \frac{l}{2} \times \frac{F \times l^2}{6 \times E \times I} \times (-2 \times 0.1429 \times 0.5 + 0.5 \times 0.1607) = 4.2 \times 10^{-6} \approx 0.$

To create diagrams of transverse (the shear) forces Q (Fig. 3.5.4 a) the differential dependence of transverse forces and bending moments: $Q = M'$ and the diagram of the bending moments M (Fig. 3.5.2 f) is used.

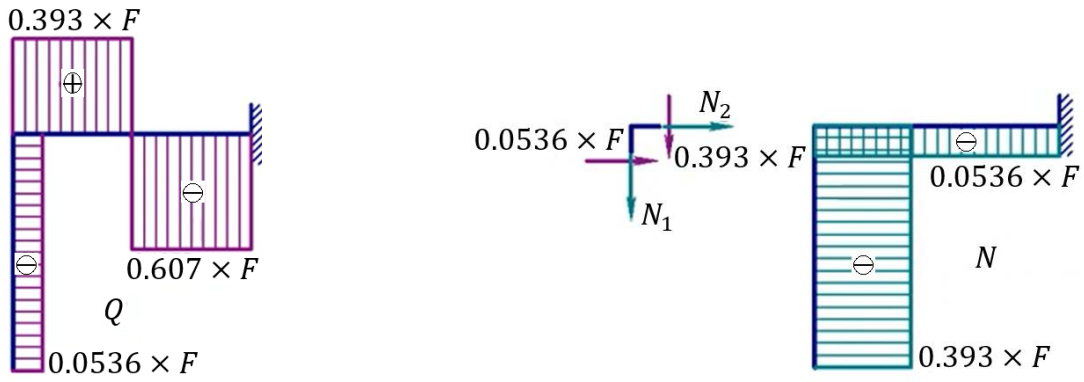


Fig. 3.5.4

For the condition of a linear shape of the moment diagram M (Fig. 3.5.2 f) by the sections i of the mechanical system the values of transverse (the shear) Q force used to determine the diagram (Fig. 3.5.4 a) are: $Q_i = \frac{M_{Ri} - M_{Li}}{l_i}$, where M_{Ri} and M_{Li} – the values of moments on the right and on the left boundaries of section i of the mechanical system within the linearity of the diagram of bending moments, l_i – the length of section of the mechanical system.

Values of transverse force Q, indicated on the diagram (Fig. 3.5.4 a) dependent on applied force F are: $Q_1 = \frac{-0.0536 \times F \times l - 0}{l} = -0.0536 \times F$,

$Q_2 = \frac{0.1429 \times F \times l - (-0.0536 \times F \times l)}{0.5 \times l} = 0.393 \times F$

and $Q_3 = \frac{-0.1607 \times F \times l - 0.1429 \times F \times l}{0.5 \times l} = -0.607 \times F$.

To determine the diagram of longitudinal forces N (Fig. 3.5.4 b) the nodes are cut out and the equations of balance are drawn up for all internal and external forces.

The values of longitudinal forces in the elements of the mechanical system are equal: $N_1 = -0.393 \times F$ and $N_2 = -0.0536 \times F$.

Questions and comments:

3.5.1. The solving of system of canonical equations of the forces method.

3.5.2. The use of diagrams of bending moments under unit forces and found values of reactions in supports (the forces in replaced connections) to determine diagrams of bending moments in elements of given statically indefinable system.

3.5.3. Checking the correctness of the final diagram of bending moments by finding the displacements in the replaced connections of a statically indefinable mechanical system, which must be equal to zero.

3.5.4. A universal check for the equality of conditional displacement to zero, which is determined using a diagram of bending moments, which is the sum of diagrams of bending moments from each of the found forces in the replaced connections.

3.5.5. The use of formula of trapezoid to find the integral of a product of functions.

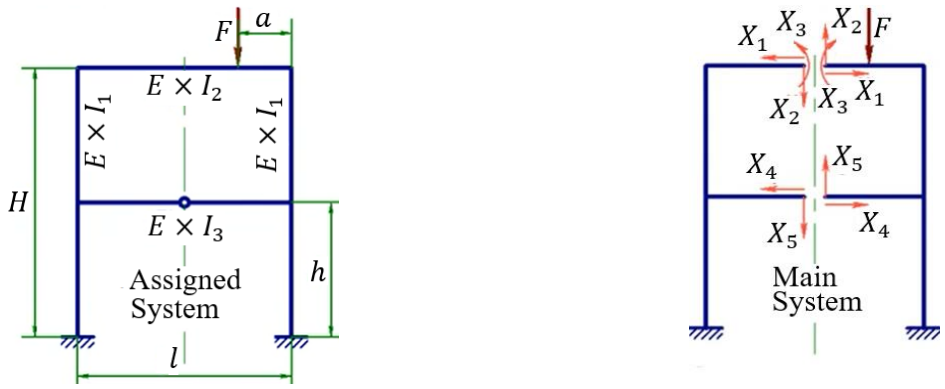
3.5.6. The shear forces diagram determining by differentiating the diagram of bending moments.

3.5.7. The longitudinal diagram determining of by cutting out nodes and drawing up equations of balance of internal and external forces.

3.6. Simplification of the canonical equations of the forces method under the condition of symmetry of the mechanical system

A mechanical system is symmetrical about an axis, if using a straight–line coinciding with the axis of symmetry, this system can be divided into two parts, each of which is a mirror image of the other. A statically indefinable mechanical system is symmetrical if not only the symmetry of the axes and support directions is satisfied, but also the symmetry of the rigidity indexes of the system elements.

One can simplify a statically indefinable symmetrical system (Fig. 3.6.1 a) for the calculation if chooses a symmetrical main system. The symmetrical main system is shown on the scheme (Fig. 3.6.1 b). The degree of static indefinability for considered mechanical system are equal to five, as well as the required number of canonical equations of the forces method.



(a) (b)

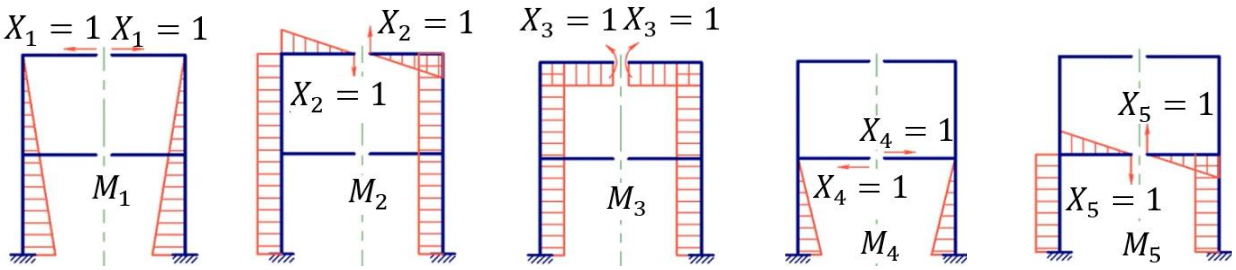
Fig. 3.6.1

The system of canonical equations of forces method for the main system under consideration has the form:

$$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \delta_{13} \times X_3 + \delta_{14} \times X_4 + \delta_{15} \times X_5 + \Delta_{1F} = 0,$$
$$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \delta_{23} \times X_3 + \delta_{24} \times X_4 + \delta_{25} \times X_5 + \Delta_{2F} = 0,$$
$$\delta_{31} \times X_1 + \delta_{32} \times X_2 + \delta_{33} \times X_3 + \delta_{34} \times X_4 + \delta_{35} \times X_5 + \Delta_{3F} = 0,$$

$\delta_{41} \times X_1 + \delta_{42} \times X_2 + \delta_{43} \times X_3 + \delta_{44} \times X_4 + \delta_{45} \times X_5 + \Delta_{4F} = 0,$
$\delta_{51} \times X_1 + \delta_{52} \times X_2 + \delta_{53} \times X_3 + \delta_{54} \times X_4 + \delta_{55} \times X_5 + \Delta_{5F} = 0.$

The diagrams of unit forces moments in connections, which are replaced during the formation of the main system for the given symmetrical statically indefinable mechanical system, are shown on the schemes (Fig. 3.6.2 a – e). Moment diagrams under action of unit forces shown on the schemes are mirror symmetrical or obliquely symmetrical.



(a) (b) (c) (d) (e)

Fig. 3.6.2

When finding the displacements δ_{ik} of each k unit forces in the direction i of each of other unit forces in the connections that are replaced during formation of the main system, the Mohr's integral is determined by product of moment diagrams: $\Delta_{ik} = \sum \int \frac{M_i \times M_k}{E \times I} dx$, wherein, the integrals of products of a pairs of moment diagrams: the mirror–symmetric and skew–symmetric ones are equal to zero.

Therefore, the following displacements are equal to zero due to these symmetries:
$\delta_{12} = \delta_{21} = \delta_{15} = \delta_{51} = \delta_{23} = \delta_{32} = \delta_{24} = \delta_{42} = \delta_{35} = \delta_{53} = \delta_{45} = \delta_{54} = 0.$

As a result, the original system of canonical equations splits into two independent subsystems, each of which contains either unknown X_i, diagrams of moments, which are either only mirror symmetric or only skew–symmetric:

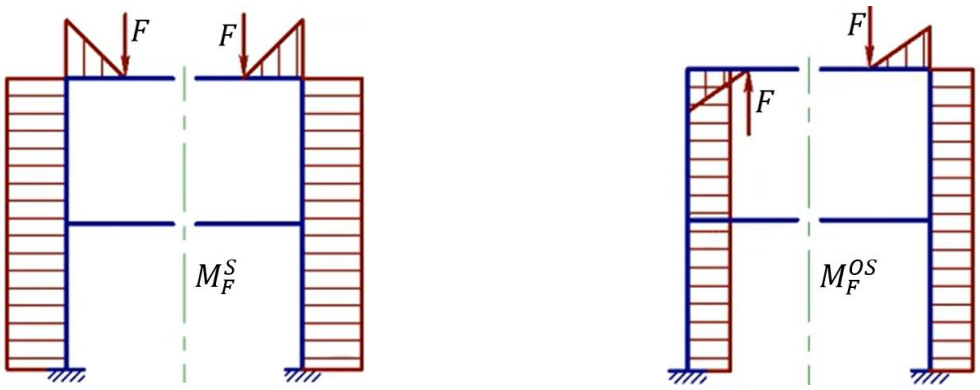
$\delta_{11} \times X_1 + \delta_{13} \times X_3 + \delta_{14} \times X_4 + \Delta_{1F} = 0,$
$\delta_{31} \times X_1 + \delta_{33} \times X_3 + \delta_{34} \times X_4 + \Delta_{3F} = 0,$
$\delta_{41} \times X_1 + \delta_{43} \times X_3 + \delta_{44} \times X_4 + \Delta_{4F} = 0.$

$\delta_{22} \times X_2 + \delta_{25} \times X_5 + \Delta_{2F} = 0,$
$\delta_{52} \times X_2 + \delta_{55} \times X_5 + \Delta_{5F} = 0.$

The independent terms of the canonical equations are the displacements Δ_{iF} in each of the directions i of unit forces in the connections, which are replaced during the formation of the main system from a given symmetrical statically indefinable mechanical system, under influence of an external load F.

If the external load is symmetrical (Fig. 3.6.3 a), then the displacements under external load Δ_{iF} in each direction i of the unit forces in the connections that are replaced when forming the main system from a given statically indefinable mechanical system, in equations with unknowns having skew–symmetric diagrams of moments, and these unknowns themselves are equal to zero, that is: $\Delta_{2F} = \Delta_{5F} = 0$ and $X_2 = X_5 = 0$.

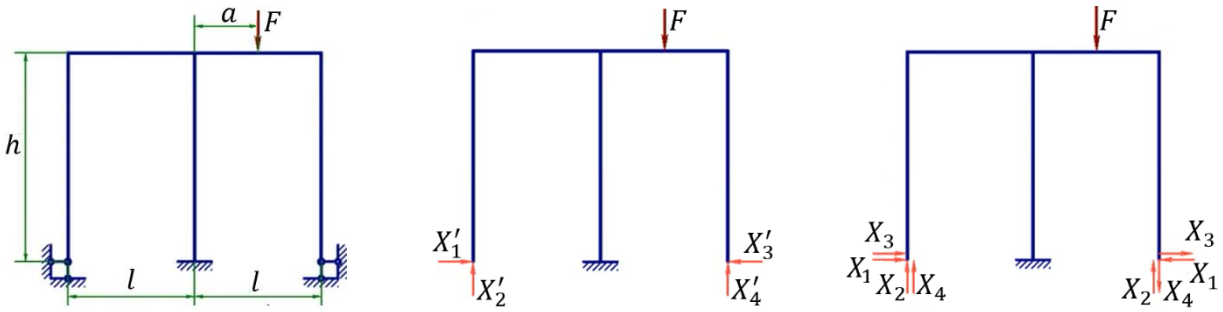
If the external load is skew–symmetric (Fig. 3.6.3 b), then the displacements under external load Δ_{iF} in each direction i of the forces in the connections that are replaced during the drawing up the main system, in equations with symmetric unknowns and the symmetric unknowns themselves are equal to zero, that is: $\Delta_{1F} = \Delta_{3F} = \Delta_{4F} = 0$ and $X_1 = X_3 = X_4 = 0$.



(a) (b)

Fig. 3.6.3

The grouping of unknowns when calculating statically indefinable symmetric system is explained using diagrams (Fig. 3.6.4 a). When choosing the main system, in cases when the connections that do not lie on the axis of symmetry are discarded, and the diagrams of moments from unit forces become asymmetrical, one can use the grouping of unknowns, which consists in the fact that the forces in symmetrically discarded connections are represented in the form of two groups of forces, one of which is symmetrical (Fig. 3.6.4 b), and the second is obliquely symmetrical (Fig. 3.6.4 c).



(a) (b) (c)

Fig. 3.6.4

The relationships between the main unknowns of the canonical equations of the forces method before and after grouping the unknowns according to the criteria of symmetry and skew–symmetry have the form:

$X_1' = X_1 + X_3$, $X_2' = X_2 + X_4$, $X_3' = X_1 - X_3$ and $X_4' = X_2 - X_4$,

$X_1 = \frac{1}{2}(X_1' + X_3')$, $X_2 = \frac{1}{2}(X_2' + X_4')$, $X_3 = \frac{1}{2}(X_1' - X_3')$ and $X_4 = \frac{1}{2}(X_2' - X_4')$.

When finding displacements (the coefficients in the canonical equations of the method of forces) and when grouping unknowns, the diagrams of moments from unit forces in connections are used, which are replaced when forming the main system from given symmetrical statically indefinable mechanical system (Fig. 3.6.5).

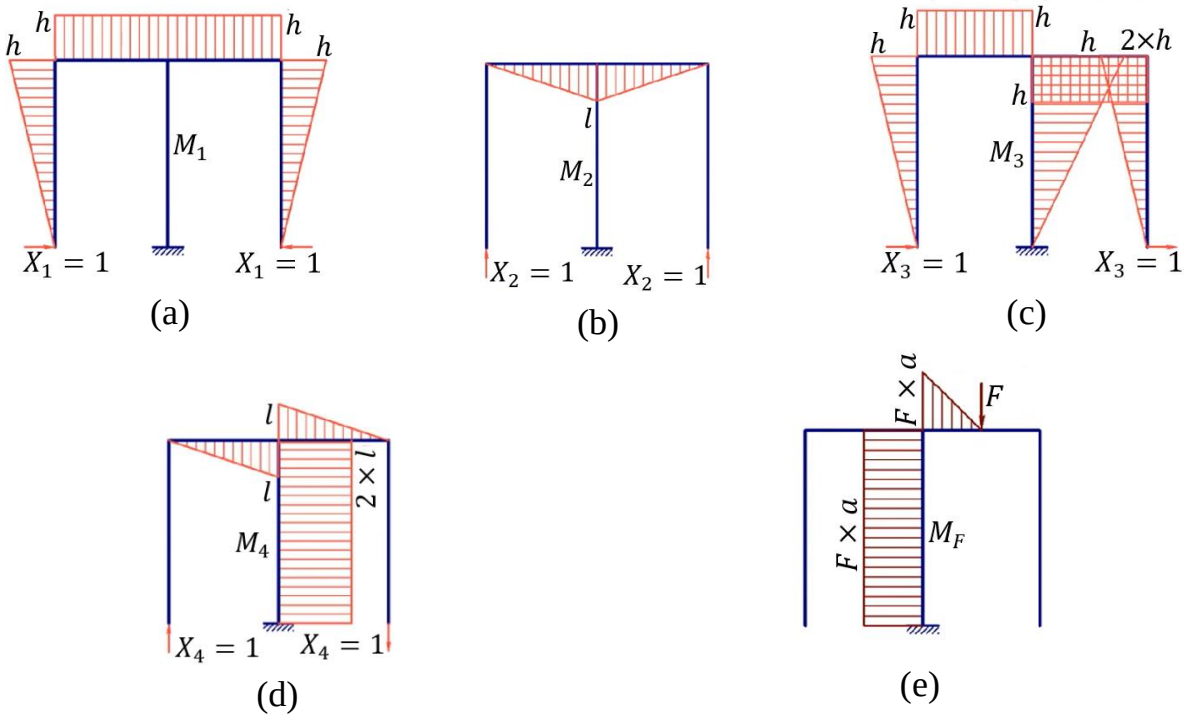


Fig. 3.6.5

Mohr's integrals when multiplying each pair of moment diagrams: the mirror symmetric and skew–symmetric one are equal to zero, therefore the following displacements are zeros: $\delta_{13} = \delta_{31} = \delta_{14} = \delta_{41} = \delta_{23} = \delta_{32} = \delta_{24} = \delta_{42} = 0$.

Considering the grouping of terms, the system of canonical equations of the forces method takes the form of two independent systems of equations:

$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \Delta_{1F} = 0$, $\quad\quad$ $\delta_{33} \times X_3 + \delta_{34} \times X_4 + \Delta_{3F} = 0$,

$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \Delta_{2F} = 0$. $\quad\quad$ $\delta_{43} \times X_3 + \delta_{44} \times X_4 + \Delta_{4F} = 0$.

Having these two independent systems of equations obtained and having found the unknowns (X_1, X_2, X_3 and X_4), one can determine the reactions of the supports (X_1', X_2', X_3' and X_4') of given statically indefinable symmetric system (Fig. 3.6.4 a), using the relations of the unknowns in the canonical equations of the forces method before and after their grouping specified above.

Questions and comments:

3.6.1. The symmetry qualities of statically indefinable mechanical system.

3.6.2. Simplifying the calculation of statically indefinable symmetric system by choosing a symmetric main system.

3.6.3. The mirror–symmetric and skew–symmetric diagrams of moments in the elements of main system formed for a given symmetric statically indefinable mechanical system.

3.6.4. Equality to zero of Mohr's integral values of the product of moment diagrams, one of which is symmetric and the second one is skew–symmetric.

3.6.5. The simplification of canonical equations system of the forces method, in which the system splits into two systems of equations containing unknowns, which diagrams of moments are either only mirror symmetric or only the skew–symmetric.

3.6.6. Equality to zero of displacements of a symmetric external load (the free terms of canonical equations of the forces method) in equations with unknowns having skew–symmetric diagrams of moments, and equality to zero of these unknowns themselves.

3.6.7. Asymmetrical diagrams of moments from efforts in discarded connections of a static indefinable system during the formation of the main system, if these connections do not lie on the axis of symmetry.

3.6.8. Grouping of unknowns when calculating a statically indefinable symmetric system using the forces method: the replacing of discarded constraints with forces that are included in one of two groups, one of which is symmetric and the second is skew–symmetric.

3.6.9. Relationships between the main unknowns of canonical equations of the forces method before and after grouping the unknowns according to the criteria of symmetry and skew–symmetry.

3.6.10. Solving independent systems of canonical equations of the forces method to find unknown forces having symmetric moment diagrams and the skew–symmetric ones.

3.6.11. The use of the relationships between the main unknowns of canonical equations of the forces method before and after grouping the unknowns according to the criteria of symmetry and skew–symmetricity.

3.7. Application of Mohr's integral method to determine the displacements of nodes during deformation of an elastic mechanical system

Mohr's integral used to find the displacements of points of a mechanical system, associated with Castigliano's theorem, showing that the partial derivative of the potential

energy of a mechanical system with respect to the value of the force is equal to the displacement of the point of force application in the direction of its action, which allows us to find the displacements of the points of the application of external forces when exposed to them.

When using Mohr's integral to find the displacement of any point in any direction under the influence of an external load on a mechanical system, a unit force is applied in this direction, the diagrams of forces are determined in the system elements of applied unit force and, over sections of the mechanical system, the integrals of the products of diagrams of moment under the external load and the diagrams of moment under unit forces applied in the directions of the displacements sought at the points of these displacements are determined in the general case as: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx + \sum \int \mu \times \frac{Q \times Q_i}{G \times A} dx + \sum \int \frac{N \times N_i}{E \times A} dx$, where M, Q and N – the diagrams of bending moments, transverse (the shear) forces and longitudinal forces of the specified load acting on a mechanical system, M_i, Q_i and N_i – the diagrams of bending moments, transverse forces and longitudinal forces from a unit load (a unit force or moment), in the direction of which displacement from the specified load is sought, E – Young's modulus of elasticity, G – the shear modulus of elasticity, I – the moment of inertia of the section of an element of a loaded mechanical system, A – the cross–sectional area of the system element, μ – the section shape factor (dimensionless coefficient) taken into account when finding the shear displacement.

In general, the section shape factor μ (Fig. 3.7.1), taken into account when finding deformation and the shear displacement, one can determine as: $\mu = \frac{A}{I_X^2} \times \int_A \frac{M_{AX}^2}{b^2} \times dA$, where A and dA – the cross–sectional area of a mechanical system element and differential area increment, M_{AX}^2 – the square of static moment of increment of the cross–sectional area relative to the axis $0X$, I_X^2 – square of the moment of inertia of the section relative to the axis $0X$, b – width of incremented sectional area element dA.

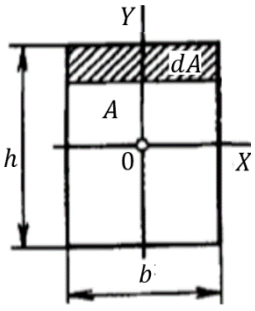


Fig. 3.7.1

In particular cases, for a solid rectangular section this coefficient is equal to relative to any of the two axes (X or Y): $\mu = 6/5$, for a solid circular section: $\mu = 10/9$, for a thin–walled circular profile (the pipe): $\mu = 2$ and etc.

The sequence of determining the displacements of mechanical system nodes:

(1) determining the diagrams of the specified load M, Q and N;

(2) determining the diagrams of unit forces M_i, Q_i or N_i, applied in the direction of the desired movement;

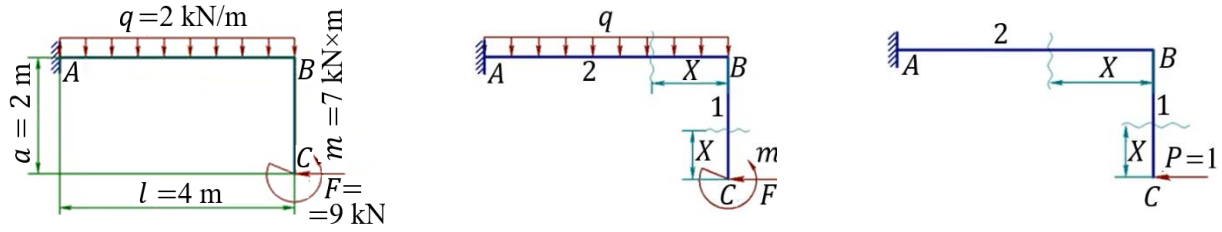
(3) use of the diagrams (M, Q, N, M_i, Q_i and N_i) and the Mohr's formula to find the nodes displacements in the options:

(a) of truss, whose elements are subject to compression and tension:

$\Delta_i = \sum \int \frac{N \times N_i}{E \times A} dx$;

(a) of beams and frames, that is the mechanical systems and elements, predominantly undergoing the bending: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx$.

Exercise. Calculate the horizontal displacement of the point C of the frame (Fig. 3.7.2 a) using Mohr's integral by direct integration by multiplying the force diagrams of the specified load and unit forces in the direction of the specified displacement.



(a) (b) (c)

Fig. 3.7.2

When solving this problem for a frame with rigid corners, the Mohr's integral is used in its version for mechanical system predominantly undergoing the bending:

$\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx$.

This frame is considered in sections in two states:

(a) in a state loaded with a given force (F =9 kN), moment (m =7 kN×m) and distributed load (q =2 kN/m);

(b) and in a state, loaded with a unit force (P =1) at the point C in the direction of its horizontal movement sought.

Section 1 (CB): $0 \le x \le 2$ m. The diagram of the bending moment of external load in the first section is determined by the formula: $M = m - F \times x = 7 - 9 \times x$.

The diagram of the bending moment in the first section under unit force (P =1) at the point C in the direction of the point horizontal displacement sought is determined by the formula: $M_i = -P \times x = -x$.

Section 2 (BA): $0 \le x \le 4$ m.

The diagram of bending moment of external load in the second section is determined as: $M = m - F \times a - \frac{q \times x^2}{2} = 7 - 9 \times 2 - \frac{2 \times x^2}{2} = -11 - x^2$.

The diagram of the bending moment in the second section under unit force (P =1) at the point C in the direction of the horizontal displacement is determined by the formula: $M_i = -P \times a = -a = -2$.

Substituting of the indicated values into Mohr's integral one can find the horizontal displacement of point C as: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx = \int_0^2 \frac{(7-9\times x)\times(-x)}{E \times I} dx + \int_0^4 \frac{(-11-x^2)\times(-2)}{E \times I} dx$, $\Delta_i = \int_0^2 \frac{9\times x^2 - 7\times x}{E \times I} dx + \int_0^4 \frac{2\times x^2 + 22}{E \times I} dx = \frac{1}{E \times I} \times \left[\left(9 \times \frac{2^3}{3} - 7 \times \frac{2^2}{2}\right) + \left(2 \times \frac{4^3}{3} + 22 \times 4\right)\right]$, $\Delta_i = \frac{140.67}{E \times I}$.

Questions and comments:

3.7.1. Castigliano's theorem, showing the dependence of displacement of points of application of forces and the energy of deformation of an elastic body by these forces.

3.7.2. Using Mohr's integral to find the displacement of any point in the direction of the applied unit force.

3.7.3. Finding the cross–section shape factor when determining the displacement under the action of transverse force using Mohr's integral.

3.7.4. The sequence of determining the displacements of nodes of a mechanical system using Mohr's integral and multiplying diagrams of forces (moments or forces).

3.7.5. Determination of the Mohr's integral by the method of direct integration by multiplying diagrams of forces from a given load and unit forces.

3.8. Finding the Mohr's integral using the Vereshchagin's method when determining the displacements of nodes in deformed elastic system

Accurate to the denominator – the constant value of stiffness ($E \times I$) for the Mohr's integral formula, one can calculate by the product of diagrams: $\int M \times M_i \times dx$, by multiplying them according to Vereshchagin's rule, which can be used if one of diagrams, as a rule, of a unit load M_i, are rectilinear. Another diagram of the moment, usually from the action of an external load M, can have any shape (rectilinear, curvilinear, in the form of a figure with a broken boundary).

Let consider the product of diagrams in the Mohr's integral: $\int M \times M_i \times dx$, taking into account the notation used: ω – the area of nonlinear diagram of the moment M from the influence of external load (Fig. 3.8.1), $d\omega = M \times dx$ – the elementary area of the nonlinear diagram M.

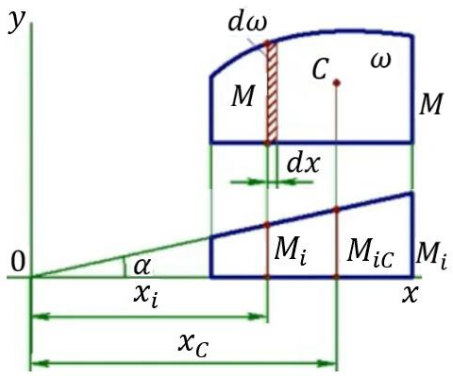


Fig. 3.8.1

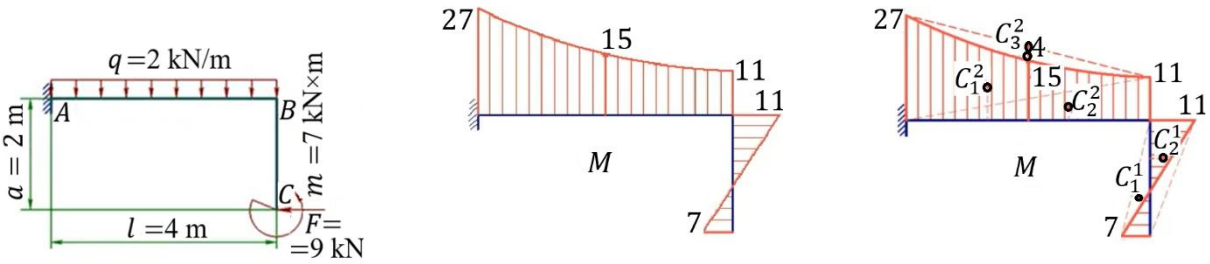
The ordinate of linear diagram M_i of the unit load is: $M_i = x_i \times tg\ \alpha$.

Taking into account the introduced notation, the product of diagrams in the Mohr's integral takes the form: $\int M \times M_i \times dx = tg\ \alpha\ \times \int x \times d\omega$, where: $\int x \times d\omega$ static moment ($x_C \times \omega$) of area of nonlinear diagram of the bending moment M of external load relative to the vertical axis $0y$: $\int M \times M_i \times dx = tg\ \alpha\ \times x_C \times \omega$.

That is: $\int M \times M_i \times dx = tg\ \alpha\ \times x_C \times \omega = \omega \times M_{iC}$, because: $tg\ \alpha\ \times x_C = M_{iC}$.

Consequently, the mathematical expression under the Mohr's integral corresponds to the product of area of the nonlinear diagram of bending moment M under external load impact and the diagram of the moment of unit load M_i in projection under the center of gravity of the moment diagram of the external load impact M: $\int M \times M_i \times dx = \omega \times y_C$.

Exercise. For frame calculation with specification conditions given above (Fig. 3.7.2 a and 3.8.2 a) by the method of direct integration when multiplying the diagrams of the efforts of given load and the unit efforts in the direction of the displacements sought find the displacements of the point C and the section turning using the Vereshchagin's method of the diagrams multiplying and the Mohr's integral method. Mechanical system (the frame) is considered in sections in four states: in a state loaded with a given force (F =9 kN), moment (m =7 kN×m) and the distributed load (q =2 kN/m), and in a state, loaded with unit force (P_1 =1) at the point C in the direction of it horizontal displacement sought, with unit force (P_2 =1) at the point C in the direction of it vertical displacement sought and with unit moment (P_3 =1) in the direction sought of the section turn at the point C. For a given frame, The exercise assigns one to determine the horizontal and vertical displacements of the point C, as well as the angle of the section turn. The rigidities of all sections are set constant and identical: $E \times I = const$.



(a) (b) (c)

Fig. 3.8.2

The diagram of bending moments of the assigned load action (Fig. 3.8.2 b) is determined, and the values of bending moments in the diagram are searched by the sections.

The section 1 (CB): $0 \le x \le 2$ m, $M = 7 - 9 \times x$, $M_C = 7$ kN×m and $M_B = 7 - 9 \times 2 = -11$, kN×m.

The section 2 (BA): $0 \le x \le 4$ m, $M_B = -11$ kN×m, $M_A = -11 - 4^2 = -27$, kN×m and $M_{1/2} = -11 - 2^2 = -15$, kN×m at $x = \frac{l}{2} = 2$.

The diagram of the bending moment under external load in the second section is described by the formula: $M = m - F \times a - \frac{q \times x^2}{2} = 7 - 9 \times 2 - \frac{2 \times x^2}{2} = -11 - x^2$.

Transformation of nonlinear moment diagram M of the external load impact is shown on scheme (Fig. 3.8.2 c). On the horizontal section 2 (BA) of the frame one supplements the diagram to a trapezoid with a center line 19 kN×m. One divides the trapezoid into two triangles with bases 27 and 11 kN×m and height 4 m.

Height of parabolic segment of nonlinear diagram of the bending moment M under external load impact on horizontal section 2 (BA) of the frame is:

$h = \frac{q \times l^2}{8} = \frac{2 \times 4^2}{8} = 4$, kN×m.

The area of this segment of bending moment diagram M is: $\omega_3 = \frac{2}{3} \times h \times l$, kN×m^2.

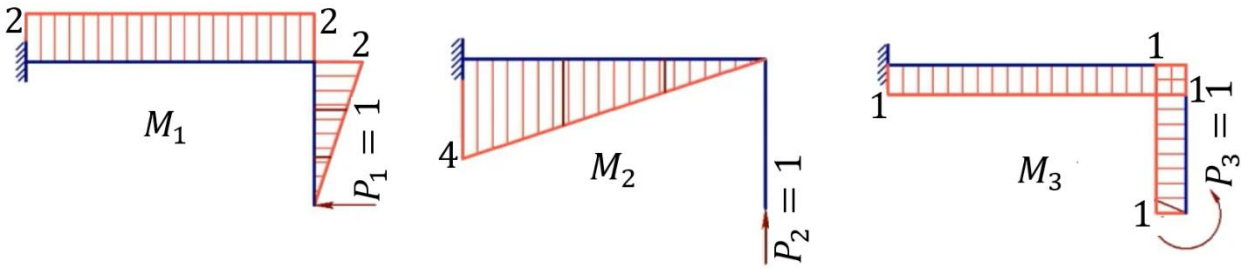
On the vertical section 1 (CB) of the frame, the bending moment diagram, which looks like a twisted trapezoid, can be replaced by two triangles with legs 7 and 11 kN×m and the common leg 2 m.

The creating of diagrams of bending moments of unit forces are performed in the schemes (Fig. 3.8.3 a, b and c).

Determination is done of displacements of the point C using Mohr's integral and the multiplying of diagrams by Vereshchagin's method. Finding the horizontal displacement of the point C (Fig. 3.8.2 c and Fig. 3.8.3 a): $\Delta_1 = \sum \int \frac{M \times M_1}{E \times I} dx = \sum \frac{\omega \times y_{C1}}{E \times I}$,

$$\Delta_1 = \frac{1}{E \times I} \times \left[\left(\frac{1}{2} \times 27 \times 4\right) \times 2 + \left(\frac{1}{2} \times 11 \times 4\right) \times 2 - \left(\frac{2}{3} \times 4 \times 4\right) \times 2\right] +$$

$$+ \frac{1}{E \times I} \times \left[-\left(\frac{1}{2} \times 7 \times 2\right) \times \frac{1}{3} \times 2 + \left(\frac{1}{2} \times 11 \times 2\right) \times \frac{2}{3} \times 2\right] = \frac{140.67}{E \times I}.$$



(a) (b) (c)

Fig. 3.8.3

The determining of vertical displacement of the point C (Fig. 3.8.2 c and Fig. 3.8.3 b) is done as: $\Delta_2 = \sum \int \frac{M \times M_2}{E \times I} dx = \sum \frac{\omega \times y_{C2}}{E \times I}$,

$\Delta_2 = \frac{1}{E \times I} \times \left[-\left(\frac{1}{2} \times 27 \times 4\right) \times \frac{2}{3} \times 4 - \left(\frac{1}{2} \times 11 \times 4\right) \times \frac{1}{3} \times 4 + \left(\frac{2}{3} \times 4 \times 4\right) \times 2 \right]$,

$\Delta_2 = -\frac{152}{E \times I}$.

The angle of the section turn of the frame at the point C (Fig. 3.8.2 c and Fig. 3.8.3 c) is calculated as: $\Delta_3 = \sum \int \frac{M \times M_3}{E \times I} dx = \sum \frac{\omega \times y_{C3}}{E \times I}$,

$\Delta_3 = \frac{1}{E \times I} \times \left[-\left(\frac{1}{2} \times 27 \times 4\right) \times 1 - \left(\frac{1}{2} \times 11 \times 4\right) \times 1 + \left(\frac{2}{3} \times 4 \times 4\right) \times 1 \right] +$

$+ \frac{1}{E \times I} \times \left[\left(\frac{1}{2} \times 7 \times 2\right) \times 1 - \left(\frac{1}{2} \times 11 \times 2\right) \times 1 \right] = -\frac{69.33}{E \times I}$.

Questions and comments:

3.8.1. The conditions under which it is possible the use of Vereshchagin's rule to determine Mohr's integral by multiplying the diagrams of forces when finding the displacements of points or the nodes of a mechanical system.

3.8.2. Mathematical form of the product of diagrams in the Mohr's integral using the Vereshchagin's rule and at the linearity of shape of diagram of unit force (or the bending moment).

3.8.3. Determining the diagram of bending moment of the specified load by sections of the mechanical system.

3.8.4. The determining by sections of the mechanical system of bending moments diagrams under the unit efforts applied at the points and in directions of the displacements sought.

3.8.5. Finding linear and angular displacements of specified point of the mechanical system using Mohr's integral and Vereshchagin's method to product the diagrams in the Mohr's integral.

3.9. Determination of the Mohr's integral using Simpson's formula and the method of trapezoid

Accurate to the denominator – the constant value of stiffness ($E \times I$) for the Mohr's integral formula, one can calculate the product of diagrams by multiplying them using Simpson's rule as: $\int M \times M_i \times dx = \frac{l}{6} \times (M_1 \times M_{i,1} + 4 \times M_{1/2} \times M_{i,1/2} + M_2 \times M_{i,2})$.

Wherein the sought displacement is equal to:

$\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx = \sum \frac{l}{6 \times E \times I} \times (M_1 \times M_{i,1} + 4 \times M_{1/2} \times M_{i,1/2} + M_2 \times M_{i,2})$,

To use Simpson's formula, it is not necessary to determine the area of the diagram and the location of its center of area. For both used diagrams (Fig. 3.9.1) it is enough to know the ordinates in three points of the section under consideration – in the initial point (M_1 and $M_{i,1}$), in the middle ($M_{1/2}$ and $M_{i,1/2}$) and at the end (M_2 and $M_{i,2}$).

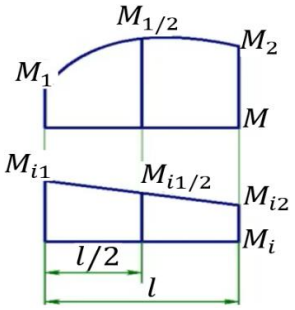


Fig. 3.9.1

If both used diagrams are the linear, then one can use the formula of trapezoid to multiply effort diagrams (the bending moments) in the Mohr's integral.

According to the formula of trapezoid, to multiply diagrams, it is enough to know the ordinates at the ends of diagrams in two points – at the initial point (M_1 and $M_{i,1}$, Fig. 3.9.2 a, b and c) and at the end (M_2 and $M_{i,2}$):

$\int M \times M_i \times dx = \frac{l}{6} \times (2 \times M_1 \times M_{i,1} + 2 \times M_2 \times M_{i,2} + M_1 \times M_{i,2} + M_2 \times M_{i,1})$.

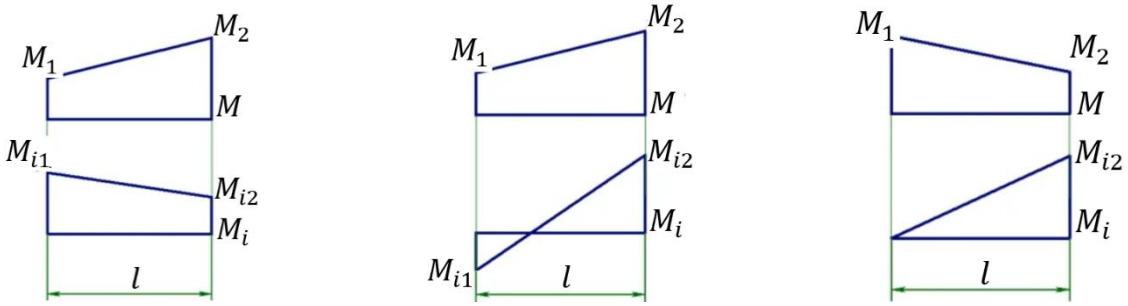


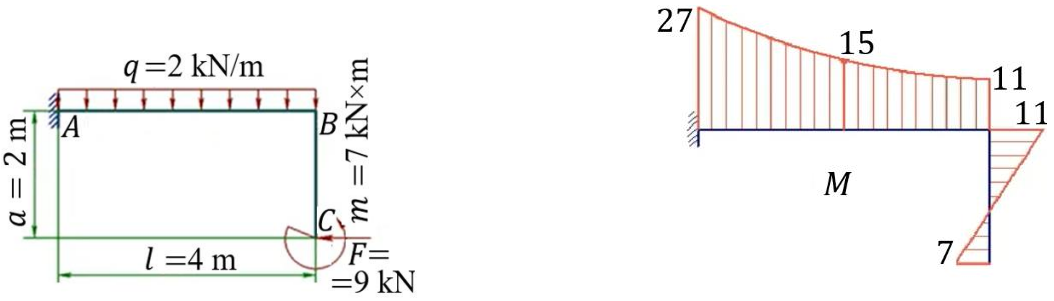
Fig. 3.9.2

Wherein the displacement sought is equal to: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx =$

$= \sum \frac{l}{6 \times E \times I} \times (2 \times M_1 \times M_{i,1} + 2 \times M_2 \times M_{i,2} + M_1 \times M_{i,2} + M_2 \times M_{i,1})$.

Exercise. For the given conditions of the frame calculation (Fig. 3.9.3 a), performed earlier by the Mohr's integral use (by direct multiplication of diagrams and the Vereshchagin's method of multiplying diagrams, Fig. 3.7.2 a and 3.8.2 a, items 3.7 and 3.8) The exercise assigns one to find the displacement of point C and the rotation angle using the Simpson's method and the method of trapezoid.

The mechanical system (the frame) is considered by the sections in four states: in the state loaded with given force (F =9 kN), the moment (m =7 kN×m) and the distributed load (q =2 kN/m), and in the state, loaded with unit force (P_1 =1) at the point C in the direction sought of it horizontal displacement, with unit force (P_2 =1) at the point C in the direction of it of the vertical displacement and with unit moment (P_3 =1) in the angular direction of the section turn at the point C.



(a) (b)

Fig. 3.9.3

For the given frame, one needs to determine the horizontal and vertical displacements of the point C, as well as the angle of the section turn. The rigidities of all sections are set constant and identical: $E \times I = const$.

The determination of the diagrams by the sections (Fig. 3.9.3 b) of bending moments of their values in the extreme and in the middle sections of the section 1 (CB) is:

$0 \le x \le 2$ m, $M = 7 - 9 \times x$, $M_C = 7$ kN×m, $M_B = 7 - 9 \times 2 = -11$, kN×m

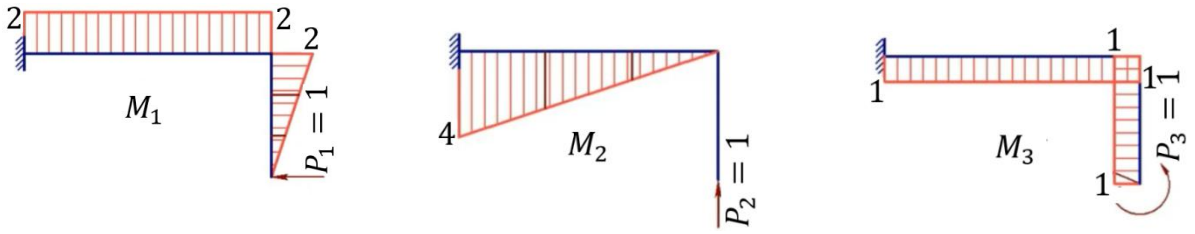
and $M_{1/2} = 7 - 9 \times 1 = -2$, kN×m at $x = \frac{a}{2} = 1$ m.

The determination of the diagrams in the section 2 (BA):

$0 \le x \le 4$ m, $M_B = -11$ kN×m, $M_A = -11 - 4^2 = -27$, kN×m

and at $x = \frac{l}{2} = 2$, $M_{1/2} = -11 - 2^2 = -15$, kN×m.

The diagrams of bending moments under unit forces are shown on schemes (Fig. 3.9.4 a, b and c).



(a) (b) (c)

Fig. 3.9.4

Horizontal displacement of the point C (Fig. 3.9.3 b and Fig. 3.9.4 a) by sections of the frame when multiplying the diagrams of bending moments using Simpson's formula is determined as: $\Delta_1 = \frac{4}{6\times E\times I} \times (27 \times 2 + 4 \times 15 \times 2 + 11 \times 2) +$

$+\frac{2}{6\times E\times I} \times (11 \times 2 + 4 \times 2 \times 1 - 7 \times 0) = \frac{140.67}{E\times I}$.

Horizontal displacement of the point C (Fig. 3.9.3 b and Fig. 3.9.4 a) determined by sections of the frame by multiplying diagrams of bending moments using the formula of trapezoid is equal to: $\Delta_1 = \frac{4}{6\times E\times I} \times (27 \times 2 + 4 \times 15 \times 2 + 11 \times 2) +$

$+\frac{2}{6\times E\times I} \times (2 \times 11 \times 2 - 2 \times 7 \times 0 + 11 \times 0 - 7 \times 2) = \frac{140.67}{E\times I}$.

Vertical displacement of the point C (Fig. 3.9.3 b and Fig. 3.9.4 b) determined by the sections of the frame by multiplying diagrams of bending moments using Simpson's or formula of trapezoids similar to the displacement sought is equal:

$\Delta_2 = \frac{4}{6\times E\times I} \times (-27 \times 4 - 4 \times 15 \times 2 + 11 \times 0) = -\frac{152}{E\times I}$.

Angle of the section C turn (Fig. 3.9.3 b and Fig. 3.9.4 c) determined by sections of the frame by multiplying diagrams of bending moments using Simpson's formula is:

$\Delta_3 = \frac{4}{6\times E\times I} \times (-27 \times 1 - 4 \times 15 \times 1 - 11 \times 1) +$

$+\frac{2}{6\times E\times I} \times (-11 \times 2 - 4 \times 2 \times 1 + 7 \times 1) = -\frac{69.33}{E\times I}$.

The angle of the section C turn (Fig. 3.9.3 b and Fig. 3.9.4 c) determined by sections of the frame by multiplying of diagrams of bending moments using the formula of trapezoid is equal: $\Delta_3 = \frac{4}{6\times E\times I} \times (-27 \times 1 - 4 \times 15 \times 1 - 11 \times 1) +$

$+\frac{2}{6\times E\times I} \times (-2 \times 11 \times 1 + 2 \times 7 \times 1 - 11 \times 1 + 7 \times 1) = -\frac{69.33}{E\times I}$.

Questions and comments:

3.9.1. Determination of the Mohr's integral by multiplying the diagrams using Simpson's rule.

3.9.2. Finding the area of the diagram and the position of it center of area using Simpson's formula.

3.9.3. Determination of Mohr's integral by multiplying the diagrams according to the rule of trapezoid, provided that the diagrams are linear.

3.9.4. Finding linear and angular displacements of specified point of the mechanical system using Mohr's integral and the Simpson's method to product the diagrams in the Mohr's integral.

3.9.5. Finding linear displacement of specified point of the mechanical system using Mohr's integral and method of trapezoid provided that the diagrams are linear for their multiplication in the Mohr's integral.

3.10. Using the forces method to determine the efforts in a statically indefinable system due to temperature effects on it

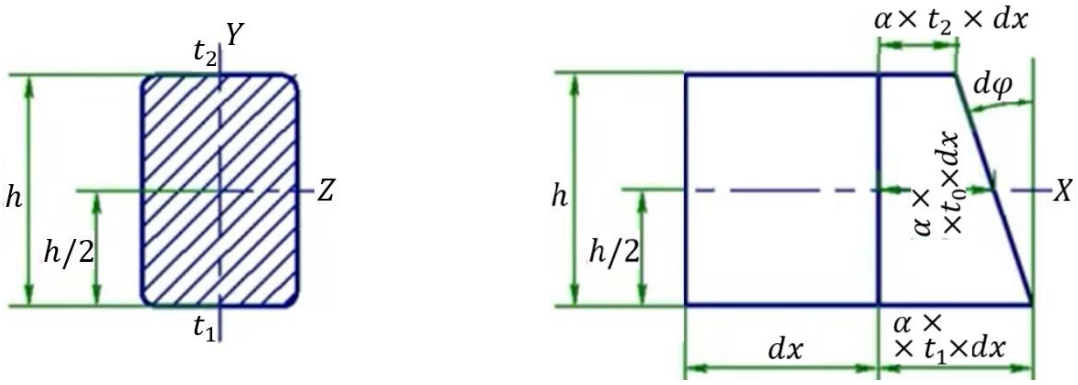
System of canonical equations of the forces method for an n–times statically indefinable mechanical system at condition, if the displacements Δ_{it}, are given, caused by a temperature effect asymmetrical on the sides of the mechanical system, has the form:

$$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \cdots + \delta_{1k} \times X_k + \cdots + \delta_{1n} \times X_n + \Delta_{1t} = 0,$$
$$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \cdots + \delta_{2k} \times X_k + \cdots + \delta_{2n} \times X_n + \Delta_{2t} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\delta_{i1} \times X_1 + \delta_{i2} \times X_2 + \cdots + \delta_{ik} \times X_k + \cdots + \delta_{in} \times X_n + \Delta_{it} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\delta_{n1} \times X_1 + \delta_{n2} \times X_2 + \cdots + \delta_{nk} \times X_k + \cdots + \delta_{nn} \times X_n + \Delta_{nt} = 0,$$

where δ_{ik} – the unit displacement in each i–th of the directions of forces in the connections that are replaced at formation of the main system by the force ($X_{k=1}$) applied in the direction of the replaced k–th connection, Δ_{it} – the displacement arising from the action of temperature in the direction i–th.

The forces in the elements of a mechanical system and the displacements of nodes and sections (the linear and angular) at an asymmetrical temperature effect on the system arise due to heat transfer and temperature differences within the cross–section of an element of this system, which is illustrated by the diagram (Fig. 3.10.1 a and b).

Under asymmetrical influence of temperature (t_1 and t_2) on both sides of an element of a mechanical system (length dx) made from a homogeneous material, a linear temperature deformation ε of this element occurs and the linear displacement is an increase in size ($\varepsilon \times dx$), which is equal to: $\varepsilon \times dx = \alpha \times t_0 \times dx = \alpha \times \frac{(t_1+t_2)}{2} \times dx$, if within the cross–section of a system element the temperature is distributed linearly, where α – is the coefficient of linear thermal expansion ($^\circ C^{-1}$) – the property of a homogeneous material of a mechanical system, and the linear temperature deformation is equal, accordingly: $\varepsilon = \alpha \times t_0 = \alpha \times \frac{(t_1+t_2)}{2}$.



(a) (b)

Fig. 3.10.1

Since the temperature effect (t_1 and t_2) on both sides of the element of length dx (which means the differential increment of the length) of the mechanical system is asymmetrical ($t_1 \neq t_2$), then due to asymmetrical temperature expansion the turning of section of the element of mechanical system (length dx) is: $d\varphi = \alpha \times \frac{(t_1 - t_2)}{h} \times dx$.

The displacement Δ_{it} in the direction i–th of the temperature influence one can find using Mohr's formula, as: $\Delta_{it} = \int \sum M_i \times d\varphi + \int \sum N_i \times \varepsilon \times dx$.

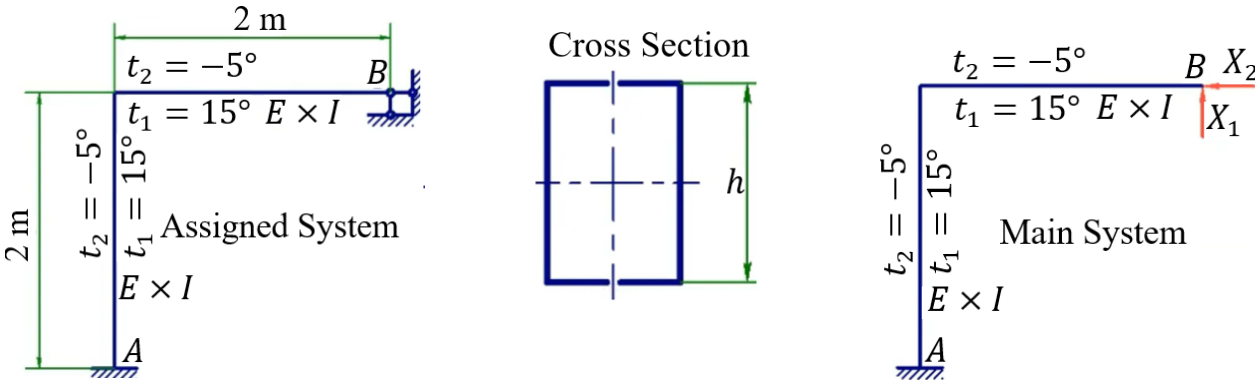
By removing from the integrals the common factors, that do not depend on the variables of integration, the formula of the Mohr's integral one can reduce to the form of dependence on the coordinate variable x along the axis of an element of mechanical system that has an uneven temperature on the sides:

$\Delta_{it} = \sum \alpha \times \frac{(t_1 - t_2)}{h} \times \int M_i \times dx + \sum \alpha \times \frac{(t_1 + t_2)}{2} \times \int N_i \times dx,$

or $\Delta_{it} = \sum \alpha \times \frac{(t_1 - t_2)}{h} \times \omega_{Mi} + \sum \alpha \times \frac{(t_1 + t_2)}{2} \times \omega_{Ni}$, where ω_{Mi} and ω_{Ni} – the values of corresponding integrals (the areas of diagrams of bending moments M_i and longitudinal forces N_i under unit forces: $X_k = 1$, $k = 1, \ldots, n$ in each i direction of connections, which are replaced by the forces when forming the main system).

Exercise. Find the reactions of the supports of a twice statically indefinable flat frame under asymmetric temperature influence (Fig. 3.10.2).

Given: the doubly statically indefinable flat frame consists of elements having a box–shaped section (Fig. 3.10.2 b), constructed by welding the flanges of steel channels # 20, having rigidity characteristics: the moment of inertia of the section of considered element of the mechanical system: $I_X = 2 \times 1520 = 3040$, cm^4, the modulus of elasticity of the material: $E = 2.1 \times 10^5$ MPa, the geometric size: $h = 0.2$ m. The coefficient of linear thermal expansion of the material is equal to: $\alpha = 12 \times 10^{-6}\ °C^{-1}$. The frame serves as a supporting part of the lightweight structure, the temperature inside, which is equal to: $t_1 = 15°$, and the temperature outside: $t_2 = -5°$.



(a) (b) (c)

Fig. 3.10.2

The main system is shown on the diagram (Fig. 3.10.2 c). The bending rigidity of a box–shaped structure consisting of two channels connected by welding is equal to: $E \times I = 2.1 \times 10^{11} \times 3040 \times 10^{-8} = 6.384 \times 10^{6}$, N×m^2.

The system of canonical equations of the forces method has the form:

$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \Delta_{1t} = 0,$

$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \Delta_{2t} = 0.$

The coefficients of the canonical equations of the forces method are displacements under the influence of unit forces. Diagrams of bending moments and shear forces under unit forces (X_1 and X_2) in connections, which were replaced during the formation of the main system, are shown on schemes (Fig. 3.10.3 b and d).

Unit displacement δ_{ik} in the direction of the i–th connection, that is displacement under unit force ($X_k = 1$) having into account the rigidity of the corresponding section of the mechanical system ($E \times I$) one can find by multiplying diagrams of bending moments (M_i and M_k) and by integrating by the sections of the mechanical system, as:

$$\delta_{ik} = \sum \int \frac{M_i \times M_k}{E \times I} dx.$$

To find numerically the displacements (δ_{ik}) the method of multiplying the area of one diagram by the ordinate of the second one under the center of the area of the first can be used, if the diagrams are located in a projection (the Vereshchagin's method). To perform this calculation, the shown diagrams are used (Fig. 3.10.3 b and d).

The coefficients of canonical equations of the forces method are determined as:

$$\delta_{11} = \sum \int \frac{M_1 \times M_1}{E \times I} dx = \frac{1}{E \times I} \times \left[\frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \times 2 + 2 \times 2 \times 2\right] =$$

$$= \frac{32}{3 \times 6.384 \times 10^6} = 1.67 \times 10^{-6},$$

$$\delta_{12} = \delta_{21} = \sum \int \frac{M_1 \times M_2}{E \times I} dx = \frac{1}{E \times I} \times 2 \times 2 \times \frac{1}{2} \times 2 = \frac{4}{6.384 \times 10^6} = 0.627 \times 10^{-6},$$

$$\delta_{22} = \sum \int \frac{M_2 \times M_2}{E \times I} dx = \frac{1}{E \times I} \times \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \times 2 = \frac{1}{6.384 \times 10^6} \times \frac{8}{6} = 0.418 \times 10^{-6}.$$

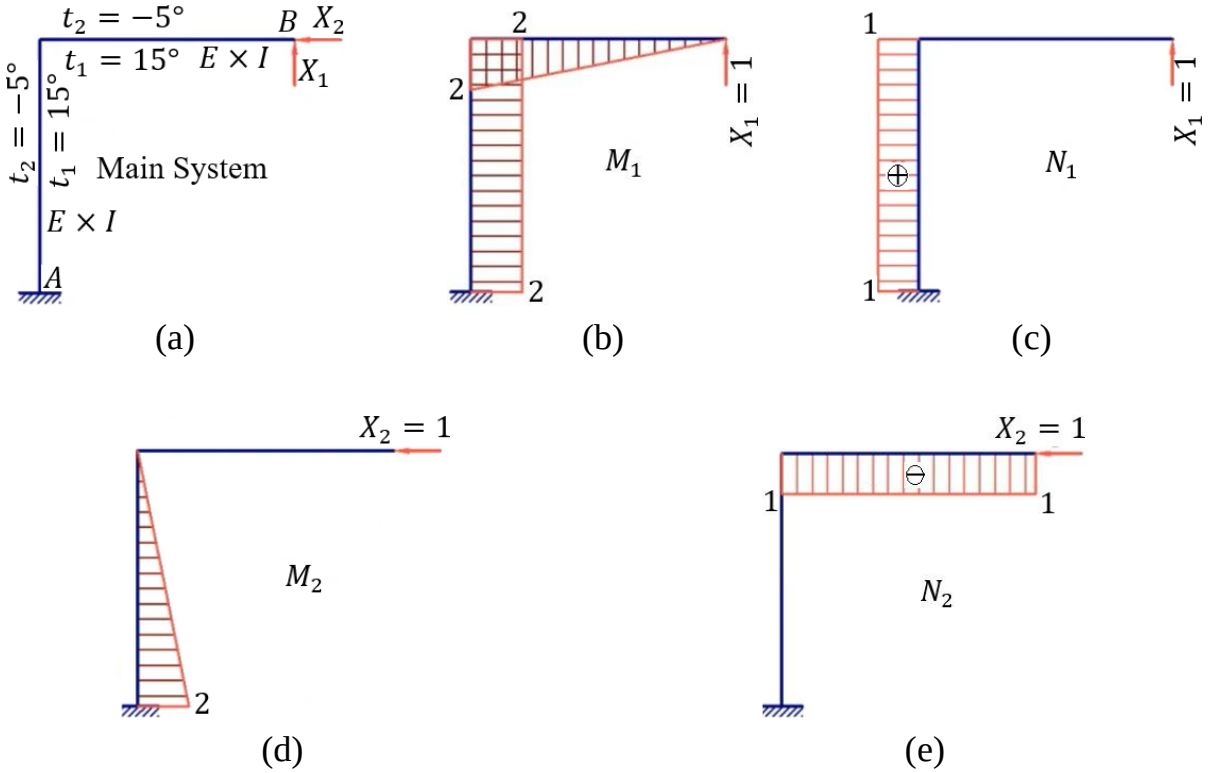


Fig. 3.10.3

The free terms of the canonical equations of the method of forces – the displacements due to temperature influences is determined as:

$$\Delta_{1t} = \alpha \times \frac{(t_1 - t_2)}{h} \times \omega_{M1} + \alpha \times \frac{(t_1 + t_2)}{2} \times \omega_{N1}$$

and $\Delta_{2t} = \alpha \times \frac{(t_1 - t_2)}{h} \times \omega_{M2} + \alpha \times \frac{(t_1 + t_2)}{2} \times \omega_{N2}$.

Using the diagrams: M_1, N_1, M_2 and N_2 (Fig. 3.10.3 b–e), one can determine displacements due to temperature influences (Δ_{1t} and Δ_{2t}) as:

$$\Delta_{1t} = 12 \times 10^{-6} \times \left[\frac{15+5}{0.2} \times \left(\frac{1}{2} \times 2 \times 2 + 2 \times 2\right) + \frac{15-5}{2} \times (1 \times 2)\right] = 7.32 \times 10^{-3}, \text{ m},$$

$$\Delta_{2t} = 12 \times 10^{-6} \times \left[\frac{15+5}{0.2} \times \left(\frac{1}{2} \times 2 \times 2\right) - \frac{15-5}{2} \times (1 \times 2)\right] = 2.28 \times 10^{-3}, \text{ m}.$$

The solution of the system of canonical equations of the method of forces is:

$1.67 \times X_1 + 0.627 \times X_2 + 7.32 \times 10^3 = 0,$

$0.627 \times X_1 + 0.418 \times X_2 + 2.28 \times 10^3 = 0,$

$X_1 = -5363$ N and $X_2 = 2565$ N.

Questions and comments:

3.10.1. System of canonical equations of the forces method, provided that the displacements caused by the temperature effect asymmetrical on the sides of the mechanical system are specified.

3.10.2. Temperature difference within the cross–section of an element of a mechanical system as a cause of forces in system elements and displacements of nodes and cross–sections.

3.10.3. Finding linear temperature deformation and displacements in nodes and sections under asymmetrical influence of temperature on both sides of a mechanical system element.

3.10.4. Reducing the formula of the Mohr's integral to the dependence on a variable coordinate along the axis of an element of a mechanical system having unequal temperature on the sides.

3.10.5. Determination of reactions of supports of a doubly statically indefinable flat frame under asymmetric temperature influence.

3.10.6. Calculation of free terms of the canonical equations of the method of forces – the displacements from temperature influences.

3.11. Finding temperature deformations and displacements of mechanical system nodes using Mohr's integral

The general formula of the Mohr's integral for flat structures is:

$$\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx + \sum \int \mu \times \frac{Q \times Q_i}{G \times A} dx + \sum \int \frac{N \times N_i}{E \times A} dx$$

With a temperature difference ($t_1 > t_2 > 0$), acting on a mechanical system, the following formulas can be used to find deformations using Mohr's integral:

(a) deformation increment of the section turn: $d\varphi = \frac{\alpha \times t_1 \times dx - \alpha \times t_2 \times dx}{h}$,

or $d\varphi = \alpha \times \frac{t_1 - t_2}{h} \times dx$, where h – the section height of a mechanical system element, α ($^\circ C^{-1}$) – the coefficient of linear thermal expansion – the property of a homogeneous material of a mechanical system;

(b) linear strain increment: $\varepsilon \times dx = \alpha \times t_0 \times dx$, where $t_0 = \frac{y_1 \times t_2 + y_2 \times t_1}{h}$ – the average temperature linearly distributed in a section between its boundaries (y_1 and y_2), in particular: $t_0 = \frac{t_2 + t_1}{2}$ at $y_1 = y_2 = \frac{h}{2}$.

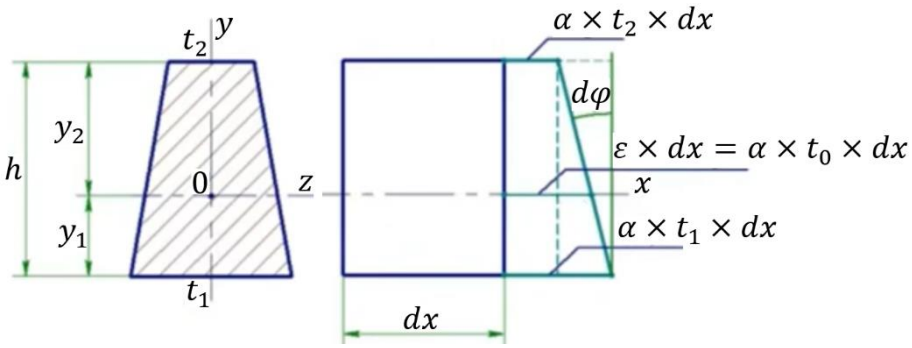


Fig. 3.11.1

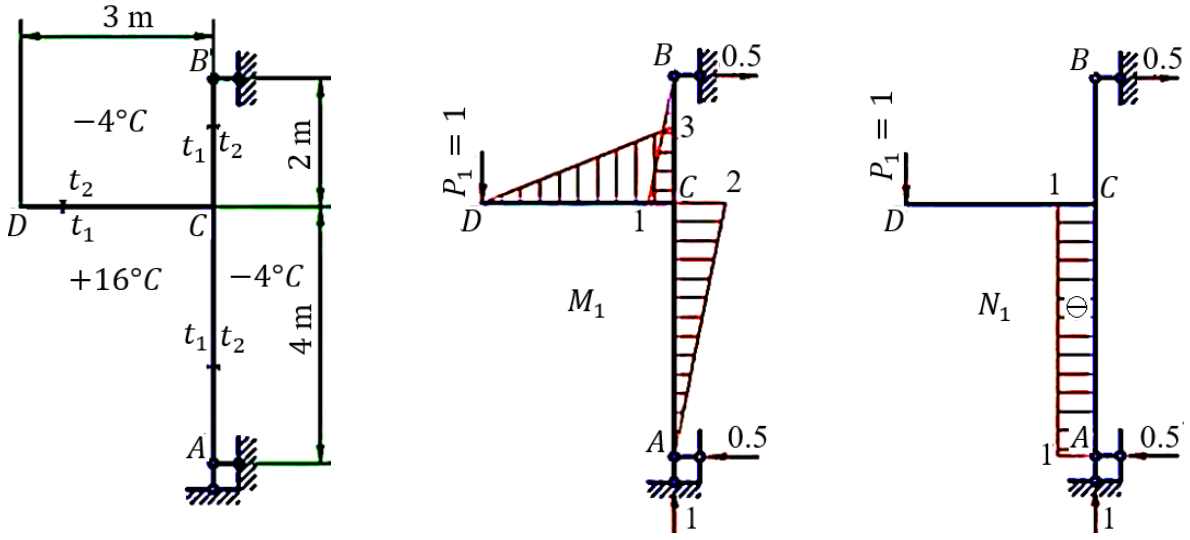
Exercise. of the free end of the console (Fig. 3.11.2 a) at the different temperature values indicated on the diagram ($-4°C$ and $+16°C$), if the height of the section at all points of the steel structure is constant and is equal to: h=0.2 m. The coefficient of linear expansion of the material is equal to: $\alpha = 1.2 \times 10^{-5}\ °C^{-1}$.

The deformations of the mechanical system at the temperature difference ($t_1 > t_2 > 0$) using Mohr's integral are found as:

$\Delta_{it} = \sum \alpha \times \frac{(t_1 - t_2)}{h} \times \int M_i \times dx + \sum \alpha \times t_0 \times \int N_i \times dx,$

or $\Delta_{it} = \sum \alpha \times \frac{(t_1 - t_2)}{h} \times \omega_{Mi} + \sum \alpha \times t_0 \times \omega_{Ni},$

where: $\omega_{Mi} = \int M_i \times dx$ and $\omega_{Ni} = \int N_i \times dx$ – the area of diagrams: the unit moment and unit longitudinal force in the direction of displacements at temperature differences ($t_1 > t_2 > 0$), acting on the mechanical system.



(a) (b) (c)

Fig. 3.11.2

The diagrams are used of bending moments M_1 (Fig. 3.11.2 b) and longitudinal forces N_1 (Fig. 3.11.2 c) under unit force ($P_1 = 1$), applied in the direction of movement of the point D of free end of the console.

Displacement of the point D of free end of the console at different temperatures indicated on the diagram ($-4°C$ and $+16°C$) equals to:

$$\Delta_{1t}=\frac{12\times10^{-5}}{0.2}\times\left[-(16+4)\times\left(\frac{1}{2}\times3\times3\right)-(16+4)\times\left(\frac{1}{2}\times4\times2\right)\right]-$$

$$-\frac{12\times10^{-5}}{2}\times(16-4)\times(4\times1)=-(10.2+0.282)\times10^{-3}\text{m}=-10.482,\text{ mm}.$$

Questions and comments:

3.11.1. Determination of the deformation when changing the temperature of a mechanical system element.

3.11.2. Determining of the displacement of a mechanical system assembly when temperature changes.

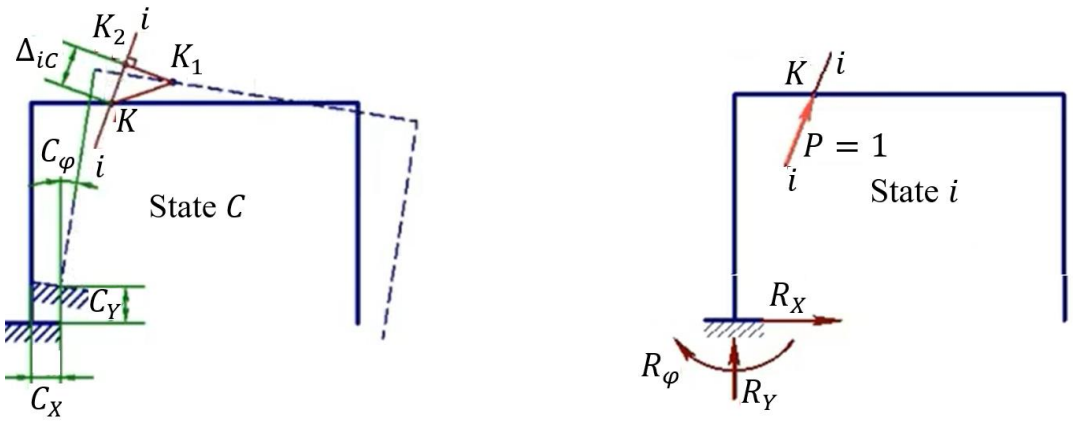
3.11.3. Using the Mohr's integral and diagrams of bending moments and longitudinal forces when finding the deformation of a mechanical system.

3.12. Determination of displacements due to subsidence of supports of a mechanical system using the principle of possible displacements

In statically definable structures, when supports settle the internal forces and deformations of elements, as well as changes in support reactions, are equal to zero.

When formulating the calculated dependence for determining the displacements of the cross–sections and nodes of a mechanical system on the subsidence of supports, the principle of possible displacements is used.

To find the displacements from the subsidence of the supports, two loading options for the mechanical system are considered (Fig. 3.12.1 a and b). The first state (one denominates «C») is characterized by support displacements: horizontal (C_X), vertical (C_Y) and angular (C_φ).



(a) (b)

Fig. 3.12.1

In the second state (one can denominate «i») a unit force is applied ($P = 1$) in the direction ($i - i$) of the displacement sought Δ_{iC}. The result of the application of unit forces is then the determination of the support reactions sought: R_X, R_Y and R_φ in the directions of unit forces.

The sequence of displacements finding under the subsidence of supports using the principle of possible displacements. In accordance with the principle of possible displacements the work of all acting external efforts W_{iC} (the moments and forces) and all internal efforts U_{iC} (which are equal to zero if the supports in statically determinate systems move: $\sum U_{iC} = 0$), equals zero in total: $\sum W_{iC} + \sum U_{iC} = 0$.

One has to determine the work of the forces of the second state «i» of the mechanical system on the displacements of the first state «C» for the statically determinable system: $\sum W_{iC} = P \times \Delta_{iC} + R_X \times C_X + R_Y \times C_Y + R_\varphi \times C_\varphi = 0$.

The displacement Δ_{iC} in the direction ($i - i$) equals:
$\Delta_{iC} = -R_X \times C_X - R_Y \times C_Y - R_\varphi \times C_\varphi$.

And in the general case the displacement Δ_{iC} in the direction ($i - i$) equals:
$\Delta_{iC} = -\sum R_{ji} \times C_j$, where C_j – the displacement when the support settles j ($j = 1, 2, \dots, n$), R_{ji} – the reaction in the direction j, corresponding to the displacement component C_j, under unit force ($P = 1$), applied in the direction ($i - i$).

The sequence of calculation of displacements arising due to supports shifting is:

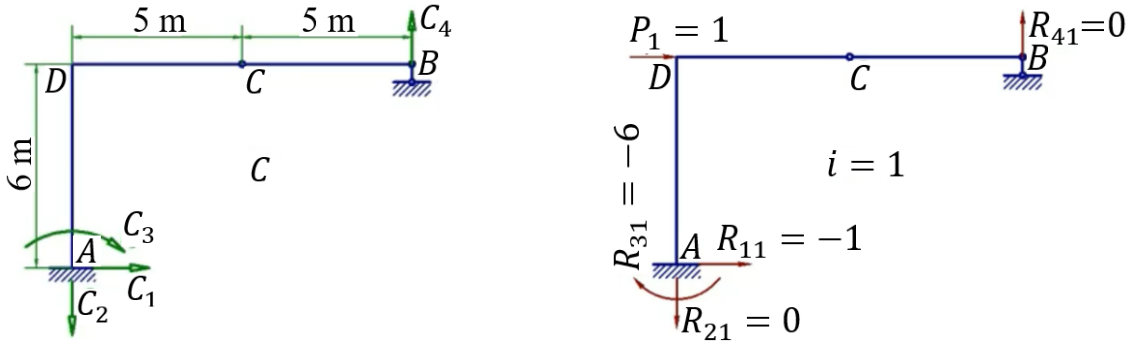
1. Make a diagram of the mechanical system indicating the directions and spaces of displacement of the supports C_j.

2. Assuming the shifting support links to be stationary, make a diagram of a mechanical system under the action of the unit force ($i - i$), in which the displacement sought caused due to supports shifting.

3. When a mechanical system is exposed to a unit load ($P = 1$) to find the reactions of the supports R_{ji} in the directions of their displacements C_j (it is recommended to take the corresponding directions of displacements of the supports C_j as the positive direction of the reactions R_{ji}).

4. Using the formula: $\Delta_{iC} = -\sum R_{ji} \times C_j$ one has to find the displacement of the point of the mechanical system caused by the displacement of the supports in the direction ($i - i$).

Exercise. The displacements of the supporting connections of the frame structure in the directions shown on the diagram (Fig. 3.12.2 a) are equal: $C_1 = 0.1$ m, $C_2 = 0.08$ m, $C_3 = 0.05$ radian and $C_4 = 0.06$ m. The exercise assigns one to determine horizontal displacement Δ_{XD} of the point D, the turn angle $\Delta_{\varphi B}$ of supporting section B and mutual turn $\Delta_{\varphi C}$ of horizontal sections: BC and DC, connected at a hinge C.



(a) (b)

Fig, 3.12.2

The horizontal displacement Δ_{XD} (Fig. 3.12.2 b) of the point D is determined as:

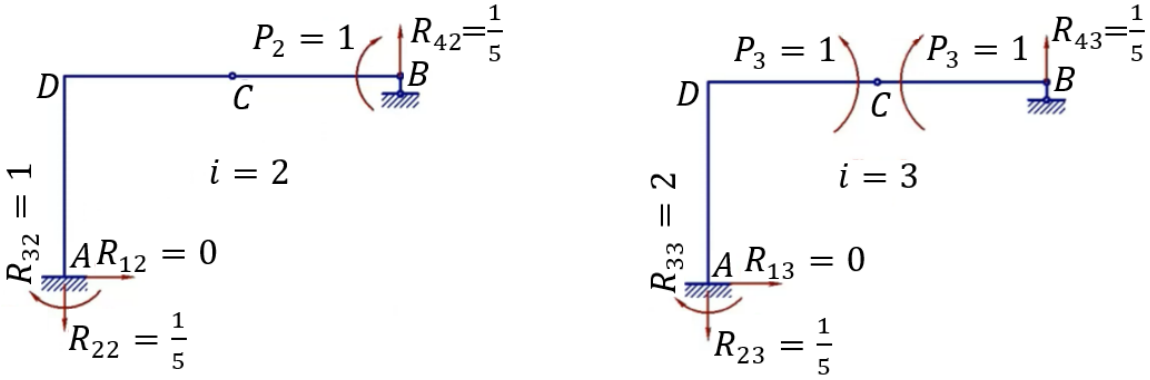
$\Delta_1 = -\sum R_{j1} \times C_j = -(R_{11} \times C_1 + R_{21} \times C_2 + R_{31} \times C_3 + R_{41} \times C_4)$,

$\Delta_1 = -(-1 \times 0.1 + 0 \times 0.08 - 6 \times 0.05 + 0 \times 0.06) = 0.4$ m.

The turn angle $\Delta_{\varphi B}$ (Fig. 3.12.3 a) of the supporting section B is equal:

$\Delta_2 = -\sum R_{j2} \times C_j = -(R_{12} \times C_1 + R_{22} \times C_2 + R_{32} \times C_3 + R_{42} \times C_4)$,

$\Delta_2 = -\left(0 \times 0.1 + \frac{1}{5} \times 0.08 - 1 \times 0.05 + \frac{1}{5} \times 0.06\right) = -0.078$ radian.



(a) (b)

Fig. 3.12.3

Mutual turn $\Delta_{\varphi C}$ (Fig. 3.12.3 b) of horizontal sections: BC and DC connected at hinge C is equal: $\Delta_3 = -\sum R_{j3} \times C_j = -(R_{13} \times C_1 + R_{23} \times C_2 + R_{33} \times C_3 + R_{43} \times C_4)$,

$\Delta_3 = -\left(0 \times 0.1 + \frac{1}{5} \times 0.08 - 2 \times 0.05 + \frac{1}{5} \times 0.06\right) = -0.128$ radian.

Questions and comments:

3.12.1. The equality to zero of internal forces and deformations during subsidence of supports of statically definable systems.

3.12.2. The sequence of using the principle of possible displacements and the sequence of calculating the displacements of cross–sections and nodes of a mechanical system by the subsidence of supports.

3.12.3. Two options for the state of a mechanical system considered when finding displacements from support subsidences: the position of system with the displacements of supports and the position under the assumption that displacements are replaced by unit forces to determine changes in the support reactions of a statically indefinable system.

3.12.4. Finding linear and angular displacements in the nodes of a statically indefinable mechanical system when the supports are displaced.

3.13. Using forces method finding the forces in statically indefinable frame

Exercise. Calculate the flat statically indefinable frame (Fig. 3.13.1 a), using the force method, to create the diagrams of internal forces (flexural stiffness in all sections of the frame is considered constant: $E \times I$=const).

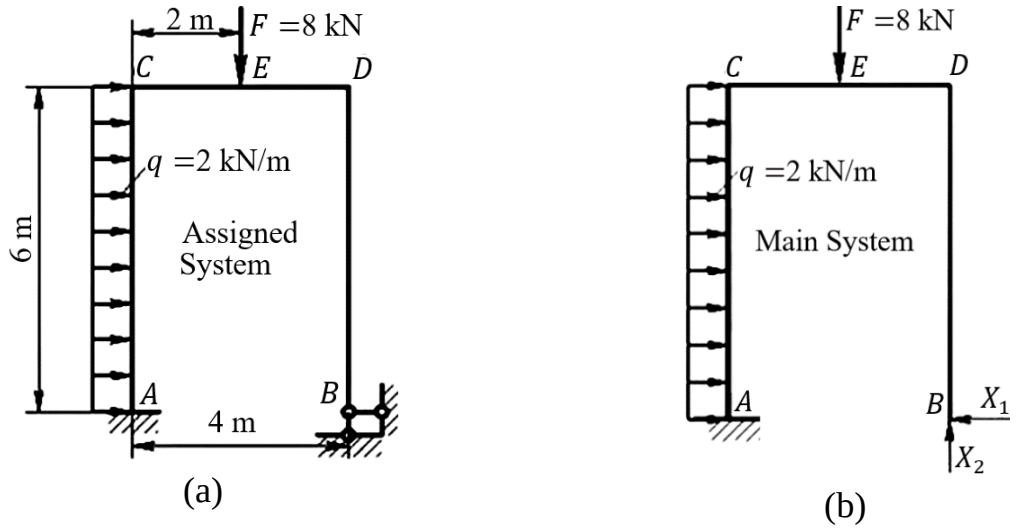


Fig. 3.13.1

The main system has the selected form shown on the scheme (Fig. 3.13.1 b). The system of canonical equations of the force method for a doubly statically indefinable mechanical system has the form:

$\delta_{11} \times X_1 + \delta_{12} \times X_2 + \Delta_{1F} = 0,$

$\delta_{21} \times X_1 + \delta_{22} \times X_2 + \Delta_{2F} = 0.$

To calculate the displacements δ_{ij} under the influence of unit forces (X_1 and X_2) in the connections that are replaced during formation of the main system, and under the external load (F and q) Δ_{iF} the Mohr's integral can be used – the sum of found integrals by

sections of the mechanical system from the products of bending moments diagrams under unit forces, as well as external load: $\delta_{ij} = \sum \int \frac{M_i \times M_j}{E \times I} dx$ and $\Delta_{iF} = \sum \int \frac{M_i \times M_F}{E \times I} dx$.

The Mohr's integral of the product of two diagrams ($M_i \times M_j$) can be determined sing Vereshchagin's method as the product of the area ω_i of one diagram by the ordinate y_j of another diagram, taken under the center of the area of the first diagram (provided that the second diagram is linear), when the diagrams are in a projection relationship: $\int M_i \times M_j \times dx = \omega_i \times y_j$.

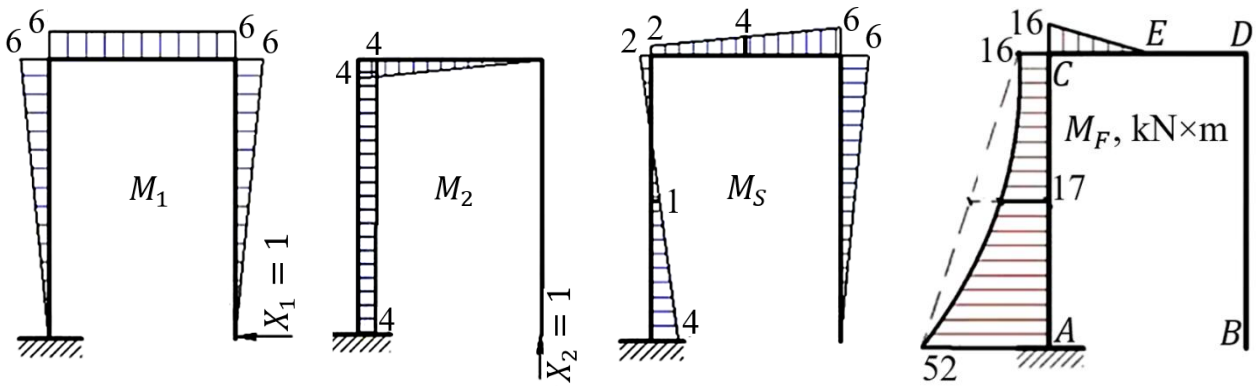
Diagrams of moments under unit forces in the main system (Fig. 3.13.2 a and b) and of external load (Fig. 3.13.2 d) are shown on the schemes.

The coefficients of canonical equations of the method of forces (the diagrams of moments shown on the schemes are used) are determined as:

$$\delta_{11} = \sum \int \frac{M_1 \times M_1}{E \times I} dx = \frac{1}{E \times I} \times \left[\left(\frac{1}{2} \times 6 \times 6 \times \frac{2}{3} \times 6\right) \times 2 + 6 \times 4 \times 6\right] = \frac{288}{E \times I} = \frac{864}{3 \times E \times I},$$

$$\delta_{12} = \delta_{21} = \sum \int \frac{M_1 \times M_2}{E \times I} dx = \frac{1}{E \times I} \times \left[-6 \times 4 \times 2 - \frac{1}{2} \times 6 \times 6 \times 4\right] = -\frac{120}{E \times I} = -\frac{360}{3 \times E \times I},$$

$$\delta_{22} = \sum \int \frac{M_2 \times M_2}{E \times I} dx = \frac{1}{E \times I} \times \left[\frac{1}{2} \times 4 \times 4 \times \frac{2}{3} \times 4 + 4 \times 6 \times 4\right] = \frac{352}{3 \times E \times I}.$$



(a) (b) (c) (d)

Fig. 3.13.2

To verify the correctness of the total displacements under unit forces ($X_1 = 1$ and $X_2 = 1$) finding the diagram of moments shown on the scheme is used (Fig. 3.13.2 e):

$$\delta_{ss} = \sum \int \frac{M_S \times M_S}{E \times I} dx = \frac{1}{E \times I} \times \left(\frac{1}{2} \times 6 \times 6 \times \frac{2}{3} \times 6 + \frac{4}{6} \times (2 \times 6 \times 6 + 2 \times 2 \times 2 + \right.$$

$$\left. + 2 \times 6 + 6 \times 2) + \frac{6}{6} \times (2 \times 2 \times 2 + 2 \times 4 \times 4 - 2 \times 4 + 4 \times 2)\right) = \frac{496}{3 \times E \times I}.$$

On the other hand, the sum of the coefficients of the canonical equations is equal to: $\delta_{11} + \delta_{12} + \delta_{21} + \delta_{22} = \frac{864}{3 \times E \times I} - \frac{360}{3 \times E \times I} - \frac{360}{3 \times E \times I} + \frac{352}{3 \times E \times I} = \frac{496}{3 \times E \times I}$.

In addition, when finding the coefficients of the canonical equations of the method of forces a «line–by–line verification» of the correctness of determining these coefficients can be performed, which is a comparison of the sum of the coefficients for each one of the

canonical equations of method of forces: $\delta_{is} = \delta_{i1} + \delta_{i2} + \cdots + \delta_{in}$, where n – the number of unknowns, that is the degree of static indefinability of a given mechanical system, and displacement δ_{is} in the i–th direction of the sum s of unit forces in the direction of each i–th connection discarded during the formation of the main system, determined using the formula: $\delta_{is} = \sum \int \frac{M_i \times M_s}{E \times I} dx$, where $M_s = M_1 + M_2 + \cdots + M_n$ – the sum of diagrams of moments of unit forces in the direction of each i–th connection.

In the problem being solved, it is not necessary to check line by line the correctness of the coefficients of the canonical equations, because the universal verification has been carried out (the finding of δ_{ss}) by integrating the product of the sum of diagrams of moments of unit forces: $\delta_{ss} = \sum \int \frac{M_s \times M_s}{E \times I} dx$ and by summing the found displacements δ_{ik} under unit forces, and the correctness of finding of displacements is thus confirmed.

To find the independent terms of the canonical equations of method of forces – the displacements under the action of an external load (Δ_{1F} and Δ_{2F}) independent on reactions in the replaced connections, the diagrams of moments are used, shown on the schemes (Fig. 3.13.2 a, b and d):

$\Delta_{1F} = \sum \int \frac{M_1 \times M_F}{E \times I} dx = \frac{1}{E \times I} \times \left[\frac{1}{2} \times 16 \times 2 \times 6 + \frac{6}{6} \times (16 \times 6 + 4 \times 17 \times 3)\right] = \frac{1188}{3 \times E \times I}$,

$\Delta_{2F} = \sum \int \frac{M_2 \times M_F}{E \times I} dx = \frac{1}{E \times I} \times \left[-\frac{1}{2} \times 16 \times 2 \times \left(\frac{2}{3} \times 4 + \frac{1}{3} \times 2\right) - \frac{6}{6} \times (52 \times 4 + 4 \times 17 \times 4 + 16 \times 4)\right] = -\frac{1792}{3 \times E \times I}$.

Column–by–column verification of the correctness of determining the terms of the canonical equations of the method of forces of external load independent action on reactions in the replaced connections (Δ_{1F} and Δ_{2F}) is:

$\Delta_{SF} = \Delta_{1F} + \Delta_{2F} = \frac{1188}{3 \times E \times I} - \frac{1792}{3 \times E \times I} = -\frac{604}{3 \times E \times I}$, on the other hand, using diagrams of moments shown on the scheme (Fig. 3.13.2 c and d) one can have:

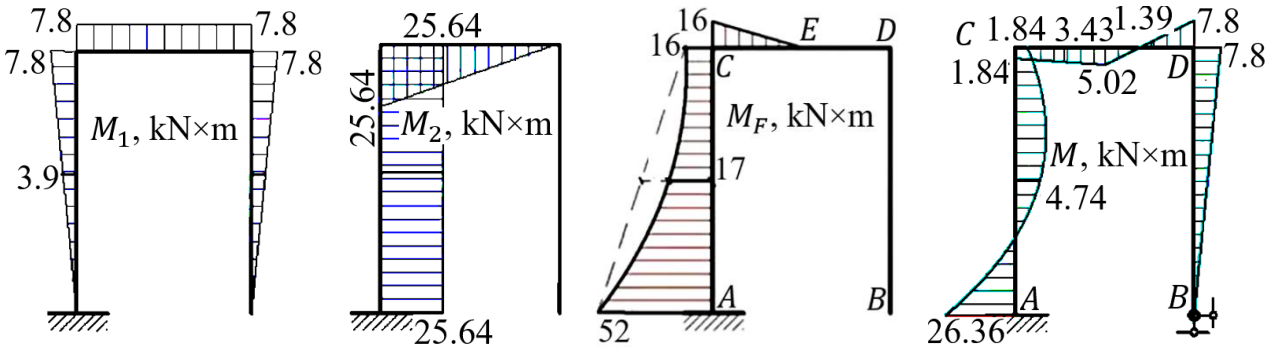
$\Delta_{SF} = \sum \int \frac{M_S \times M_F}{E \times I} dx = \frac{1}{E \times I} \times \left[\frac{1}{2} \times 16 \times 2 \times \left(\frac{2}{3} \times 2 + \frac{1}{3} \times 4\right) + \frac{6}{6} \times (-52 \times 4 + 4 \times 17 \times 1 + 16 \times 2)\right] = -\frac{604}{3 \times E \times I}$, the verification confirms the correctness of finding the terms of the canonical equations of the method of forces under the action of an external load (Δ_{1F} and Δ_{2F}).

Substitution of the found coefficients (the displacements of unit forces: δ_{11}, δ_{12}, δ_{21} and δ_{22} and the displacements of given external load: Δ_{1F} and Δ_{2F}) into the canonical equations of forces method, the solution of the system of equations and the determination of forces in connections (the support reactions), discarded during formation of the main statically definable system from given statically indefinable one has the form:

$864 \times X_1 + 360 \times X_2 + 1188 = 0$,

$-360 \times X_1 + 352 \times X_2 - 1792 = 0$, then $X_1 = 1.30$ kN and $X_2 = 6.41$ kN.

The resulting diagram of bending moments in the frame (Fig. 3.13.3 d) is determined by summing the diagrams: $M = M_F + M_1 \times X_1 + M_2 \times X_2$. For each section of the frame, the total values M of bending moments (M_F, $M_1 \times X_1$ and $M_2 \times X_2$) were found and the diagram is shown in three points: the initial, final and average, except for the BD frame section, for which only two values were determined.



(a) (b) (c) (d)

Fig. 3.13.3

For example, in the point A of the section AC the moment is equal:

$M_A = -52 + 0 + 25.64 = -26.36$, kN×m.

In the middle point of the section AC the moment is equal:

$M_{AC} = -17 - 3.9 + 25.64 = 4.74$, kN×m.

At the end of the section AC (the point C) the moment is equal:

$M_C = -16 - 7.8 + 25.64 = 1.84$, kN×m and etc.

The diagram of transverse (the shear) forces is determined using the bending moments diagram. Assuming the linear shape of the diagram of the bending moments M by the sections of the mechanical system, the values of the transverse (the shear) force used to determine it diagram Q (Fig. 3.13.4 a), are found as: $Q_i = \frac{M_{Ri} - M_{Li}}{l_i}$, where M_{Ri} and M_{Li} – the values of moments on the right and on the left ends of i–th section of the mechanical system within the linearity of the bending moments diagram, l_i – the length of mechanical system i–th section.

In case of the evenly distributed load action q the shear forces are found as:

$Q_{Li} = \frac{q \times l}{2} + \frac{M_{Ri} - M_{Li}}{l_i}$ – for the left end, and $Q_{Ri} = -\frac{q \times l}{2} + \frac{M_{Ri} - M_{Li}}{l_i}$ – for the right.

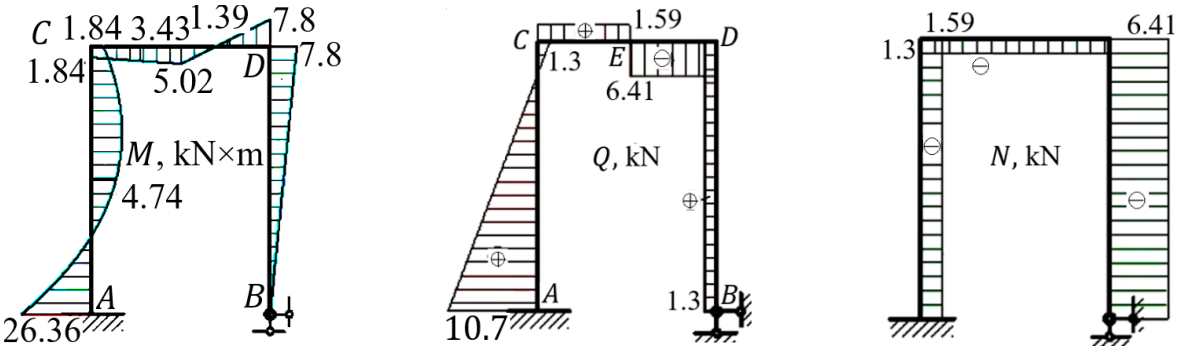
Values of transverse (the shear) force Q are shown in the diagram (Fig. 3.13.4 b).

The transverse force, determined by the sections of the frame, for the section AC is equal: $Q_A = \frac{2 \times 6}{2} + \frac{1.84 - (-26.36)}{6} = 6 + 4.7 = 10.7$, kN, $Q_C = -6 + 4.7 = -1.3$, kN,

for the section CE: $Q_{CE} = \frac{5.02 - 1.84}{2} = 1.59$, kN,

for the section ED: $Q_{ED} = \frac{-7.8 - 5.02}{2} = -6.41$, kN,

for the section BD: $Q_{BD} = \frac{7.8-0}{6} = 1.3$, kN.



(a) (b) (c)

Fig. 3.13.4

For diagrams determining of the longitudinal forces (Fig. 3.13.4 c) the diagram of transverse forces is used. The nodes are sequentially cut out (Fig. 3.13.5) and according to the condition of each node balance under the action of transverse forces Q the longitudinal forces N are determined.

$N_{AC} = -1.59$ kN
$N_{CD} = -1.3$ kN

(a)

$N_{CD} = -1.3$ kN
$N_{BD} = -6.41$ kN

(b)

Fig. 3.13.5

Questions and comments:

3.13.1. The system of canonical equations method of forces for a doubly statically indefinable mechanical system.

3.13.2. Finding the coefficients of the canonical equations of the method of forces – the displacements under the action of unit forces in the direction of replaced connections during the formation of the main statically definable mechanical system.

3.13.3. Determination of the Mohr's integral by multiplying the diagrams using the Vereshchagin's method – as the product of the area of one diagram by the ordinate of another diagram (the rectilinear) in projection under center of the area of the first diagram.

3.13.4. The universal verification of the correctness of finding the total of displacements under the action of unit forces and the line–by–line verification of the correctness of determining the coefficients of canonical equations of method of forces.

3.13.5. Finding the terms of canonical equations of the method of forces independent on reactions in the replaced connections – the displacements under the action of an external load.

3.13.6. Column–by–column verification of the correctness of determining the terms of the canonical equations of method of forces under the action of an external load independent on reactions in the connections replaced during the formation of the main system.

3.13.7. Solving the canonical equations of the method of forces, determining the reactions in the connections replaced during the formation of the main system and determining the resulting diagram of bending moments as the sum of bending moment diagrams under the influence of an external load and under the influence of reactions in the replaced connections.

3.13.8. Determining diagram of transverse (the shear) forces using numerical differentiation of the diagram of the bending moments.

3.13.9. The use of diagram of transverse (the shear) forces and the balance conditions of the mechanical system at any node to determine the diagram of longitudinal forces.

3.14. Determination of forces in elements of a statically indefinable mechanical system, using the displacements method

The method of displacements when finding forces in a statically indefinable mechanical system provides for the primary determination of number of the displacements in the connections of the system, corresponding to the degree of it static indefinability. The found values of displacements make it possible to determine forces and displacements in other sections of the mechanical system. This is achieved due to the fact that the stress–strain state of a simply connected element of a mechanical system is completely determined by the load applied to it and displacements in those sections in which this element is connected to other elements of the mechanical system.

For elements of a statically indefinable mechanical system the relationship between stresses, strains and displacements one needs to determine initially. This relationship is easiest to find for straight elements in the form of rods of constant stiffness (Fig. 3.14.1).

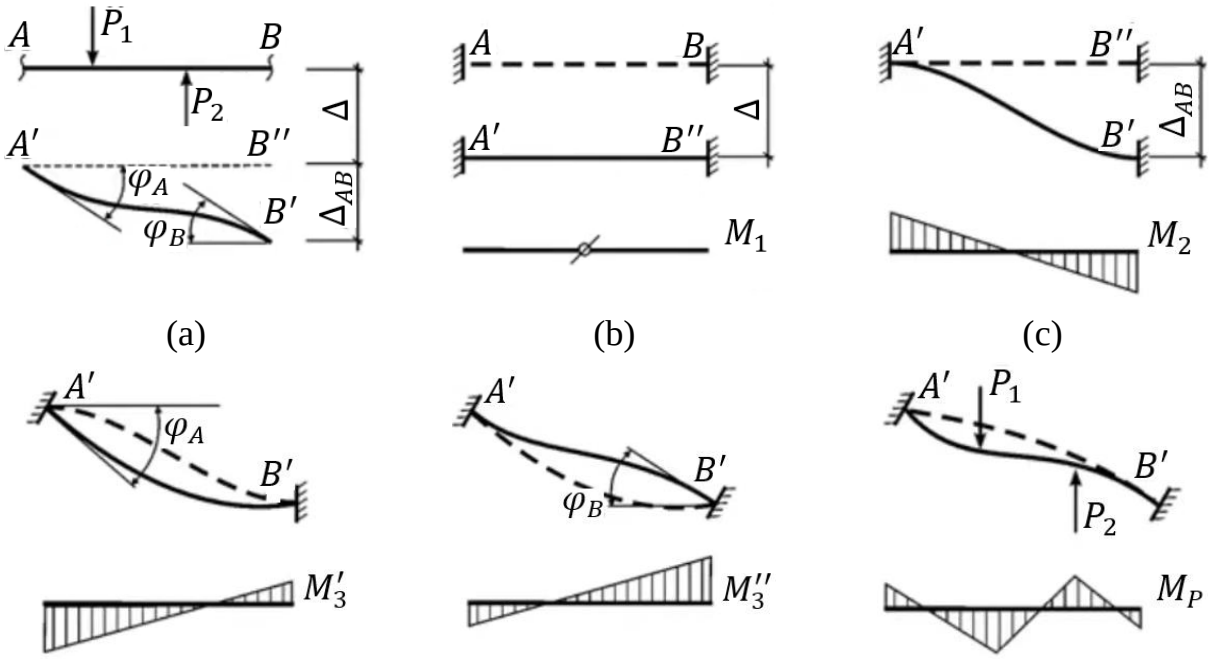


Fig. 3.14.1

For any fastenings of the ends of the rods the forces and transverse movements in all sections are determined by the angles of rotation and relative linear displacements of the ends of the rods, as well as external loads acting on the rods. Therefore, in the method of displacements a rectilinear rod of constant stiffness is selected as the main element.

To find the minimum number of nodal displacements necessary to determine the stress–strain state of a statically indefinable mechanical system to the ends of the fixed rod AB must be sequentially assigned the linear displacements Δ and Δ_{AB}, the angular displacements φ_A and φ_B, and within rod AB apply an external load. Wherein from the linear displacement Δ of the entire rod the internal forces and deformations do not occur.

Internal forces and deformations due to local load acting within the limits of rod AB, one can find separately. Therefore, to determine the stress–strain state of the entire rod, it is enough to know three unknown displacements – two angular displacements of its ends φ_A and φ_B and one translational displacement – mutual displacement of the ends of the rod Δ_{AB}. Consequently, the degree of kinematic indefinability of a bilaterally clamped rod is equal to three.

The basic assumptions (the hypotheses) of the method of displacements, which predetermine the number of unknowns of the method of displacements are:

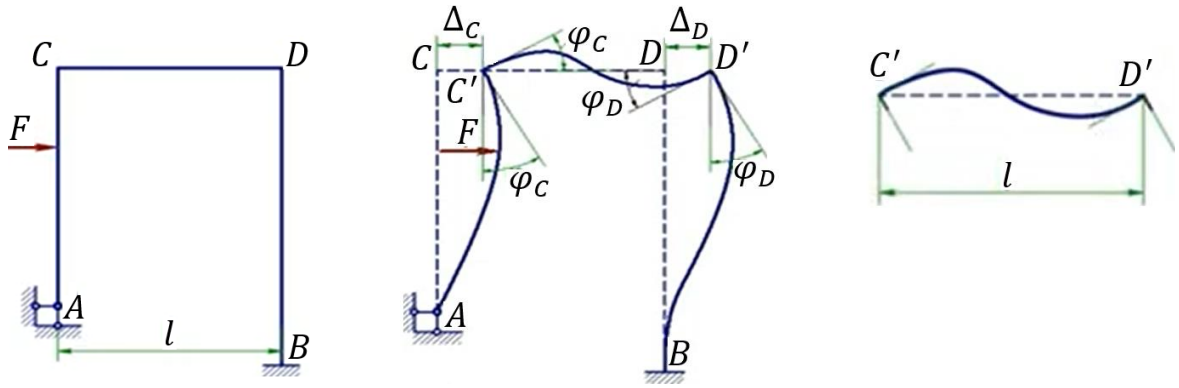
1. The angles between the rods that are connected in a rigid joint do not change when the mechanical system deforms. That is, all ends of the rods connected in a rigid unit rotate at the same angle.

2. For rods working in bending (in frames, etc.), the deformations from transverse and longitudinal forces one can neglect when finding displacements.

3. The convergence of the ends of the rod as a result of its bending is not taken into account.

4. The angles of rotation of the sections are assumed to be equal to their sines and tangents due to the smallness of these angles.

According to the accepted assumptions, the length l of the rod (Fig. 3.14.2 c) before deformation and as a result of deformation $C'D'$ is the same and equals l.



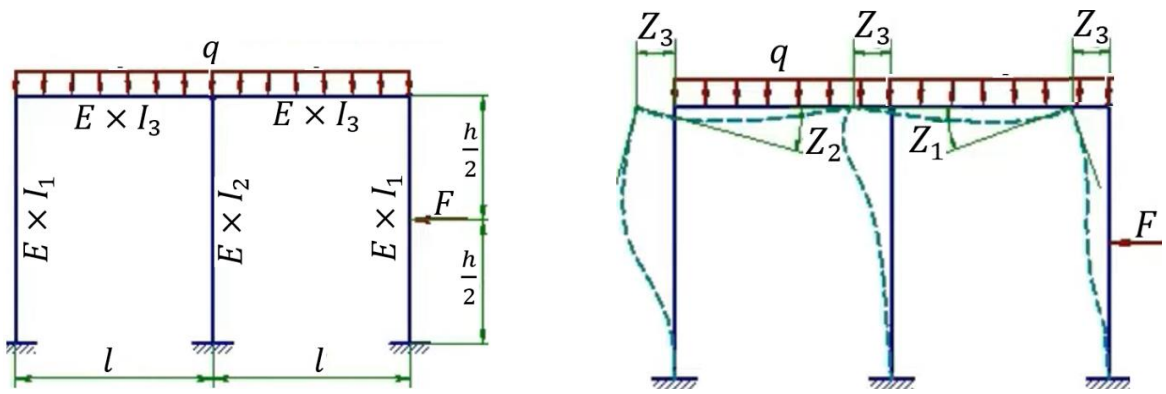
(a) (b) (c)

Fig. 3.14.2

In addition, the displacements of nodes C and D are the same, directed tangentially to their circular trajectories and perpendicular to the pillars AC and BD.

The degree of kinematic indefinability. To find the number of unknown linear displacements in all frame nodes, including the supports, one needs to introduce the hinges. Then the number of degrees of freedom w (the linear displacements) and the degree of static indefinability by linear displacements of given mechanical system one can find using the formula for the truss: $n_l = 2 \times Y - S - C_0$ (double the number of hinges Y minus the number of links S minus the number of support connections C_0).

The degree of angular mobility n_a of the mechanical system (Fig. 3.14.3 a) is equal to the number of rigid nodes.



(a) (b)

Fig. 3.14.3

Total number n of all unknown displacements (the linear n_l and angular n_a) is the degree of kinematic indefinability of mechanical system and is determined using the formula: $n = n_l + n_a$.

The unknown displacements are indicated Z_i (Fig. 3.14.3 b), that is the sequence: $Z_1, Z_2, \ldots, Z_n$.

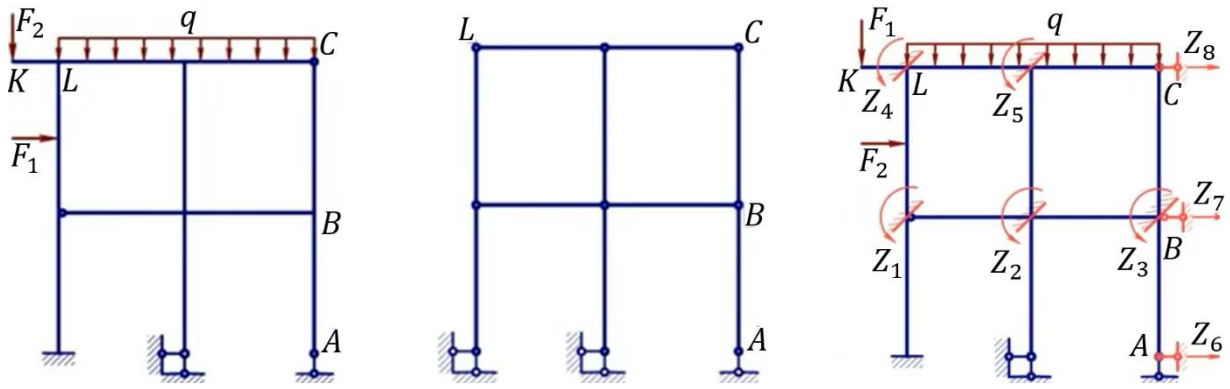
The main system in the method of displacements has to be kinematically definable. To get it, the additional connections have to be introduced into given statically indefinable mechanical system so that the ends of all rods are fixed and their displacements are prevented.

Therefore, the total number of connections entered is equal to the number of unknowns of displacements method. The number of additional connections is equal to the degree of kinematic indefinability n of the mechanical system. Main unknowns of displacements method are the displacements of the additional connections.

To obtain the main system of displacements method, the original system is transformed into a set of single–span beams of constant rigidity with fixed supports that do not allow the ends of the beams to have angular and linear displacements. This can be done by introducing additional connections of two types into the nodes: a floating embedding (movable pinching) and an additional supporting rod.

Floating connections are conditional ones that do not allow the nodes of the elements of mechanical systems to turn, but do not prevent their translational displacements.

The additional support rods must be installed only in those units that have independent linear movements. To determine the number of independent linear displacements of the system and the locations of additional support rods installation, the hinge diagram (Fig. 3.14.4 b) of original statically indefinable mechanical system is used (Fig. 3.14.4 a).



(a) (b) (c)

Fig. 3.14.4

For a statically indefinable mechanical system, which is shown on the diagram (Fig. 3.14.4 a), the displacements method to obtain the main system one needs to apply five floating embeddings (n_a=5, Fig. 3.14.4 c) and three additional support rods, so that the ends of all rods are fixed and their displacements under the influence of an applied external load (F and q) are excluded. The degree of static indefinability by linear displacements of given mechanical system is: $n_l = 2 \times Y - S - C_0$ = 2×9 – 10 – 5 = 3.

By giving the introduced connections the displacements Z_i (the main displacements), one can obtain in the main system (Fig. 3.14.4 c) exactly those forces that correspond to a given statically indefinable mechanical system (which is a given structure and a given load).

Questions and comments:

3.14.1. The priority of finding the degree of static indefinability of the mechanical system, corresponding to the number of displacements in its connections and the number of variables in the canonical equations of the method of displacements.

3.14.2. The unique correspondence between the stress–strain state of the elastic element of the mechanical system and the load applied to the system element at the nodes of its connection with other elements of the system.

3.14.3. Preliminary determination of the deformations of a statically indefinable mechanical system used in the compatibility equations and in the canonical equations of the stiffness characteristics of its linear elastic elements.

3.14.4. The selection as the main element of the mechanical system of a straight rod of constant stiffness, subject to external load, as well as linear and angular displacements at the ends.

3.14.5. The independence of internal forces acting in an elastic rod on it general linear displacement, i.e. the relativity of movement of ends of the rod.

3.14.6. The use hinge scheme to determine nodes of the mechanical system that have independent linear displacements and where additional support rods need to be installed.

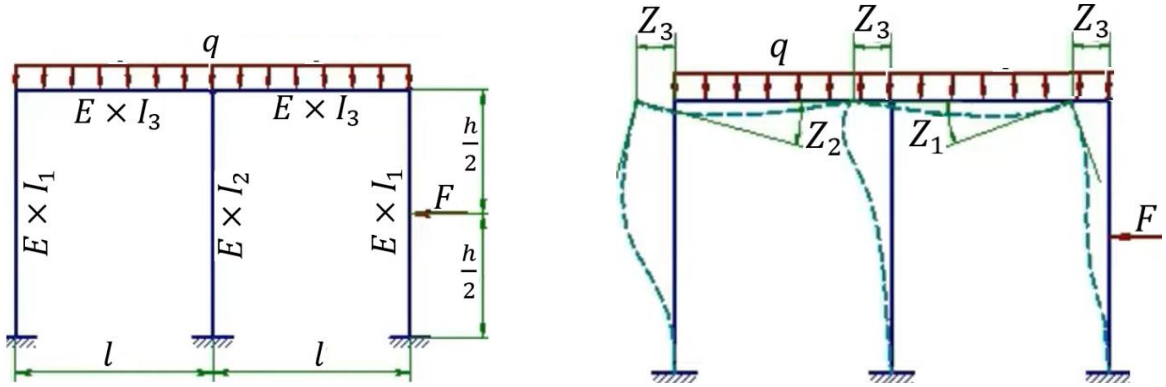
3.14.7. Formation of a statically definable basic system of the method of displacements by using floating embeddings and additional support rods for a given statically indefinable of the mechanical system.

3.14.8. Determining the degree of kinematic indefinability of a mechanical system by the number of required floating embeddings and additional support rods when forming the main system.

3.14.9. The main displacements of the introduced connections (the floating embeddings and additional support rods) serving as variables of the canonical equations of displacements method.

3.15. The main system and canonical equations of the displacements method

To eliminate differences in the deformations of the given mechanical system (kinematically indefinable, Fig. 3.15.1 a) and the main system (Fig. 3.15.1 b), one gives displacements of the additional connections. According to the principle of superposition, the separate states of the main system are considered, in each of which the external loads act and one of the main displacements occurs.



(a) (b)

Fig. 3.15.1

The total reaction in the i–th additional connection, taking into account the influence of all these factors, considered independently within the framework of the superposition principle, is equal to: $R_i = R_{i1} + R_{i2} + \cdots + R_{ik} + \cdots + R_{in} + R_{iF}$, where R_{ik} – the reaction in the i–th additional connection at the displacement of main system in the direction Z_k, R_{iF} – the reaction in the i–th additional connection of the external load impact.

Physical meaning of the canonical equations of displacements method. Since in a given statically indefinable system there are no additional connections, then for the given statically indefinable system and the main statically definable system to be equivalent the reactions in the additional connections of the main system must be equal to zero: $R_1 = 0$, $R_2 = 0$, …, $R_i = 0$, … and $R_n = 0$.

The conditions of equivalence of a given statically indefinable system and the main statically definable system have the form of the system of equations:

$R_1 = R_{11} + R_{12} + \cdots + R_{1n} + R_{1F} = 0,$

$R_2 = R_{21} + R_{22} + \cdots + R_{2n} + R_{2F} = 0,$

……,

$R_n = R_{n1} + R_{n2} + \cdots + R_{nn} + R_{nF} = 0.$

Taking into account the reactions r_{ik} in each additional i–th connection under single displacement of the k–th connection, taking into account the proportionality of reactions and the displacements: $R_{ik} = r_{ik} \times Z_k$, one can draw up the equivalence condition of the given and main systems in the form of a system of canonical equations:

$r_{11} \times Z_1 + r_{12} \times Z_2 + \cdots + r_{1n} \times Z_n + R_{1F} = 0,$

$r_{21} \times Z_1 + r_{22} \times Z_2 + \cdots + r_{2n} \times Z_n + R_{2F} = 0,$

……,

$r_{n1} \times Z_1 + r_{n2} \times Z_2 + \cdots + r_{nn} \times Z_n + R_{nF} = 0.$

Coefficients of the canonical equations r_{ik} are the reactions in the additional main system connections from the unit displacements of these connections. The free terms R_{iF} of the canonical equations are the reactions in each i–th connection of the assigned load action in the main system.

The number of equations is equal to the number of additional connections, then there are degrees of kinematic indefinability of the assigned system, which in turn is equal to the number of main unknowns of the displacements method.

Physical meaning of each equation is that the total reaction in the corresponding additional connection main system from all unknown displacements and external load is zero due to the absence of this connection in the assigned system.

Questions and comments:

3.15.1. Attaching displacements to additional connections of the main system to eliminate differences in its deformations and the assigned statically indefinable mechanical system.

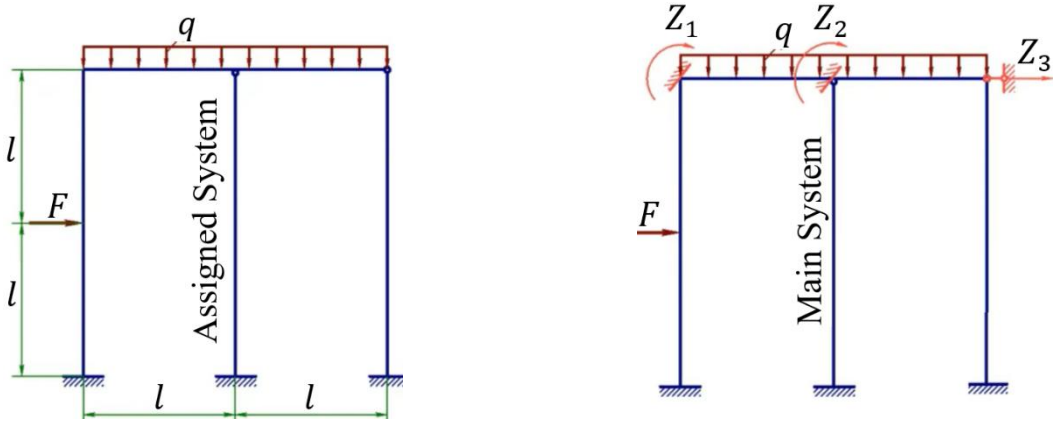
3.15.2. Using the principle of superposition when finding the main displacements (i.e., the displacements of additional connections), in which the individual states of the main system are considered, in each of which the external loads act and one of the main displacements occurs.

3.15.3. Physical meaning of the canonical equations of the displacement method is the equality of reactions in the additional connections of the main system to zero, i.e. the equivalence of specified statically indefinable mechanical system and the basic one.

3.15.4. The system of canonical equations of the displacement method. The terms of canonical equations that depend on the main displacements, corresponding to reactions in additional connections, and the terms of the equations that depend on the external load.

3.16. Finding the coefficients and free terms of canonical equations of the displacements method

Exercise. A statically indefinable mechanical system is specified (Fig. 3.16.1 a). The applied force equals: $F = 5$ kN, the distributed load acts: $q = 3$ kN/m, the length of the section is: $l = 2$ m, the linear stiffness of sections of the structure is specified as the constant value: $E \times I = const$.



(a) (b)

Fig. 3.16.1

The degree of kinematic indefinability taking into account the linear n_l and angular n_a mobilities of nodes of the mechanical system equals: $n = n_a + n_l = 2 + 1 = 3$.

To eliminate the angular mobility of the nodes the floating embeddings are considered in two rigid nodes of the assigned system, and to eliminate the linear mobility of the nodes an additional support rod is assumed. To form the main statically definable system (Fig. 3.16.1 b) three forces are replaced by connections: Z_1, Z_2 and Z_3.

To find the coefficients and free terms of the canonical equations of displacements method, the diagrams of moments are determined: M_1, M_2 (Fig 3.16.2 a, b and c) and M_3, corresponding to unit displacements, as well as the diagram of the bending moment M_F of external load (Fig. 3.16.2 d).

For diagrams creating, corresponding to unit displacements, formulas of bending moments and reactions in statically indefinable beams under typical external influences caused the unit displacements can be used.

Static method for determining coefficients and free terms. Reactions corresponding the unit movements and the load reactions are found by conditions of the balance. To determine the coefficients and free terms in the first and second canonical equations, the reactive moments in floating embeddings are determined under the conditions of balance of nodes (Fig. 3.16.3) using the created diagrams of moments for unit displacements.

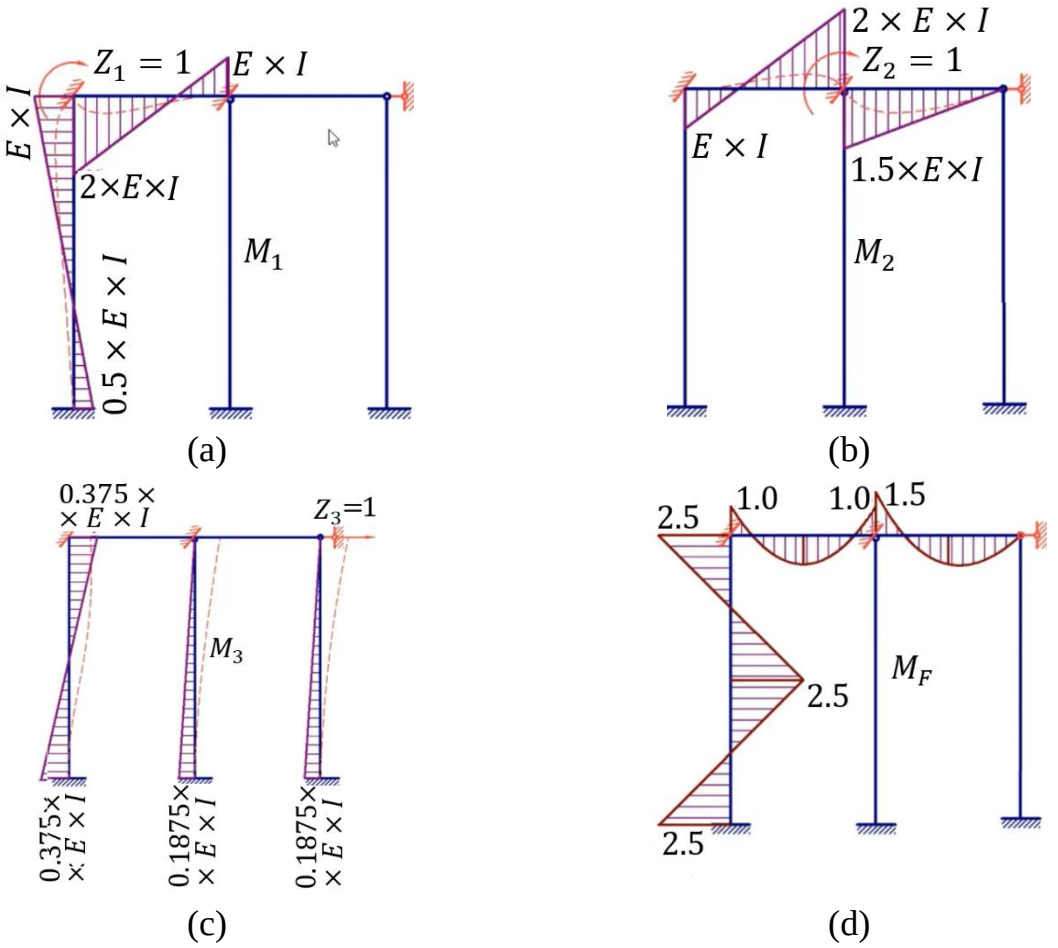
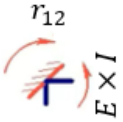


Fig. 3.16.2

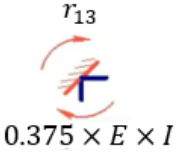
$r_{11} = 3 \times E \times I$

(a)



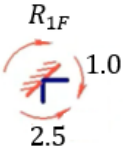
$r_{12} = E \times I$

(b)



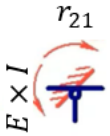
$r_{13} = -0.375 \times \times E \times I$

(c)



$R_{1F} = 1.5$

(d)



$r_{21} = E \times I$

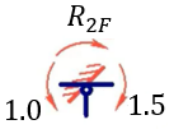
(e)

$r_{22} = 3.5 \times E \times I$

(f)

$r_{23} = 0$

(g)



$R_{2F} = -0.5$

(h)

Fig. 3.16.3

To determine the reactions in the additional support rod (the coefficients and free terms in the last canonical equation), one uses the conditions of the balance of the crossbar (Fig. 3.16.4) taking into account the transverse forces acting from the cut off racks.

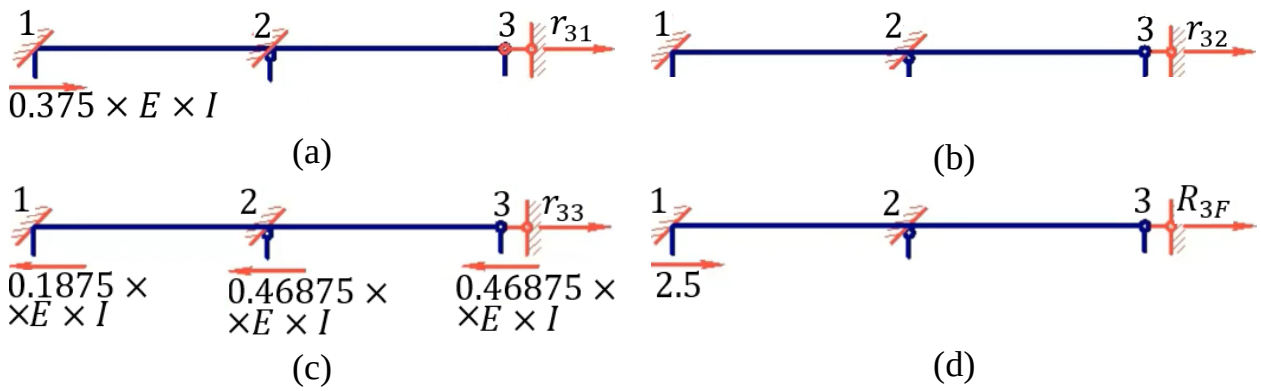


Fig. 3.16.4

Finding the coefficients for the third equation. Transverse forces from single displacement Z_1 is found by the diagram of the corresponded moments M_1 (Fig. 3.16.2 a) as: $Q_1 = \frac{-E\times I-0.5\times E\times I}{4} = -0.375 \times E \times I$, $Q_2 = Q_3 = 0$ and $r_{31} = -0.375 \times E \times I$.

Transverse (shear) forces from single displacement Z_2, found by the diagram of moments M_2 (Fig. 3.16.2 b) as: $Q_1 = Q_2 = Q_3 = 0$ and $r_{32} = 0$.

Transverse forces from single displacement Z_3, found by corresponded diagram of moments M_3 (Fig. 3.16.2 c): $Q_1 = \frac{0.375\times E\times I+0.375\times E\times I}{4} = 0.1875 \times E \times I$,

$Q_2 = Q_3 = \frac{0+0.1875\times E\times I}{4} = 0.046875 \times E \times I$ and $r_{33} = 0.28125 \times E \times I$.

Transverse forces of external load, found by the diagram of the external load moments M_F (Fig. 3.16.2 d) are equal: $Q_1 = \frac{-2.5-2.5}{2} = -2.5$, kN, $Q_2 = Q_3 = 0$ and $r_{3F} = -2.5$, kN.

The system of canonical equations of displacements method has the form:

$r_{11} \times Z_1 + r_{12} \times Z_2 + \cdots + r_{1n} \times Z_n + R_{1F} = 0,$

$r_{21} \times Z_1 + r_{22} \times Z_2 + \cdots + r_{2n} \times Z_n + R_{2F} = 0,$

……,

$r_{n1} \times Z_1 + r_{n2} \times Z_2 + \cdots + r_{nn} \times Z_n + R_{nF} = 0.$

Coefficients of the canonical equations of displacements method r_{ik} are the reactions in the additional main system connections from the unit displacements of these connections. The free terms R_{iF} of the canonical equations are the reactions in each i–th connection of the assigned load action in the main system.

When substituting the found coefficients and terms into the system of canonical equations of displacements method for the main displacements in the direction of additional connections of the main system, the system of equations takes the form:

$$3 \times Z_1 + Z_2 - 0.375 \times Z_3 = -\frac{1.5}{E \times I},$$
$$Z_1 + 3.5 \times Z_2 + 0 \times Z_3 = \frac{0.5}{E \times I},$$
$$-0.375 \times Z_1 + 0 \times Z_2 + 0.28125 \times Z_3 = \frac{2.5}{E \times I}.$$

Questions and comments:

3.16.1. Determination of the degree of kinematic indefinability of mechanical system taking into account the linear and angular mobility of nodes.

3.16.2. The use of floating embeddings to eliminate angular mobility of nodes and the use of additional support rods to eliminate linear mobility of nodes when forming the basic statically definable system of displacements method.

3.16.3. Determining the diagrams of bending moments of the main system, corresponding unit displacements of floating embeddings and the additional support rods, and diagrams of bending moments of external load.

3.16.4. Application of formulas of bending moments corresponding to unit displacements and reactions in typical statically indefinable beams for use in determining of bending moments diagrams of main system.

3.16.5. Using the conditions of balance of nodes of a mechanical system to find coefficients and free terms in the canonical equations of the displacement method.

3.16.6. Assessing the balance of forces on the side of the cut–off posts to determine the reactions in the additional support rod.

3.16.7. The reactions in additional connections of main system corresponding the unit displacements serving as coefficients of the canonical equations of displacement method.

3.16.8. Free terms of the canonical equations of displacement method, which are reactions in each additional connection of main system from the action of specified load.

3.16.9. The form of the system of equations of the displacement method when substituting the found coefficients and free terms.

3.17. The formulas for bending moments and reactions finding in statically indefinable beams under standard external influences caused the unit displacements

To create canonical equations of displacements method the formulas of bending moments and reactions in statically indefinable beams under typical external influences with unit displacements can be used.

The common examples of statically indefinable beams that can serve as elements of mechanical systems when calculated using the canonical equations of displacements method are considered below.

A clamped beam with a hinged support is shown on the schemes (Fig 3.17.1 a and b). Forces in the beam supports corresponding to the assumption of displacements method about unit turn of clamp ($\varphi = 1$) are equal: $M_A = \frac{3 \times E \times I}{l}$ and $R_A = R_B = \frac{3 \times E \times I}{l^2}$.

(a) (b)

Fig. 3.17.1

A doubly statically indefinable beam clamped at the ends is shown on the schemes (Fig 3.17.2 a and b). The reactions in the clamped beam corresponding to the assumption of displacements method of unit turn of clamped end ($\varphi = 1$) are equal: $M_A = \frac{4 \times E \times I}{l}$, $M_B = \frac{2 \times E \times I}{l}$ and $R_A = R_B = \frac{6 \times E \times I}{l^2}$.

(a) (b)

Fig. 3.17.2

A doubly statically indefinable beam clamped at the ends is shown on the schemes (Fig 3.17.3 a and b). The reactions in clamping a beam, corresponding to the assumption of displacements method about symmetrical unit turns of both clamps ($\varphi_A = 1$ and

$\varphi_B = 1$) are equal: $M_A = M_B = \frac{2 \times E \times I}{l}$ and $R_A = R_B = 0$ (the supports reactions are equal to zero due to the symmetry of unit turns).

(a) (b)

Fig. 3.17.3

A doubly statically indefinable beam clamped at the ends is shown on the schemes (Fig 3.17.4 a and b). Reactions in pinched beams corresponding to the assumption of displacements method about skew–symmetric unit turns of both clamps ($\varphi_A = 1$ and $\varphi_B = 1$) are equal: $M_A = M_B = \frac{6 \times E \times I}{l}$ and $R_A = R_B = \frac{12 \times E \times I}{l^2}$.

(a) (b)

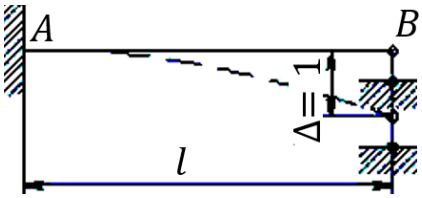
Fig. 3.17.4

A doubly statically indefinable end–clamped beam is shown on the schemes (Fig 3.17.5 a and b). Reactions in pinching of the beam, corresponding to the assumption of the displacements method about a unit shear of pinches ($\Delta = 1$) are equal:
$M_A = M_B = \frac{6 \times E \times I}{l^2}$ and $R_A = R_B = \frac{12 \times E \times I}{l^3}$.

(a) (b)

Fig. 3.17.5

A clamped beam with a hinged support is shown on the schemes (Fig 3.17.6 a and b). The forces in the supports of the beam, corresponding to the assumption of displacements method about a unit shift of the simply supported end of the beam ($\Delta = 1$) are equal: $M_A = \frac{3 \times E \times I}{l^2}$ and $R_A = R_B = \frac{3 \times E \times I}{l^3}$.



(a)

(b)

Fig. 3.17.6

Questions and comments:

3.17.1. Values of reactions in statically indefinable beams under standard external influences with unit displacements, used in finding the coefficients of the canonical equations of displacements method.

3.17.2. Examples of statically indefinable beams with a variety of forms of applied external load, as well as options for supporting or pinching the ends.

Chapter 4. The use of computer algebra Mathcad and the Microsoft EXCEL program for mathematical operations on matrices and vectors, to determine the displacements of nodes of a mechanical system using Mohr's integral, to find reactions, forces and displacements using the finite element method

4.1. Rules for multiplying matrices, finding the determinant of a matrix and the inverse of a matrix, rules for multiplying vectors used in calculations of mechanical systems

When performing calculations and when determining the elements of statically indefinable mechanical systems, in particular, when finding stresses and strains within the framework of various methods the mathematical operations on matrices and vectors are used, including those discussed below.

It is also necessary to mention that for operations on matrices and vectors the convenient and effective the means are: the computer algebra system Mathcad and the EXCEL program in the standard Microsoft Office application package, which are also used in documenting the results obtained, in particular, for writing WORD documents and for they archiving, as well as for preparing graphical representations of the results of calculations of mechanical systems and for the mathematical operations on matrices and vectors.

If two rectangular matrices A and B of dimensions $l \times m$ and $m \times n$ are given, that is: $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{l1} & a_{l2} & \dots & a_{lm} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$, then the product of these matrices A and B is a matrix C of dimension $l \times n$: $C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{l1} & c_{l2} & \dots & c_{ln} \end{bmatrix}$, having the elements: $c_{ij} = \sum_{r=1}^{m} a_{ir} \times b_{rj}$, for $i = 1, 2, \dots l$ and $j = 1, 2, \dots n$, the definition of which is shown on the diagram (Fig. 4.1.1).

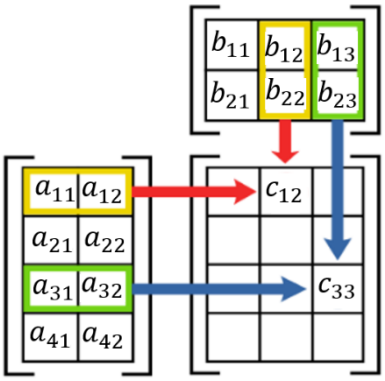


Fig. 4.1.1

The multiplication of two matrices A and B is possible only if the number of columns m of matrix A is equal to the number of rows m of the matrix B.

The matrix determinant, denoted by: $\det A$, ΔA or Δ, is a scalar value that makes sense only for square matrices and characterizes the oriented «stretching» or «compression» of the multidimensional Euclidean space on which the scalar product of matrices is introduced, after transforming this space by the matrix A of size: $n \times n$, consisting of elements a_{ij} $(i = 1, \dots, n$ and $j = 1, \dots, n)$ the determinant is calculated using the formula: $\det A = \sum_{\alpha_1, \alpha_2, \dots, \alpha_n} (-1)^{N(\alpha_1, \alpha_2, \dots, \alpha_n)} \times a_{1\alpha_1} \times a_{2\alpha_2} \times \dots \times a_{n\alpha_n}$.

The summation when finding the determinant is carried out over all permutations $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of numbers: $1, 2, \dots, n$, and the exponent $N(\alpha_1, \alpha_2, \dots, \alpha_n)$ at minus one (-1), means the number of inversions in the permutation: $\alpha_1, \alpha_2, \dots, \alpha_n$.

To count the number of inversions in a permutation, it is necessary to count for each element α_i of the permutation $(\alpha_1, \alpha_2, \dots, \alpha_n)$ how many elements larger than it are in front of it and add all these numbers (or count how many elements smaller than it are behind it and add all these numbers). Thus, the determinant includes: $n!$ its terms.

For a square matrix 2×2: $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ the determinant is equal: $\Delta = a \times d - b \times c$.

This matrix can be considered as a linear mapping that transforms a square on a plane with vertices: (0, 0), (1, 0), (0, 1) and (1, 1) into a parallelogram (Fig. 4.1.2) with vertices: (0, 0), (a, b), $(a + b, b + d)$ and (c, d). The magnitude of the determinant modulo: $|a \times d - b \times c|$ equal to the area of the parallelogram, which is the scaling factor for the area of the square, and also shows whether the scaling is a reflection, taking into account the sign of this determinant Δ.

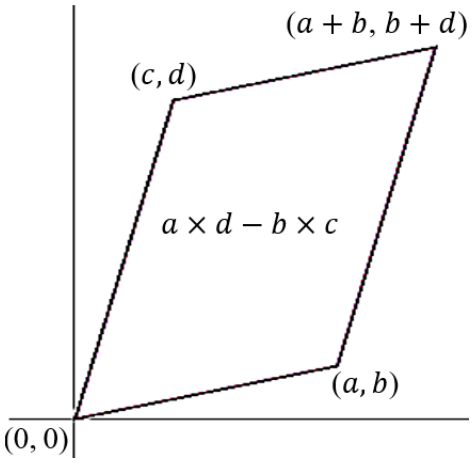


Fig. 4.1.2

When transposing a matrix, that is at a symmetric reflection of the matrix relative to the main diagonal it determinant does not change.

For a square matrix A of dimension 3×3:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ the determinant is equal:

$\det A = a_{11} \times a_{22} \times a_{33} + a_{12} \times a_{23} \times a_{31} + a_{21} \times a_{32} \times a_{13} -$

$-a_{13} \times a_{22} \times a_{31} - a_{21} \times a_{12} \times a_{22} - a_{32} \times a_{23} \times a_{11}.$

Finding the determinant $\det A$ of a square matrix A of dimension 3×3 one can show schematically (Fig. 4.1.3) as the difference between two sums of products of matrix elements a_{ij}, multiplied in a certain sequence, which is shown in the diagram by lines connecting the matrix elements.

$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Fig. 4.1.3

To calculate the determinant of a square matrix A of dimension: $n \times n$, consisting of elements a_{ij} ($i = 1, \dots, n$ and $j = 1, \dots, n$), one can use the row decomposition formula for matrix A. For example, by the decomposition from the first row of the matrix A it determinant one can find as: $\det A = \sum_{j=1}^{n} (-1)^{1+j} \times a_{1j} \times M_{1j}$, where M_{1j} – the additional minor of matrix A to it element a_{1j} along the first row when decomposing the matrix along this row ($i = 1$).

The additional minor M_{ij} of matrix A to it element a_{ij} is the determinant of the matrix remaining when deleting from the original matrix A of it i–th row and j–th column.

The algebraic complement of element a_{ij} of matrix A is the numerical value:

$A_{ij} = (-1)^{i+j} \times M_{ij}$.

Unit matrix E has the form: $E = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$.

Matrix A^{-1} of dimension: $n \times n$ is the inverse of matrix A of dimension: $n \times n$, if, when multiplying the original matrix A by its inverse matrix A^{-1} the unit matrix E of this dimension: $n \times n$ is obtained: $A^{-1} \times A = A \times A^{-1} = E$.

To find the inverse matrix A^{-1} the adjoint (the reciprocal) matrix A_M^T (that is the transposed matrix composed of algebraic additions: $(-1)^{i+j} \times A_{ji}$ of element A_{ij} of the original matrix A) is divided by the determinant $\det A$ of the original matrix: $A^{-1} = \frac{A_M^T}{\det A}$.

Adjoint (the reciprocal) matrix A_M^T has the form: $A_M^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$.

The mathematical operations: the multiplication of matrices and vectors, as well as the determination of the inverse matrix, the adjoint matrix, etc., widely used in calculations of mechanical systems, are comprehensively automated in the Mathcad computer algebra system and in the EXCEL program, as well as in other programs of the standard Microsoft Office applications.

The vector product of vectors: $\vec{a}$ and $\vec{b}$ is determined by a rule that is similar to that discussed above (Fig. 4.1.3), used to find the determinant of a square matrix of dimension: 3×3 , and the first row of the matrix is a vector with unit vectors (i, j, k). In addition, the vector product of vectors: $\vec{a}$ and $\vec{b}$ one can find using the formula for defining of the determinant of this matrix by it row decomposition, which is also considered above.

The formula for finding the vector product of vectors ($\vec{a}$ and $\vec{b}$):

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} =$$

$$= i \times \left(a_y \times b_z - a_z \times b_y\right) - j \times (a_x \times b_z - a_z \times b_x) + k \times \left(a_x \times b_y - a_y \times b_x\right).$$

Value (the modulus) of the vector product: $\left|\vec{a} \times \vec{b}\right|$ is equal to the product of the lengths of the vectors ($|\vec{a}|$ and $\left|\vec{b}\right|$) by the sine of the angle ($\widehat{\vec{a}, \vec{b}}$) between them:

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \times \left|\vec{b}\right| \times \sin(\widehat{\vec{a}, \vec{b}}).$$

The scalar product of vectors: $\vec{a} = \{a_x, a_y, a_z\}$ and $\vec{b} = \{b_x, b_y, b_z\}$ when calculated by coordinates is equal: $\vec{a} \cdot \vec{b} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z$.

The scalar product found by the angle between vectors, equals:

$$\left(\vec{a} \times \vec{b}\right) = |\vec{a}| \times \left|\vec{b}\right| \times \cos(\widehat{\vec{a}, \vec{b}}).$$

The angle between the vectors ($\widehat{\vec{a}, \vec{b}}$), calculated using the scalar product by finding the cosine of this angle, is: $\cos(\widehat{\vec{a}, \vec{b}}) = \frac{(\vec{a} \times \vec{b})}{|\vec{a}| \times |\vec{b}|} = \frac{a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \times \sqrt{b_x^2 + b_y^2 + b_z^2}}$.

Questions and comments:

4.1.1. The scope of application of operations on matrices and vectors in the calculation of mechanical systems.

4.1.2. Convenience of use the Microsoft Office operating system are for operations on matrices and vectors: the Mathcad computer algebra system and the EXCEL program.

4.1.3. Matrix product, the dimension of a matrix serving as a product of matrices. The equality of the number of columns of the first matrix in the product and the number

of rows of the second matrix, which serves as a necessary condition for the product of matrices.

4.1.4. Square matrix, the determinant of square matrix, the mathematical meaning of the determinant as an oriented «stretching» or «compression» of a multidimensional Euclidean space on which the scalar product of matrices is introduced, after transforming this space by the square matrix.

4.1.5. The geometric meaning of a square matrix of dimension: 2×2, as the scale of the area of a square, converted by a square matrix on a plane into a parallelogram.

4.1.6. The determinant of a two–by–two elements square matrix.

4.1.7. Transposing a square matrix and preserving the determinant of a square matrix when transposing it.

4.1.8. The determinant of a square matrix of dimensions: 3×3, a sequence of mathematical operations on the elements of a square matrix of dimensions three by three to find the determinant.

4.1.9. Formula for decomposing a square matrix by row when finding the determinant. The minor of matrix element and the algebraic complement of the matrix element.

4.1.10. The identity matrix is a square matrix of zero elements, on the diagonal of which there are the elements of units.

4.1.11. An inverse matrix, the product of which by the original matrix is equal to the matrix of the elements of units. A transposed matrix. Reciprocal (the adjoint) matrix – a trans-posed matrix consisting of algebraic complements of the original matrix. Finding the inverse matrix by dividing the reciprocal matrix by the determinant of the original matrix.

4.1.12. The vector product of vectors. The analogy of the vector product of vectors in a Cartesian coordinate system in three–dimensional space and finding the determinant of a square matrix of size three by three, if the first row of the matrix is the unit orts of the coordinate system, and the second and third rows are projections on the coordinate axes of the multiplied vectors.

4.1.13. Magnitude (the modulus) of the vector product of vectors.

4.1.14. Determining the angle between vectors and the scalar product of vectors.

4.2. Operators of computer algebra Mathcad and EXCEL programs used for operations on matrices and vectors in calculations of mechanical systems

To calculate mechanical systems and elements, as well as to perform the operations on matrices and vectors, it is convenient to use the Mathcad computer algebra system developed by the company PTC, Inc (Parametric Technology Corporation).

The worksheet in the Mathcad has a grid that resembles a checkered notebook sheet. This grid in the Mathcad settings can be changed or deleted. The top panel of the Mathcad window has tabs and option symbols, the functions of which are intuitive.

On the Mathematics tab there are options: Operators (the Mathcad panel is shown on the diagram Fig. 4.2.1), Symbols (for example, the Greek alphabet), Constants, Symbolic operations and other less commonly used options. It is possible to specify units and systems of measurement.

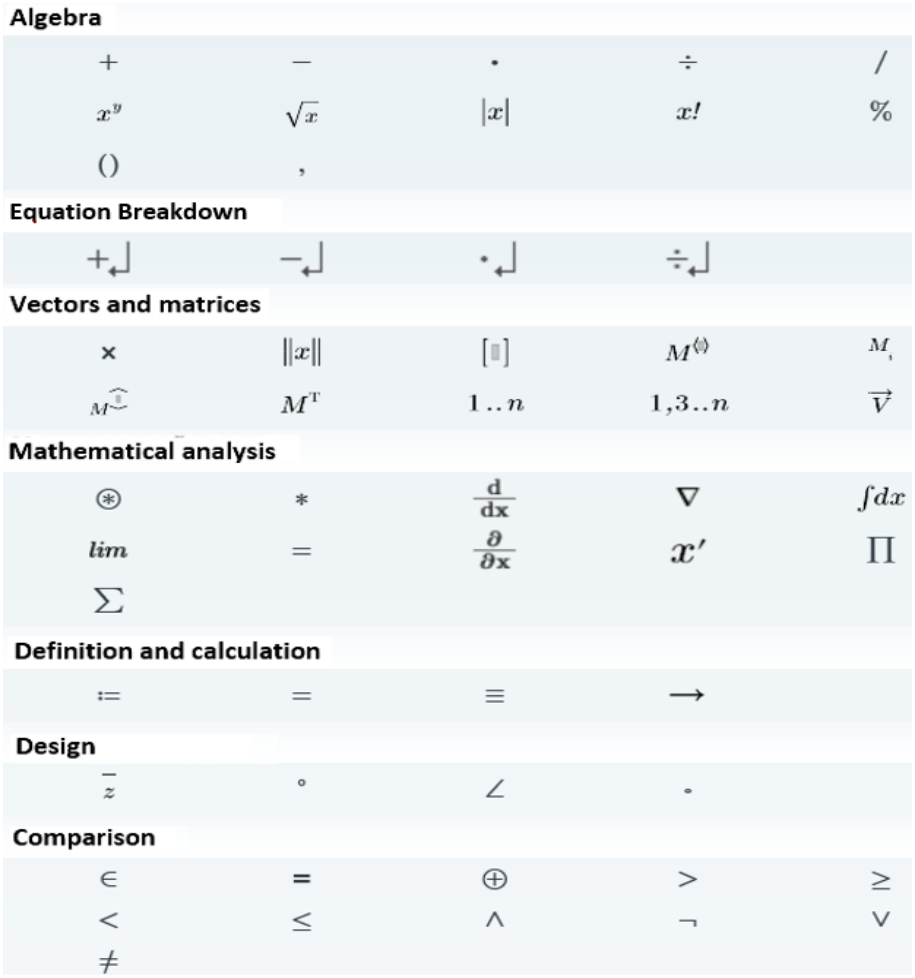


Fig. 4.2.1

The Input\Output tab offers options that, in particular, may be required when entering data from the EXCEL program.

Options on the Functions tab are used in various calculations, and all functions are combined into option «f_x». The use of the Matrices/Tables tab. On this tab there are options when performing operations on matrices: Insert matrix, Insert table, Operators with vectors/matrices and Functions with vectors/matrices. When performing actions on matrices and vectors, one can use the options on the Calculation tab. Typically, actions on matrices and vectors begin by setting the option on this tab ORIGIN:=1, what does the

number of the first element mean 1 for operations on vectors or (1,1) for operations on matrices.

For the exercise on using Mathcad, it is proposed to type two matrices (M and S, fragment of the Mathcad page, Fig. 4.2.2, if you have sufficient experience in working with matrices in the Mathcad computer algebra system, the exercises in question are not required), using exercise operation «:=» and the Matrices/Tables tab on the top panel of Mathcad, matrix transposition M into matrix C, finding the square root of the sum using the Mathematics tab and finding the cosine of angle, taking into account the dimension of the angle and rounding digits of the result.

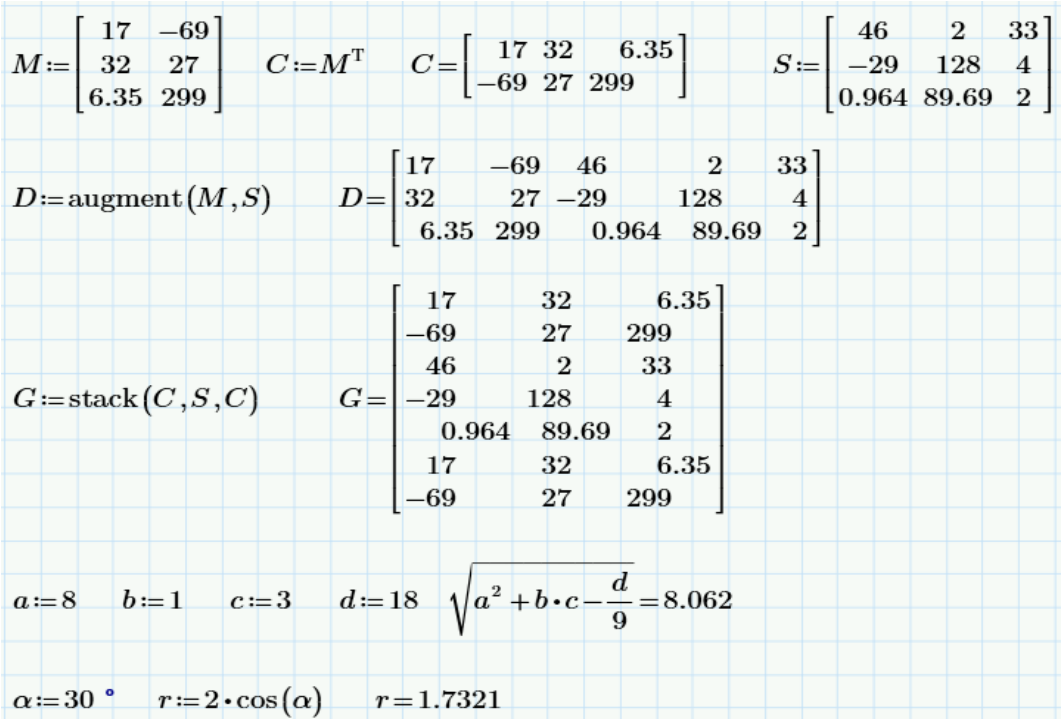


Fig. 4.2.2

The Document tab allows one to format the page (to assign orientation, spacing, etc.). There is a Resources section containing the Mathcad educational capabilities, as well as an automatic link «» to access the PTC University website (learningconnector.ptc.com), where online courses are available. In addition, the top panel of Mathcad has tabs: Graphs, Formula Formatting, Text Formatting.

Common actions in the Mathcad computer algebra system, used to perform mathematical operations on matrices, when solving systems of algebraic equations, and when calculating forces, displacements and deformations in rod mechanical systems are:

S:=submatrix(M, x_1, y_1, x_2, y_2) – the selecting the matrix S in the matrix M, in the parameters of the «submatrix» operation: the addresses of initial (x_1 and y_1) and final (x_2 and y_2) cells of the source matrix M;

M:=putregion(P, S, x_1, y_1) – the inclusion of the matrix S in the matrix P and obtaining the resulting matrix M, in the parameters of the «putregion» operation: the address of initial (x_1 and y_1) cells of the original matrix P for insertion of the matrix S;

M:=augment(P, S, …) – the matching the columns of matrices: P, S …;

M:=stack(P, S, …) – the matching the rows of matrices: P, S …

The discussed and other commands in the Mathcad are located on the Matrices/Tables tab in the Functions with vectors/matrices panel.

To transfer tabular data from the EXCEL program in the form of a matrix to the Mathcad computer algebra system, one can use the READEXCEL function located on the Mathcad panel on the Input/Output tab. Wherein the EXCEL file used must be located in the same directory as the Mathcad file.

Common commands when working in the EXCEL program to perform mathematical operations on matrices, used in calculating of forces, displacements and deformations in rod mechanical systems. In the EXCEL program, to transpose a matrix (for example, a cosine matrix), one needs to select the placement area on the EXCEL page taking into account the dimensions of the transposed matrix and enter a command in the address bar indicating the ranges (__:__) of the cells of the original matrix: = TRANSPOSE(__:__) and use a combination of keystrokes to execute the command: control shift enter.

To multiply matrices, for example, to multiply a transposed cosine matrix by the local stiffness matrix of a finite element bar when obtaining a matrix of a discrete model, one needs to select the area where the product of matrices is placed, taking into account the size of their product, and enter a command in the address bar indicating the ranges of cells of the original matrices (__:__;__:__), observing the sequence of matrices in the product: = MMULT(__:__;__:__) and use a combination of keystrokes to execute the command: control shift enter.

The inverse matrix (for example, the inverse matrix with respect to the stiffness matrix of mechanical system) is found using the function: = MINVERSE(__:__), to execute this command in EXCEL one needs to use keystroke: control shift enter.

Questions and comments:

4.2.1. The tabs located on the top panel of the Mathcad window and options located on these tabs.

4.2.2. The use of the Matrices/Tables tab and its options to set matrices and to perform operations on matrices.

4.2.3. Use of the Math tab to find values of functions and perform math operations.

4.2.4. The use of the Document tab to format a Mathcad page.

4.2.5. The common actions in the Mathcad computer algebra system, used to perform mathematical operations on matrices, when solving systems of algebraic equations, and when calculating forces, displacements and deformations in rod mechanical systems.

4.2.6. The action «submatrix» (the selecting one matrix from another) in the Mathcad computer algebra system, the parameters of the «submatrix» operation.

4.2.7. «Putregion» action (the inclusion of one matrix into another and obtaining the resulting matrix) in the Mathcad system, the parameters of the «putregion» operation.

4.2.8. The action «augment» (the combining matrices by columns) in the Mathcad system, the parameters of the operation «augment».

4.2.9. The action «stack» (the combining matrices along rows) in the Mathcad system, the parameters of the «stack» operation.

4.2.10. The READEXCEL function, located on the Mathcad panel on the Input/Output tab, used to transfer tabular data in matrix form from the EXCEL program to the Mathcad computer algebra system.

4.2.11. Common commands when working in the EXCEL program to perform mathematical operations on matrices, used in calculating of forces, displacements and deformations in rod mechanical systems.

4.2.12. The transposing a matrix in the EXCEL.

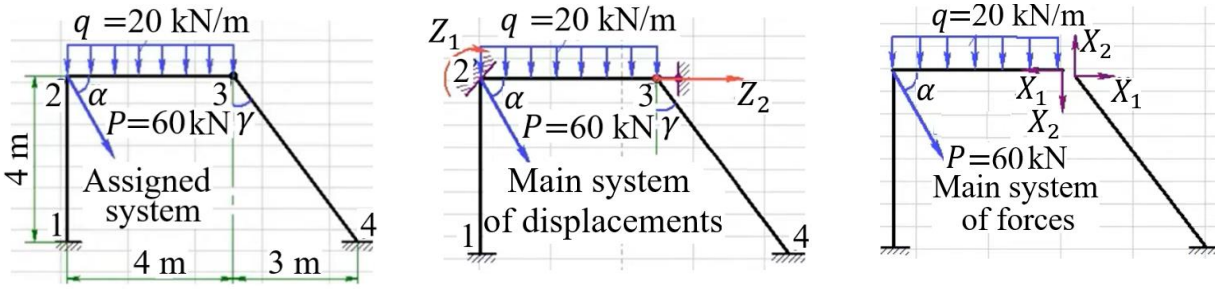
4.2.13. The product of matrices in the EXCEL program, for example, the product the transposed cosine matrix by the local stiffness matrix of a finite element to obtain the stiffness matrix of a discrete model.

4.2.14. Finding an inverse matrix in EXCEL.

4.3. Application of computer algebra system Mathcad to determine forces in a statically indefinable mechanical system, using the displacements method. Use of the Villiot's diagram to find displacements of nodes taking into account the angles of hinge connection of incompressible rods

Exercise. Statically indefinable mechanical system with the non–parallel struts includes hinge. The dimensions and loads are shown on the diagram (Fig. 4.3.1 a). Concentrated force (P =60 kN) acts on rigid node 2 and is applied at an angle α to the horizontal crossbar of the mechanical system. Distributed load (q =20 kN/m) acts on horizontal section 2–3. The sections 1–2 and 2–3 of the mechanical system are rigidly connected. The rod 3–4 is hinged at the node 3. The exercise assigns one to find support reactions by the method of displacements using the multiplication of diagrams of moments using the Mathcad computer algebra system.

The linear stiffness of all elements of the mechanical system is equal. In exercise it is recommended to accept the conditional value of linear stiffness: $E \times I = 10^3$ N×m^2.



(a) (b) (c)

Fig. 4.3.1

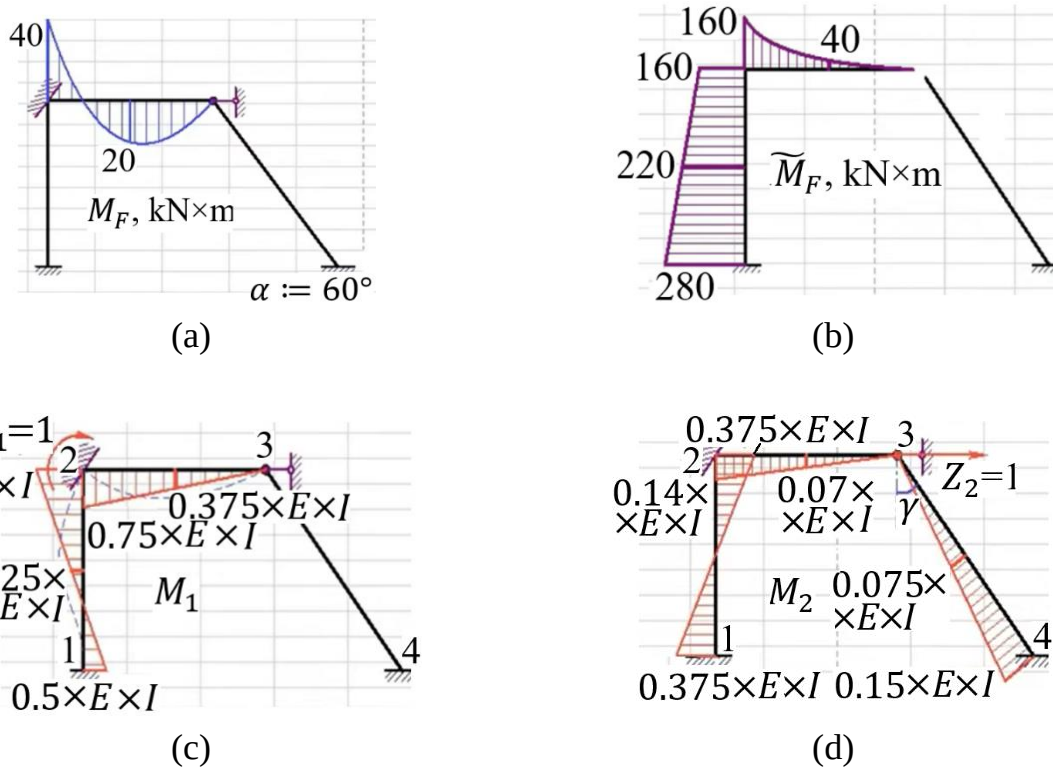
The degree of kinematic indefinability ($n = n_l + n_a$) is determined by the summation of unknown displacements – the linear n_l and the angular n_a. The number of angular displacements corresponds to the number of rigid nodes, such as one node 2, that is: $n_a = 1$.

The degree of linear mobility (the number of unknown linear displacements) one can determine by considering the hinge diagram (Fig. 4.3.1 b) as the number of degrees of freedom of the hinge system, which is twice the number of nodes (4 nodes, with supports) minus the number of connecting rods (3) and minus twice the number of reference connections (2): $n_l = 4 \times 2 - 3 - 2 \times 2 = 1$. The degree of kinematic indefinability is equal to two: $n = 1 + 1 = 2$.

To eliminate the linear mobility within the assumptions of displacements method one needs to add a horizontal support rod at node 3 (or at node 2). To eliminate the angular movement at node 2 one needs to add floating clamp (the floating clamp that does not allow turn displacement of node 2).

When eliminating linear and angular mobility in accordance with the hypotheses of the displacements method, a statically definable main system (Fig. 4.3.1 b) is obtained, corresponding to the assigned statically indefinable system (Fig. 4.3.1 a).

Using known diagrams of bending moments and reactions in statically indefinable beams under typical external influences (section 3.17), one can create the diagrams of bending moments of the external load by the sections: 1–2, 2–3 and 3–4, acting on each of these sections (q =20 kN/m and P =60 kN, Fig. 4.3.2 a) and of moments corresponding to unit displacements (Z_1=1 and Z_2=1, Fig. 4.3.2 c and d).



(a) (b)

(c) (d)

Fig. 4.3.2

When calculating statically indefinable mechanical system using the displacement method, the diagrams of bending moments in the main system due to the action of an external load are used (Fig. 4.3.2 a) and the diagrams of moments corresponding to single movements of nodes of the main system supported by additional rods and floating embeddings (Fig. 4.3. 2 c and d), which can be created using standard diagrams of moments, acting in statically indefinable beams under the condition of single angular or linear displacements of their ends (section 3.17, Fig. 3.17.1–3.17.6, etc.), as well as typical diagrams of the displacements acting in statically indefinable beams at specified load (Fig. 4.3.3 a–d).

Provided that one end of the upper horizontal crossbar is pinched and the second end is hinged, which corresponds to the basic system of the displacement method, the shape of the diagram (Fig. 4.3.2 a) and the values of the moments at characteristic points can be found from a typical diagram of the moments acting in statically indefinable beams at given load (Fig. 4.3.3 a).

When creating diagrams of moments corresponding to unit displacements, using data on typical diagrams of moments of statically indefinable beams, each beam of the main system of the displacement method is considered separately. In addition, the dimensions of the mechanical system shown in the diagram (Fig. 4.3.1 a) are taken into account: 4 m, 4 m and 3 m.

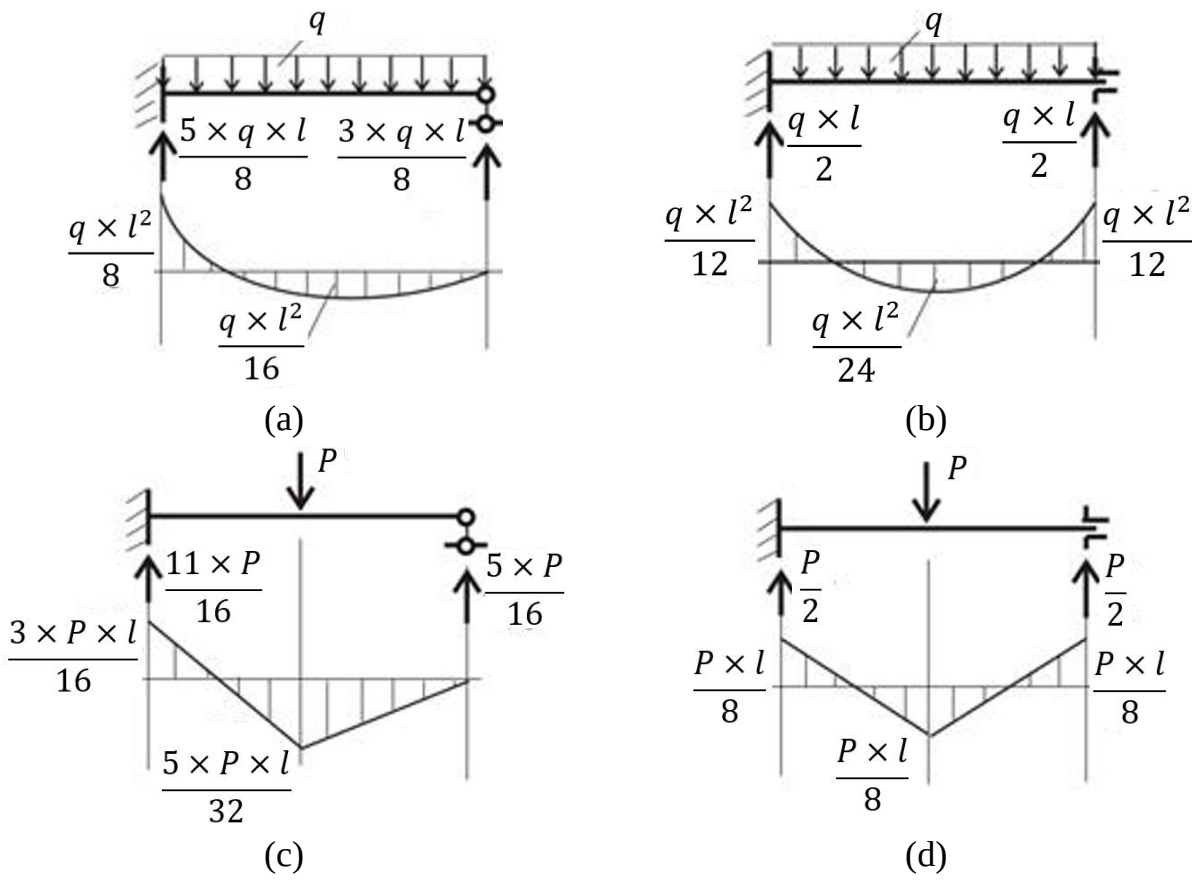


Fig.4.3.3

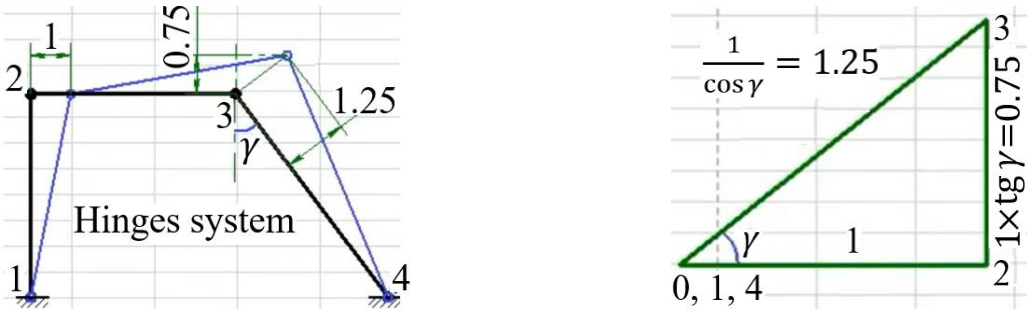
It would be mentioned that the force applied in node 2: P =60 kN, acting at an angle ($\alpha = 60°$), does not affect the shape of the moments diagram for the main system of displacements method, because the force is applied in the node and its action does not extend to the area 2–3, although the force P applied in node 2 affects the diagram of moments for the main system of forces method, if the main system is formed by replacing the hinge of the assigned mechanical system with the forces acting in the hinge, as shown below.

When determining the diagrams, the action of transverse forces, transverse distributed load and bending moments, which are in the differential and integral mutual dependencies, is taken into account, and as an assumption the longitudinal forces acting in the elements of the system are not taken into account.

Since, within the framework of this assumption, the rods in the main system of displacements method are considered as incompressible, then the displacements in the hinged rods 2–3 and 3–4 must be coordinated taking into account the angles of the hinged connection of the incompressible rods.

This can be conveniently done using the hinge diagram (Fig. 4.3.4 a) and the D.V. Villiot's diagram (Fig. 4.3.4 b), which shows that the hinged end 3 of the incompressible rod 2–3 moves in the same direction as the hinged end 2 of this rod (on a hinge diagram), at the rotation of the incompressible rod 1–2 relative to the hinge, which replaces support 1, wherein the incompressible rod 3–4 rotates relative to the hinge, which replaces support

4, wherein hinge 3 moves along the intended circle at the rotation of rod 3–4 relative to assumed hinge in support 4.



(a) (b)

Fig. 4.3.4

Thus, the displacement of hinge 3 in rod 2–3, directed along the circle of rotation of rod 2–3 relative to hinge 2, and the displacement of this hinge in rod 3–4, directed along the circle when rotating rod 3–4 relative to hinge 4, one can find as the proportion of the leg and hypotenuse: a right triangle with an angle γ at one of the vertices, under which the rod is inclined to the vertical 3–4. Hinges 1 and 4 are supporting when creating the triangle of the Villiot's diagram. Therefore, the diagram of moments in rod 3–4 is created not of a single displacement of node 3 along the axis of incompressible rod 2–3, but of the displacement: $\frac{1}{\cos\gamma} = 1.25$, corresponding to the hypotenuse of a right triangle according to the Villiot's diagram. And the displacement of node 3 in rod 2–3 is find out not by one, but by 0.75, which is in the proportion of the legs of a right triangle: $1 \times \mathrm{tg}\,\gamma = 0.75$. The indicated ratios of displacements and their values are used in creating of the diagrams (Fig. 4.3.2 d) of moments in rods of main system of displacements method under the influence of a single displacement: $Z_2 = 1$ in the node 3.

The data of determined diagrams are entered into the EXCEL table (see fragment of the table on the scheme, Fig. 4.3.5), including both the data of the diagram of moments of external load for the main system of displacements method M_F, and the data of the diagram of moments corresponding to unit displacements (Z_1 and Z_2): M_1 and M_2, as well as diagram data of moments of external load for the main system of method of forces $\widetilde{M}_F$, for their further use in the calculation of mechanical systems by the method of displacements using the Mathcad computer algebra system.

The values of moments corresponding to the specified diagrams are transferred to the table at three points of each element of the mechanical system – at the starting point of each rod, at the midpoint and at the end point, which makes it possible to take into account the nonlinear shape of some diagrams of moments of external load when executing mathematical operations of displacements method using these diagrams.

	A	B	C	D	E
1		Data for calculation using			
2		the displacement method			
3		M_F	M_1	M_2	$\widetilde{M}_F$
4	1	0	0.5	-0.375	-280
5	2	0	-0.25	0	-220
6	3	0	-1	0.375	-160
7	4	-40	0.75	0.14	-160
8	5	20	0.375	0.07	-40
9	6	0	0	0	0
10	7	0	0	0	0
11	8	0	0	0.075	0
12	9	0	0	0.15	0

Fig. 4.3.5

To import data, which is convenient when performing calculations in the Mathcad computer algebra system, the EXCEL file has to be in the same folder as the Mathcad file.

The system of canonical equations of displacements method is used:

$$r_{11} \times Z_1 + r_{12} \times Z_2 + \cdots + r_{1n} \times Z_n + R_{1F} = 0,$$
$$r_{21} \times Z_1 + r_{22} \times Z_2 + \cdots + r_{2n} \times Z_n + R_{2F} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$r_{n1} \times Z_1 + r_{n2} \times Z_2 + \cdots + r_{nn} \times Z_n + R_{nF} = 0,$$

which has a matrix notation: $\begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1m} \\ r_{21} & r_{22} & \ldots & r_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ r_{n1} & r_{n2} & \ldots & r_{nn} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \ldots \\ Z_n \end{bmatrix} = - \begin{bmatrix} R_{1F} \\ R_{2F} \\ \ldots \\ R_{nF} \end{bmatrix}$,

or: $[r][Z] = -[R_F]$, where $[r]$ – the stiffness matrix, which is a symmetrical matrix of forces that arise during alternate unit displacement of each additionally introduced connection, $[Z]$ – the matrix of unknown displacements of additional connections, $[R_F]$ – the matrix of efforts in these connections under the external load action.

The stiffness matrix (the matrix of coefficients of the equations of displacements method) $[r]$ one can define as: $[r] = [M_i]^T[B][M_i]$, where $[M_i]$ – the matrix of bending moments corresponding to unit displacements of main system of the displacements method, $[B]$ – the pliance matrix of the mechanical system, with the help of which a connection is established between the deformations of the system and its internal forces.

Vector (the matrix column) of forces in the connections of the main system from the external load action (i.e., the vector of free terms in the canonical equations of the displacement method) one can determine as: $[R_F] = -[M_i]^T[B][\widetilde{M}_F]$, where $[\widetilde{M}_F]$ – the vector (matrix column) of bending moments under the external load action in the main system method of forces obtained from the assigned system, or in the main system of displacements method by eliminating the unnecessary connections, including the connection whose reaction is determined.

Matrix solution of the system of canonical equations of displacements method has the form of the vector (matrix columns) of the main displacements corresponding to the additional rods and floating embeddings supplemented to a given statically indefinable mechanical system during the formation of the main system: $[Z] = -[r]^{-1}[R_F]$.

Values of resulting diagram of bending moments, which is defined as: $[M] = [M_i][Z] + [M_F]$, where $[M_F]$ – the vector (matrix column) of bending moments of the external load action in a given system, which is compiled according to the values of bending moments in the main system of displacements method, in contrast to the vector (matrix column) $[\widetilde{M}_F]$ of bending moments of the external load action in the main statically definable system of forces method (Fig. 4.3.1 c), obtained from the assigned system by replacing discarded «extra» connections with forces in them, or in contrast to the statically definable main system of the displacements method, using which one can determine a given statically indefinable mechanical system by eliminating unnecessary connections (the additional rods and floating embeddings, Fig. 4.3.1 b).

Matrix method of calculating frames using the computer algebra system Mathcad and the method of displacements. The values of diagrams of moments of the external load and the diagrams of moments of unit forces, applied in the direction of the required displacements, are copied into the Mathcad from the EXCEL program (Fig. 4.3.5), what is shown on the fragment of the Mathcad page (Fig. 4.3.6). The designation of moments of external load for main system of the method of forces $\widetilde{M}_F$ has been replaced with a designation more convenient for Mathcad MC_F.

M_F	M_1	M_2	MC_F
0	0.5	−0.375	−280
0	−0.25	0	−220
0	−1	0.375	−160
−40	0.75	0.14	−160
20	0.375	0.07	−40
0	0	0	0
0	0	0	0
0	0	0.075	0
0	0	0.15	0

Fig. 4.3.6

$$M_F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -40 \\ 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad M_1 = \begin{bmatrix} 0.5 \\ -0.25 \\ -1 \\ 0.75 \\ 0.375 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} -0.375 \\ 0 \\ 0.375 \\ 0.14 \\ 0.07 \\ 0 \\ 0 \\ 0.075 \\ 0.15 \end{bmatrix} \quad MC_F = \begin{bmatrix} -280 \\ -220 \\ -160 \\ -160 \\ -40 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Fig. 4.3.7

The data from the table shown on the diagram of the Mathcad working page fragment can be presented in vector form (Fig. 4.3.7).

It is more convenient to combine the values of diagrams of bending moments corresponding to unit displacements (Z_1 and Z_2): M_1 and M_2 (Fig. 4.3.2 c and d) for calculations of the mechanical system using the method of displacements into a common

matrix and take into account the linear stiffness multiplier ($E \times I = 10^3$ N×m^2) to the values of these vectors (M_1 and M_2, fragment of the Mathcad working page is shown on the scheme, Fig. 4.3.8).

$$EI := 10^3 \qquad MI^{\langle 1 \rangle} := EI \cdot M_1 \qquad MI^{\langle 2 \rangle} := EI \cdot M_2$$

$$MI = \begin{bmatrix} 500 & -375 \\ -250 & 0 \\ -1 \cdot 10^3 & 375 \\ 750 & 140 \\ 375 & 70 \\ 0 & 0 \\ 0 & 0 \\ 0 & 75 \\ 0 & 150 \end{bmatrix}$$

Fig. 4.3.8

$$f(i,j) := 0 \qquad N := \mathrm{matrix}(9,9,f)$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.3.9

Finding the pliance matrix $[B]$ in the Mathcad, with the help of which a connection is established between the deformations of the system and its internal forces. Since the mechanical system has three sections on which diagrams of bending moments are created, and in calculation of the mechanical system using the method of displacements three values of bending moments are used in each section, then the dimension of the pliance matrix is: 9×9.

Initially, one needs to generate a zero matrix in the Mathcad with dimension: 9×9 (the fragment of the Mathcad working page is shown on the scheme, Fig. 4.3.9).

Vector of the sections lengths (l), determined according to a given scheme of the mechanical system, and the pliance matrix for each of the sections $b(k)$, determined using the Simpson's formula, taking into account the equality of linear stiffness of all sections, have the form shown in a fragment of the Mathcad working page (Fig. 4.3.10).

$$n := 3 \qquad k := 1..n \qquad l := \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} \qquad b(k) := \frac{l_k}{6\ EI} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b(1) = \begin{bmatrix} 6.667 \cdot 10^{-4} & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 6.667 \cdot 10^{-4} \end{bmatrix}$$

$$b(2) = \begin{bmatrix} 6.667 \cdot 10^{-4} & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 6.667 \cdot 10^{-4} \end{bmatrix} \qquad b(3) = \begin{bmatrix} 8.333 \cdot 10^{-4} & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 8.333 \cdot 10^{-4} \end{bmatrix}$$

Fig. 4.3.10

The combination of the pliance matrices found using Simpson's formulas for each of the sections $b(k)$ into the pliance matrix of the entire mechanical system is carried out

using the «putregion» function available in the Mathcad (fragment of the Mathcad working page is shown on the scheme, Fig.4.3.11). The insertion indices of the pliance matrices for each of the sections are to be pre–set: i and j into the previously generated zero matrix $[N]$, having the dimension: 9×9.

$$i := \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad j := \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad B(k) := \text{putregion}\left(N, b(k), i_k, j_k\right) \qquad B := \sum_{k=1}^{n} B(k)$$

Fig. 4.3.11

The matrix $[B]$ of pliance of the mechanical system as a fragment of the Mathcad working page is shown on fragment of the Mathcad working page (Fig. 4.3.12).

$$B = \begin{bmatrix} 6.667 \cdot 10^{-4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.667 \cdot 10^{-4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.667 \cdot 10^{-4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.003 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.667 \cdot 10^{-4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8.333 \cdot 10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.003 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.333 \cdot 10^{-4} \end{bmatrix}$$

Fig. 4.3.12

The coefficients of the canonical equations of displacements method are sought in matrix form: $[r] = [M_i]^T [B][M_i]$ (fragment of the Mathcad working page is shown on the scheme, Fig. 4.3.13).

$$r := MI^{\mathrm{T}} \cdot B \cdot MI \qquad r = \begin{bmatrix} 1.75 \cdot 10^3 & -235 \\ -235 & 251.133 \end{bmatrix}$$

Fig. 4.3.13

$$R_F := -MI^{\mathrm{T}} \cdot B \cdot MC_F \qquad R_F = \begin{bmatrix} -40 \\ -7.6 \end{bmatrix}$$

Fig. 4.3.14

Free terms of the canonical equations of displacements method are sought in matrix form: $[R_F] = -[M_i]^T [B]\left[\widetilde{M}_F\right]$ (fragment of the Mathcad working page is shown on scheme, Fig. 4.3.14).

The main unknowns of displacements method are defined as: $[Z] = -[r]^{-1}[R_F]$ (fragment of the Mathcad working page is on the scheme, Fig. 4.3.15).

$$Z := -r^{-1} \cdot R_F \qquad Z = \begin{bmatrix} 0.031 \\ 0.059 \end{bmatrix}$$

Fig. 4.3.15

$$M := MI \cdot Z + M_F \qquad M = \begin{bmatrix} -6.758 \\ -7.698 \\ -8.637 \\ -8.637 \\ 35.682 \\ 0 \\ 0 \\ 4.431 \\ 8.861 \end{bmatrix}$$

Fig. 4.3.16

Bending moments in characteristic sections of a given statically indefinable system are determined using the formula: $[M] = [M_i][Z] + [M_F]$ (fragment of the Mathcad working page is on the scheme, Fig. 4.3.16).

Based on the results of calculating the values of bending moments one can determine the resulting diagram of moments by the sections (Fig. 4.3.17).

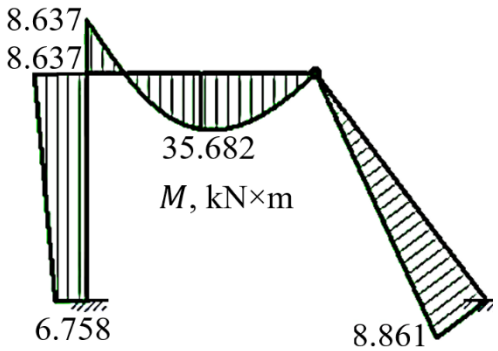


Fig. 4.3.17

Questions and comments:

4.3.1. The degree of kinematic indefinability of mechanical system.

4.3.2. The degree of linear mobility of mechanical system (the number of unknown linear displacements).

4.3.3. Adding a support rod to eliminate the linear displacement of statically indefinable mechanical system when calculating it by the displacements method.

4.3.4. Adding a floating embedding to eliminate angular movement of a statically indefinable mechanical system when calculating it by the displacements method.

4.3.5. Forming the main system of the displacements method to locate extra displacements of mechanical system, which number corresponds to the degree of the kinematic indefinability.

4.3.6. The use diagrams of bending moments and reactions in statically indefinable beams under typical external influences to determine diagrams of bending moments of external load in the sections of mechanical system.

4.3.7. The use of moments acting in statically indefinable beams subject to single angular or linear displacements of their ends to determine diagrams of bending moments of main system of the displacements method.

4.3.8. Mutual differential and integral dependencies of bending moments and transverse (the shear) forces when creating their diagrams.

4.3.9. Methodological assumption of incompressibility of rods when using the displacements method, which allows one to neglect the longitudinal forces and their diagrams when calculating with this method.

4.3.10. The coordination of displacements in hinged rods of a mechanical system, taking into account the angles of connection of the rods.

4.3.11. The use of Villiot's diagram when coordinating the displacements of hinge joints of mechanical system, correlating the displacements of hinge joints of the system as sides of a right triangle, considering the support hinges moving in circles relative to the centers in the support joints.

4.3.12. The use of EXCEL table to systematize data applied in the next in the calculation of a statically indefinable mechanical system by the displacements method in the Mathcad computer algebra system, copying data from the EXCEL to the Mathcad, converting data in table columns into vector form in the Mathcad.

4.3.13. System of canonical equations of the displacements method, the matrix form of recording of the canonical equations system.

4.3.14. The pliance matrix of a mechanical system, with the help of which a connection is established between the deformations of the system and its internal forces.

4.3.15. Finding the rigidity matrix of a mechanical system – the matrix of multipliers (coefficients) of the canonical equations of the displacements method as a product of matrices: the transposed matrix of moments corresponding to unit displacements in the main system of the displacements method on the pliance matrix of the mechanical system, with the help of which the connection is established of deformations of the system with its internal forces, and on the matrix of moments corresponding to unit displacements.

4.3.16. Definition of a vector (the matrix column) in connections under the influence of an external load (that is the vector of free terms of the canonical equations of the displacements method) as a product with a minus sign of matrices: the transposed matrix of moments corresponding to unit displacements in the main system of the displacements method on the pliance matrix of the mechanical system and in the vector of moments under the influence of an external load in main system of the displacements method.

4.3.17. Form of the matrix solution of the system of canonical equations of the displacements method.

4.3.18. The difference between the main statically definable systems: of the displacements method, in which the excess displacements of nodes are eliminated by adding the extra rods and the floating embeddings to a given statically indefinable mechanical system, and the method of forces, in which the redundant connections of a given statically indefinable system are replaced by reactions in them.

4.3.19. Determination in matrix form of the values for the final diagram of bending moments in the elements of assigned statically indefinable system as the sum of the matrix of moments under influence of an external load in a given system and the product of bending moment matrices corresponding to unit displacements in the main system and the vector (matrix columns) of the found main displacements.

4.3.20. The dimension of the square matrix of pliance as the product of number of main displacements (the number of additional rods and floating embeddings, introduced during the formation of the main system of the displacements method) and the number of sections of the mechanical system on which bending moment diagrams are determined.

4.3.21. Finding a pliance matrix in the Mathcad, using a zero square matrix of the appropriate dimension, determining diagonal pliance matrices for individual sections of mechanical system (using Simpson's formula to find the displacements), combining pliance matrices found for sections into the pliance matrix of an entire mechanical system.

4.3.22. Solution in the Mathcad computer algebra system of the system of canonical equations of the displacements method: the determination of coefficients of canonical equations (the stiffness matrix of the main statically definable system), finding the vector of free terms of the canonical equations (the efforts in the connections of main system under the external load), determination of bending moments in characteristic sections of a given statically indefinable system.

4.4. Calculation of forces in the elements of a flat truss using the finite element method in the Mathcad, specifying the geometric characteristics of the truss and nodal load, finding the stiffness matrices of finite elements, the incidence matrix and the stiffness matrix of a discrete model, determining the displacement vectors and end forces

The usual sequence for calculating beam structures using the finite element method is as follows:

1. Determination of a discrete calculation model of a given rod system (the division into finite elements, assigning the nodes) and description of the discrete model: numbering of nodes and rods, determination of their number.

2. Selecting general and local coordinate systems and determining the coordinates of nodes in the general coordinate system.

3. Drawing up a vector of displacements of nodes of the computational discrete model assigned system.

4. Identification of finite elements, their types, finding linear stiffnesses and establishing a correspondence between the numbers of rods and the numbers of the initial and final nodes for these rods.

5. Converting external loads on rods (distributed and concentrated loads) to nodal loads into a general coordinate system, determining the total nodal forces at each node of the discrete model.

6. Creation of stiffness matrices of finite bar elements in local coordinate systems, which are linked to finite elements.

7. Determination of direction sines and cosines for each finite element and transformation of the stiffness matrices of finite rod elements to the global coordinates.

8. The formation of the stiffness matrix of the entire mechanical system in a general (global) coordinate system.

9. Obtaining a system of resolving equations by taking into account boundary conditions (the availability of reference connections); wherein a diagonal matrix can be used or to delete the rows and columns corresponding to zero displacements.

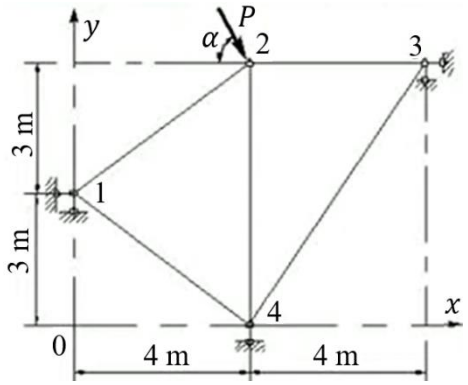
10. Solving the system of equations and finding the nodal displacements of discrete model.

11. Determination of nodal displacements for finite elements under forces acting on finite elements.

12. Finding forces and displacements in finite elements, determining the diagrams of forces in mechanical system and determining the shape of the system deformations.

Exercise. A flat truss is assigned (Fig. 44.1 a) with four hinged nodes: 1, 2, 3 and 4, rods in the truss: 1–2, 1–4, 2–3, 2–4 and 3–4 (4 (at the beginning it is relevant to assign

the smaller number i to the rod finite element, and a bigger number j to the end). The external force: P =50 kN is applied to node 2 at an angle: $\alpha = 65°$.



(a)

	A	B	C	D	E
1		Data for truss calculation using			
2		finite elements method			
3		x_i	y_i	x_j	y_j
4	1	0	3	4	6
5	2	0	3	4	0
6	3	4	6	8	6
7	4	4	6	4	0
8	5	8	6	4	0

(b)

Fig. 4.4.1

To find the lengths of rod finite elements and the elements angles to the coordinate axes in the EXCEL table (see table fragment on the diagram, Fig. 4.4.1 b). The coordinates of the ends of the rods of flat truss are entered for their further use in calculations by the finite element method with the Mathcad computer algebra system.

The coordinates of the beginning and end of each rod in the Mathcad are shown on a fragment of the working page (Fig. 4.4.2 a). Vector specification of the coordinates of beginningы and the endы of the rods is presented in another fragment in the Mathcad (Fig. 4.4.2 b).

x_i	y_i	x_j	y_j
0	3	4	6
0	3	4	0
4	6	8	6
4	6	4	0
8	6	4	0

(a)

$$x_i = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \\ 8 \end{bmatrix} \quad y_i = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 6 \\ 6 \end{bmatrix} \quad x_j = \begin{bmatrix} 4 \\ 4 \\ 8 \\ 4 \\ 4 \end{bmatrix} \quad y_j = \begin{bmatrix} 6 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

(b)

Fig. 4.4.2

The projections of finite elements – the flat truss of rods on the coordinate axes and lengths of the rods, determined in the Mathcad, are shown on a fragment of the working page (Fig. 4.4.3).

$$x := x_j - x_i \qquad x^{\mathrm{T}} = [4\ \ 4\ \ 4\ \ 0\ \ -4]$$

$$y := y_j - y_i \qquad y^{\mathrm{T}} = [3\ \ -3\ \ 0\ \ -6\ \ -6]$$

$$l := \sqrt{x^2 + y^2} \qquad l^{\mathrm{T}} = [5\ \ 5\ \ 4\ \ 6\ \ 7.211]$$

Fig. 4.4.3

Vectors of geometric characteristics. Vectors in the Mathcad of cosines and sines of angles of the rods with a coordinate axis x determining is shown on a fragment of the working page (Fig. 4.4.4).

$$C := \frac{x}{l} \qquad C^{\mathrm{T}} = [0.8\ \ 0.8\ \ 1\ \ 0\ \ -0.555] \qquad S := \frac{y}{l} \qquad S^{\mathrm{T}} = [0.6\ \ -0.6\ \ 0\ \ -1\ \ -0.832]$$

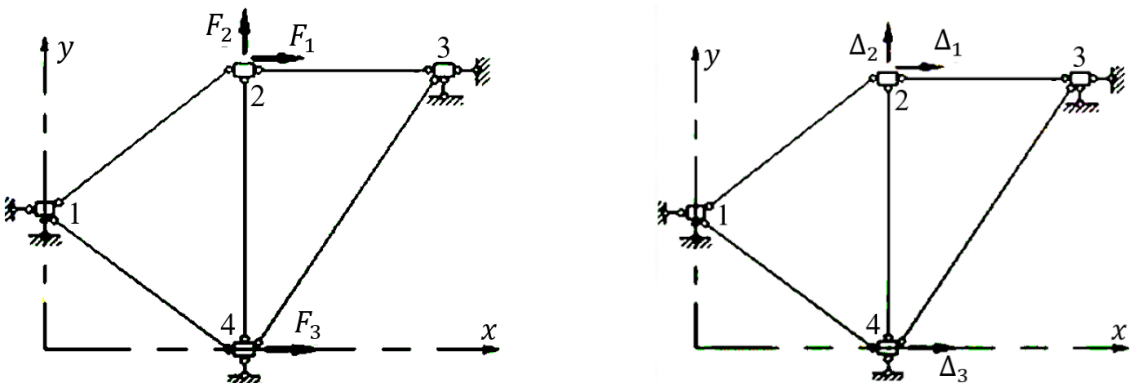
Fig. 4.4.4

The vector of linear stiffnesses f of finite elements – the rods of a flat truss determining in the Mathcad is shown on a fragment of the working page (Fig. 4.4.5).

$$f0 := 1 \qquad f := [2 \cdot f0\ \ 2 \cdot f0\ \ 3 \cdot f0\ \ f0\ \ 2 \cdot f0]^{\mathrm{T}} \qquad f^{\mathrm{T}} = [2\ \ 2\ \ 3\ \ 1\ \ 2]$$

Fig. 4.4.5

Projections of the nodal load vector F (Fig. 4.4.6 a) are sought in the directions of displacements of the discrete model (Δ_1, Δ_2 and Δ_3, Fig. 4.4.6 b) in those nodes in which an external load P is applied, which is found in the Mathcad (the fragment of the page, Fig. 4.4.7).



(a) (b)

Fig. 4.4.6

$$F_1 := P \cdot \cos(\alpha) \qquad F_2 := -P \cdot \sin(\alpha) \qquad F_3 := 0 \qquad F^{\mathrm{T}} = [21.131\ \ -45.315\ \ 0]$$

Fig. 4.4.7

The stiffness matrix of a rod finite element has the general form:

$$[k]_e = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}.$$

For each rod the finite element of a flat mechanical system (the truss or frame), the stiffness matrix $[k]_e$, having dimension: 6×6, formed by proportionality coefficients between each of six displacements: δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6 (three displacements at each end of the rod finite element: two linear and one angular) and the corresponding term of the canonical equation in each of the six forces: r_1, r_2, r_3, r_4, r_5 and r_6, arising in the direction of these displacements.

The total forces caused by each of the displacements, taking into account the rigidity of the rod finite element in the direction of each of the displacements (δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6) are balanced by forces from the attached rod finite elements, and the system of the balance equations of each rod finite element has the form:

$$k_{11} \times \delta_1 + k_{12} \times \delta_2 + k_{13} \times \delta_3 + k_{14} \times \delta_4 + k_{15} \times \delta_5 + k_{16} \times \delta_6 = r_1,$$
$$k_{21} \times \delta_1 + k_{22} \times \delta_2 + k_{23} \times \delta_3 + k_{24} \times \delta_4 + k_{25} \times \delta_5 + k_{26} \times \delta_6 = r_2,$$
$$k_{31} \times \delta_1 + k_{32} \times \delta_2 + k_{33} \times \delta_3 + k_{34} \times \delta_4 + k_{35} \times \delta_5 + k_{36} \times \delta_6 = r_3,$$
$$k_{41} \times \delta_1 + k_{42} \times \delta_2 + k_{43} \times \delta_3 + k_{44} \times \delta_4 + k_{45} \times \delta_5 + k_{46} \times \delta_6 = r_4,$$
$$k_{51} \times \delta_1 + k_{52} \times \delta_2 + k_{53} \times \delta_3 + k_{54} \times \delta_4 + k_{55} \times \delta_5 + k_{56} \times \delta_6 = r_5,$$
$$k_{61} \times \delta_1 + k_{62} \times \delta_2 + k_{63} \times \delta_3 + k_{64} \times \delta_4 + k_{65} \times \delta_5 + k_{66} \times \delta_6 = r_6.$$

Typically, the stiffness matrices of rod finite element are sparse, that is, many of the elements of these matrices are equal to zero. With uniform linear stiffness along the length of the rod finite element, the corresponding elements of the stiffness matrix depend on the type of supports at the ends. One can distinguish four main types of rod finite elements, differing in the type of supports at the ends: a rod finite element clamped in embeddings at both ends (usually designated as a rod finite element of the first type and corresponds mainly to rods in frames with rigid nodes), a rod finite element clamped in the left support and hingedly supported in the right (the second type), a rod finite element clamped in the right support and hingedly supported in the left (the third type) and a rod finite element on hinged supports at both ends – the fourth type element is typical for trusses made of hinged rods, in particular, for considered truss.

Depending on the types of rod finite elements, the coefficients in their stiffness matrices are to be predefined and available for use in the calculation of a mechanical system using the finite element method. In particular, for finite elements of the fourth type (on hinged supports at both ends), in the stiffness matrix in global (the general) coordinate

system, taking into account the inclination β of the rod finite element to the x axis, the coefficients are equal:

$k_{11} = k_{44} = -k_{14} = -k_{41} = f \times (\cos\beta)^2,$

$k_{22} = k_{55} = -k_{25} = -k_{52} = f \times (\sin\beta)^2,$

$k_{12} = k_{21} = k_{45} = k_{54} = -k_{15} = -k_{51} = -k_{24} = -k_{42} = f \times \sin\beta \times \cos\beta,$

$k_{16} = k_{26} = k_{36} = k_{46} = k_{56} = k_{61} = k_{62} = k_{63} = k_{64} = k_{65} = k_{66} =$

$= k_{13} = k_{23} = k_{33} = k_{43} = k_{53} = k_{31} = k_{32} = k_{34} = k_{35} = 0.$

The stiffness matrix of the rod finite element of the fourth type in global (the general) coordinate system on hinged supports at both ends, considering the inclination β of the rod finite element to the axis x is determined in the Mathcad (fragment of page is shown in the diagram, Fig. 4.4.8).

The resulting stiffness matrices of five rod finite elements determining in the Mathcad is shown on a fragment of the working page (Fig. 4.4.9).

$$k(i) := f_i \cdot \begin{bmatrix} C_i^2 & C_i \cdot S_i & 0 & -C_i^2 & -(C_i \cdot S_i) & 0 \\ C_i \cdot S_i & S_i^2 & 0 & -(C_i \cdot S_i) & -S_i^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -C_i^2 & -(C_i \cdot S_i) & 0 & C_i^2 & C_i \cdot S_i & 0 \\ -(C_i \cdot S_i) & -S_i^2 & 0 & C_i \cdot S_i & S_i^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.4.8

$$k(1) = \begin{bmatrix} 1.28 & 0.96 & 0 & -1.28 & -0.96 & 0 \\ 0.96 & 0.72 & 0 & -0.96 & -0.72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.28 & -0.96 & 0 & 1.28 & 0.96 & 0 \\ -0.96 & -0.72 & 0 & 0.96 & 0.72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad k(2) = \begin{bmatrix} 1.28 & -0.96 & 0 & -1.28 & 0.96 & 0 \\ -0.96 & 0.72 & 0 & 0.96 & -0.72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.28 & 0.96 & 0 & 1.28 & -0.96 & 0 \\ 0.96 & -0.72 & 0 & -0.96 & 0.72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k(3) = \begin{bmatrix} 3 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad k(4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k(5) = \begin{bmatrix} 0.615 & 0.923 & 0 & -0.615 & -0.923 & 0 \\ 0.923 & 1.385 & 0 & -0.923 & -1.385 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.615 & -0.923 & 0 & 0.615 & 0.923 & 0 \\ -0.923 & -1.385 & 0 & 0.923 & 1.385 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.4.9

Zero matrix determining in the Mathcad is shown on a fragment of the working page (Fig. 4.4.10).

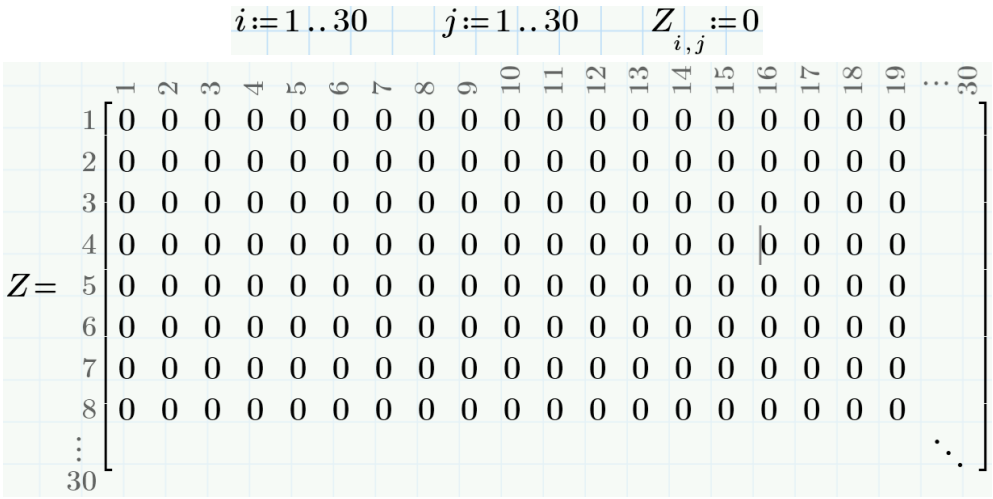


Fig. 4.4.10

The quasi-diagonal stiffness matrix of the discrete model is formed by diagonally filling the generated zero matrix with dimension: 30×30 with stiffness matrices of five rod finite elements with dimension: 6×6 each. The quasi-diagonal matrix determining in the Mathcad is shown on a fragment of the working page (Fig. 4.4.11).

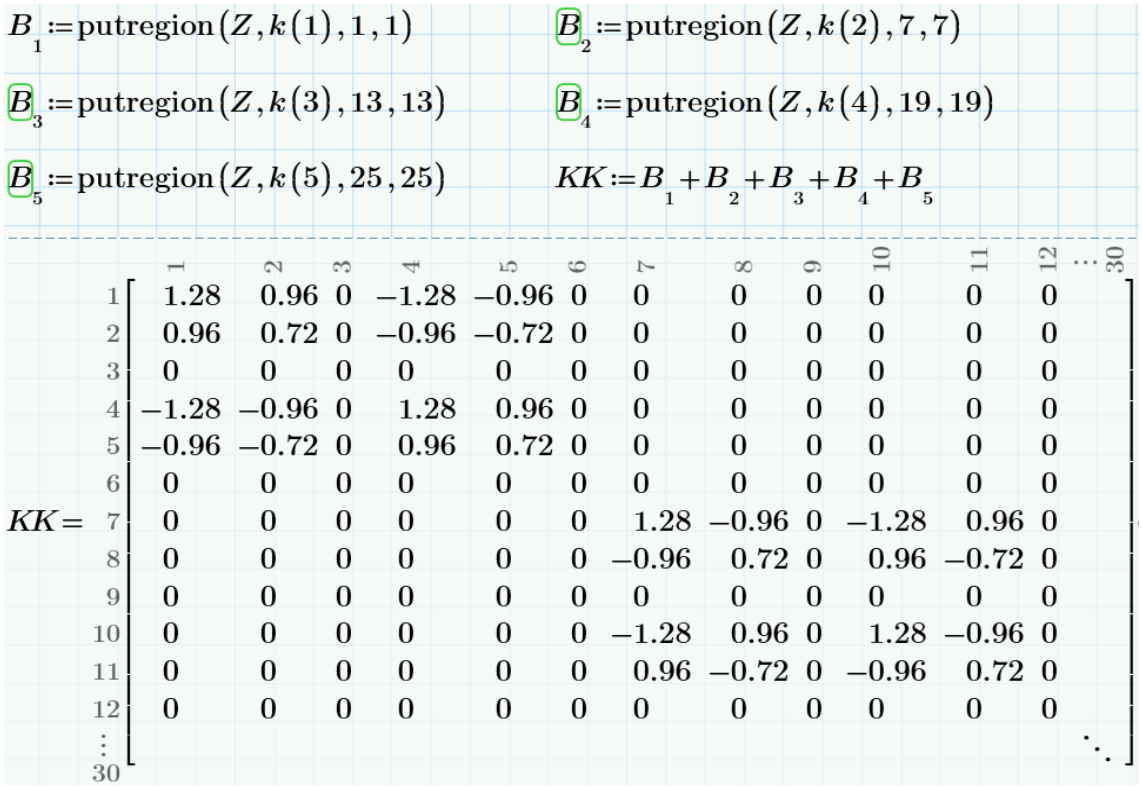


Fig. 4.4.11

Formation of incidence (the adjacency) matrix, which indicates the connections between the incident elements: the displacements of each rod finite element (δ_1, δ_2, δ_3, δ_4,

δ_5 and δ_6, Fig. 4.4.12) and the displacements of the discrete model (Δ_1, Δ_2 and Δ_3, Fig. 4.4.6 b).

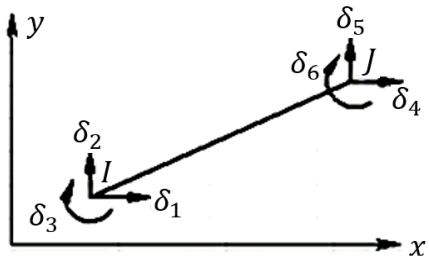


Fig. 4.4.12

A non–zero value of an element of the incidence matrix indicates the relationship between displacements of the rod finite element and the displacements of the discrete model (their incidence). It is more convenient to enter data into the incidence matrix in the EXCEL (see the table fragment on the diagram, Fig. 4.4.13).

	G	H	I	J	K	L
9				Displacements of Discrete Model		
10		Finite Element	Displacement	Δ_1	Δ_2	Δ_3
11	1	1 – 2	δ_1	0	0	0
12	2		δ_2	0	0	0
13	3		δ_3	0	0	0
14	4		δ_4	1	0	0
15	5		δ_5	0	1	0
16	6		δ_6	0	0	0
17	7	1 – 4	δ_1	0	0	0
18	8		δ_2	0	0	0
19	9		δ_3	0	0	0
20	10		δ_4	0	0	1
21	11		δ_5	0	0	0
22	12		δ_6	0	0	0
23	13	2 – 3	δ_1	1	0	0
24	14		δ_2	0	1	0
25	15		δ_3	0	0	0
26	16		δ_4	0	0	0
27	17		δ_5	0	0	0
28	18		δ_6	0	0	0
29	19	2 – 4	δ_1	1	0	0
30	20		δ_2	0	1	0
31	21		δ_3	0	0	0
32	22		δ_4	0	0	1
33	23		δ_5	0	0	0
34	24		δ_6	0	0	0
35	25	3 – 4	δ_1	0	0	0
36	26		δ_2	0	0	0
37	27		δ_3	0	0	0
38	28		δ_4	0	0	1
39	29		δ_5	0	0	0
40	30		δ_6	0	0	0

Fig. 4.4.13

The designations of rod finite elements (1–2, 1–4, 2–3, 2–4 and 3–4) and designations of displacements: δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6 for each rod finite element are entered in the EXCEL table and the incidence values of the displacements of the ends of each of the rod elements and of the displacements of the discrete mode: Δ_1, Δ_2 and Δ_3.

To import data into the Mathcad, the EXCEL file must be in the same folder as the Mathcad file. The formation of the structural matrix (the incident matrix, which is the adjacency of displacements of the discrete model and rod finite elements) is performed in the Mathcad (the fragment of page is shown, Fig. 4.4.14) by transferring data from tables in EXCEL. To transfer data one can use the READEXCEL function, located on the Mathcad panel on the Input/Output tab.

$$I := \text{READEXCEL}(\text{".\Calculation flat truss using Finite element method_ data.xlsx"}, \text{"Лист1!J11:L40"})$$

$$I^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. 4.4.14

The stiffness matrix $[K]$ of the discrete model is determined using the incidence matrix $[I]$ and quasi–diagonal matrix $[KK]$ using the formula: $[K] = [I]^T[KK][I]$ is shown the Mathcad (the fragment of page is shown, Fig. 4.4.15).

$$K := I^{\mathrm{T}} \cdot KK \cdot I \qquad K = \begin{bmatrix} 4.28 & 0.96 & 0 \\ 0.96 & 1.72 & 0 \\ 0 & 0 & 1.895 \end{bmatrix}$$

Fig. 4.4.15

The inverse matrix $[K]^{-1}$ to the stiffness matrix of the discrete model, as well as the matrix of displacements $[\Delta]$ of the discrete model, depending on the nodal load vector F and on the inverse stiffness matrix of the discrete model determining in the Mathcad is shown in the fragment of page (Fig. 4.4.16).

$$K^{-1} = \begin{bmatrix} 0.267 & -0.149 & 0 \\ -0.149 & 0.665 & 0 \\ 0 & 0 & 0.528 \end{bmatrix} \qquad \Delta := K^{-1} \cdot F \qquad \Delta = \begin{bmatrix} 12.399 \\ -33.266 \\ 0 \end{bmatrix}$$

Fig. 4.4.16

There is the proportional relationship between the end forces (the reactions) $[r']_e$, acting on the rod finite element, and end displacements $[\delta']_e$. In the matrix form, one can represent this dependence as: $[r']_e = [k']_e[\delta']_e$.

The stiffness matrix $[k']_e$ of a rod finite element in a local coordinate system has the form:

$$[k']_e = \begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} & k'_{15} & k'_{16} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} & k'_{25} & k'_{26} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} & k'_{35} & k'_{36} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} & k'_{45} & k'_{46} \\ k'_{51} & k'_{52} & k'_{53} & k'_{54} & k'_{55} & k'_{56} \\ k'_{61} & k'_{62} & k'_{63} & k'_{64} & k'_{65} & k'_{66} \end{bmatrix}.$$

End forces $[s']_e$ in the local coordinate system are the sum of end reactions $[r']_e$, caused by the displacements of nodes of the discrete model and of the end forces $[p']_e$, arising from the influence of a load distributed along the rod: $[s']_e = [r']_e + [p']_e$.

Because there is no load acting on the rods of assigned flat truss (except for the load on the nodes), then: $[p']_e = 0$, and the end forces are equal: $[s']_e = [r']_e$.

Depending on the displacements $[\delta']_e$ the end force vectors $[s']_e$ in rod finite elements one can find, as: $[s']_e = [h]_e[\delta]_e$, where $[h]_e$ – the matrix converted from the stiffness matrix $[k']_e$ of the rod finite element in the local coordinate system to the global coordinate system: $[h]_e = [k']_e[T]_e$, where $[T]_e$ – the transformation matrix of finite elements in the local coordinate system to the global system, when the displacements of finite elements $[\delta']_e$ are determined by the displacements of the discrete model (Δ_1, Δ_2 and Δ_3).

The matrix $[h]_e$, that is the stiffness matrix $[k']_e$ of rod finite element in the local coordinate system, converted to the global coordinate system, is determined by multiplying the finite element stiffness matrix $[k']_e$ and the matrix $[T]_e$ of transformation of finite elements in the local coordinate system to the global system (the fragment of the page in the Mathcad, Fig. 4.4.17).

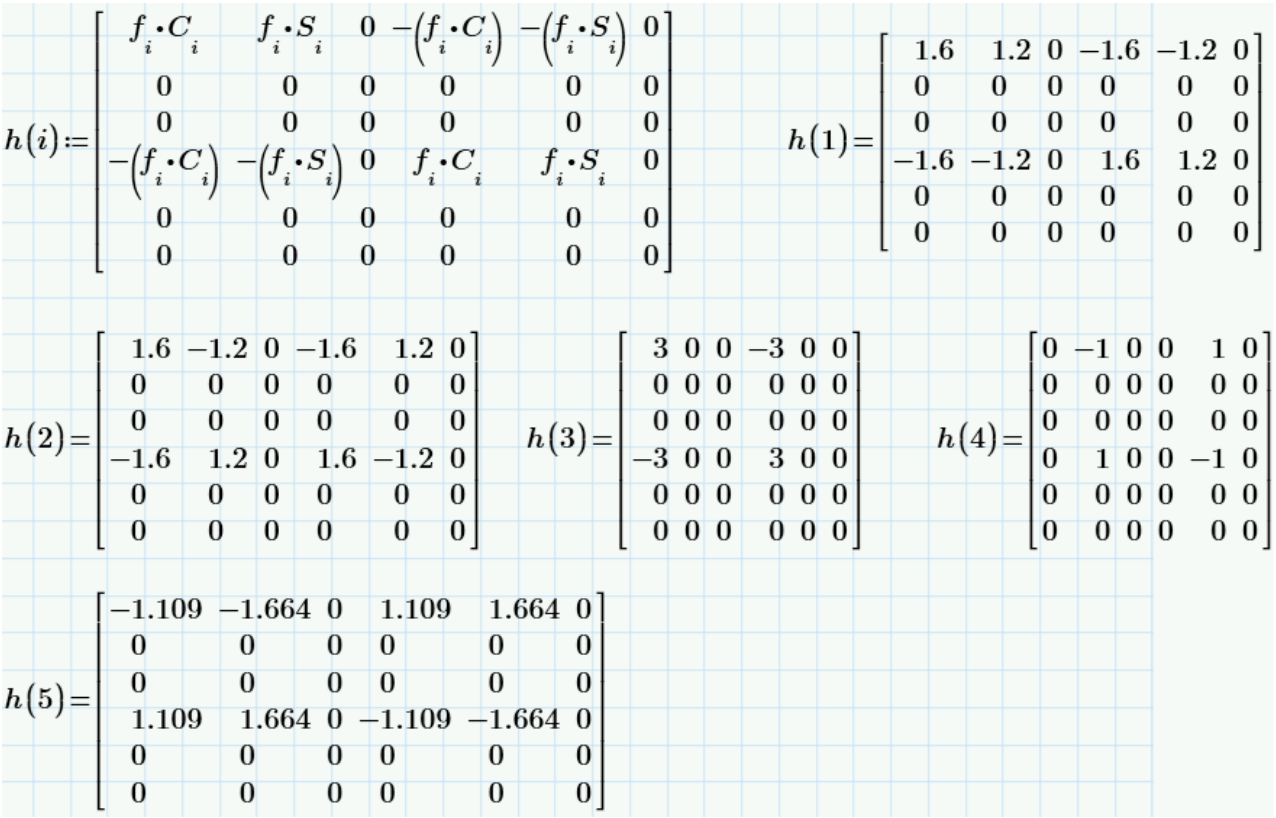


Fig. 4.4.17

The vectors of end displacements $[\delta']_e$ of rod finite elements are determined with found displacements of the discrete model in the Mathcad (fragment of page is shown in the diagram, Fig. 4.4.18).

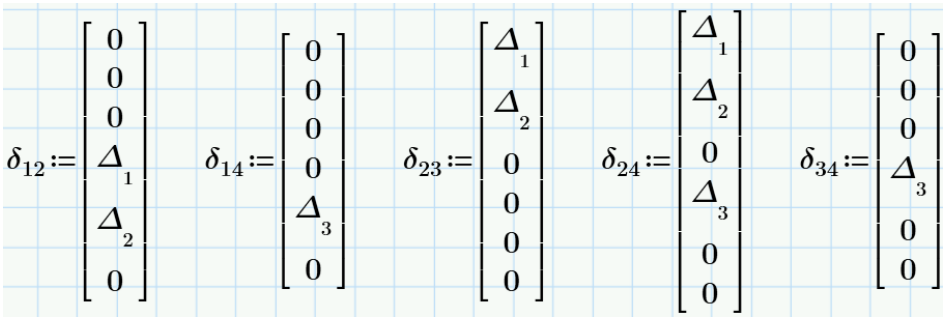


Fig. 4.4.18

The vectors of forces $[s']_e$ in rod finite elements are determined in the Mathcad (the fragment of page is shown in the diagram, Fig. 4.4.19).

$$S_{12} := h(1) \cdot \delta_{12} \quad S_{14} := h(2) \cdot \delta_{14} \quad S_{23} := h(3) \cdot \delta_{23} \quad S_{24} := h(4) \cdot \delta_{24} \quad S_{34} := h(5) \cdot \delta_{34}$$

$$S_{12} = \begin{bmatrix} 20.082 \\ 0 \\ 0 \\ -20.082 \\ 0 \\ 0 \end{bmatrix} \quad S_{14} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_{23} = \begin{bmatrix} 37.196 \\ 0 \\ 0 \\ -37.196 \\ 0 \\ 0 \end{bmatrix} \quad S_{24} = \begin{bmatrix} 33.266 \\ 0 \\ 0 \\ -33.266 \\ 0 \\ 0 \end{bmatrix} \quad S_{34} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Fig. 4.4.19

The result of calculating longitudinal forces in truss rods using the finite element method in the Mathcad is shown in the fragment of the page (Fig. 4.4.20).

FE	*N* (kN)
1 – 2	−20.082
1 – 4	0
2 – 3	−37.196
2 – 4	−33.266
3 – 4	0

Fig. 4.4.20

Questions and comments:

4.4.1. The typical sequence for finite element calculations of the rod structures.

4.4.2. The sequence of the assigning numbers to finite elements and nodes of a mechanical system.

4.4.3. Determining in the Mathcad of lengths of the finite elements by the coordinates of their ends (the connection points with supports or other finite elements), used in the form of vectors generated from a coordinate table exported from the EXCEL program, and finding the vector of projections of finite elements on the coordinate axes.

4.4.4. Determination in vector form in the Mathcad of cosines and sines of direction (the inclination) angles of finite elements with respect to coordinate axes.

4.4.5. Specifying the stiffness vector of the finite elements of a mechanical system in the Mathcad.

4.4.6. Finding in the Mathcad of the nodal load vector projections along the directions of displacement of the discrete model in those nodes at which an external load is applied.

4.4.7. The system of balance equations for a rod finite element, the general form and dimension of its stiffness matrix.

4.4.8. Stiffness matrix of a finite element rod in the Mathcad, determining the values of the elements of the stiffness matrices of finite element rods.

4.4.9. Sparse matrices of the rod finite elements stiffness.

4.4.10. An example of determining the coefficients of the stiffness matrix of a rod finite element of the fourth type (the rod on hinged supports at both ends) in global (the general) coordinate system, taking into account its inclination to the abscissa axis.

4.4.11. Forming in the Mathcad of a zero matrix, the dimension of which corresponds to the dimension of discrete model of rod finite elements of a mechanical system.

4.4.12. The creation of a quasi–diagonal stiffness matrix of discrete model in the Mathcad using a zero matrix.

4.4.13. The formation of an incidence (the adjacency) matrix, which indicates the connections between the incident elements: the displacements of the rod finite element and displacements of the discrete model, using the EXCEL program and importing data from EXCEL into the Mathcad.

4.4.14. Determination in the Mathcad of the stiffness matrix of discrete model of a mechanical system using incidence matrix and the quasi–diagonal matrix.

4.4.15. Finding in the Mathcad of inverse matrix to the stiffness matrix of the discrete model of mechanical system.

4.4.16. Determination in the Mathcad of the vector of end forces (the reactions) acting on a rod finite element.

4.4.17. Converting the stiffness matrix of a bar finite element from a local coordinate system to a global system.

4.4.18. Finding the vectors of end displacements of rod finite elements in the Mathcad using the found displacements of discrete model.

4.4.19. Determination in the Mathcad of vectors of end forces in finite element rods and entering into the resulting table the values of longitudinal forces in the truss rods found by the finite element method.

4.5. The concept of finite element method in calculations of mechanical systems

The finite element method in the issues of calculating mechanical systems when determining the forces, displacements, stresses and deformations is a mathematical form of combining into the discrete model of the standard solutions available for individual parts of the mechanical system, that is for of it finite elements, or previously found solutions. This allows one to formalize the problem of calculating the mechanical system, taking into account the fact that the finite elements themselves are quite simple and there are ready-made solutions for them and they are to be combined in a discrete model.

Combining finite elements into a discrete model is a formally logical approach, that is the mathematical one within the framework of vector algebra, while ready–made solutions used for finite elements have a physical (the mechanical) nature, for example, pre–found elastic properties of finite elements, that is the results of bringing loads and other mechanical properties to nodes (to the ends) of the finite elements, considered separately, taking into account the rigidity of the finite elements and the type of their supports and embeddings of nodes.

The result of reducing loads and other mechanical properties to nodes has a vector form in commonly used local coordinates associated with the finite element, and the vector algebraic combination of finite elements into discrete model of the mechanical system involves the transition from the local coordinate systems associated with each of the finite elements, in which the characteristics of these finite elements are determined in vector form, to the global coordinate system associated with the integral discrete model of the mechanical system, that is applied to a geometric transformation from local coordinate systems to the global coordinate system.

To combine into a single discrete model the solutions available for individual parts of the mechanical system – the finite elements, the distributed load acting on the mechanical system and its finite elements is replaced by the nodal load, and the discrete model is characterized by the connections of finite elements at nodes.

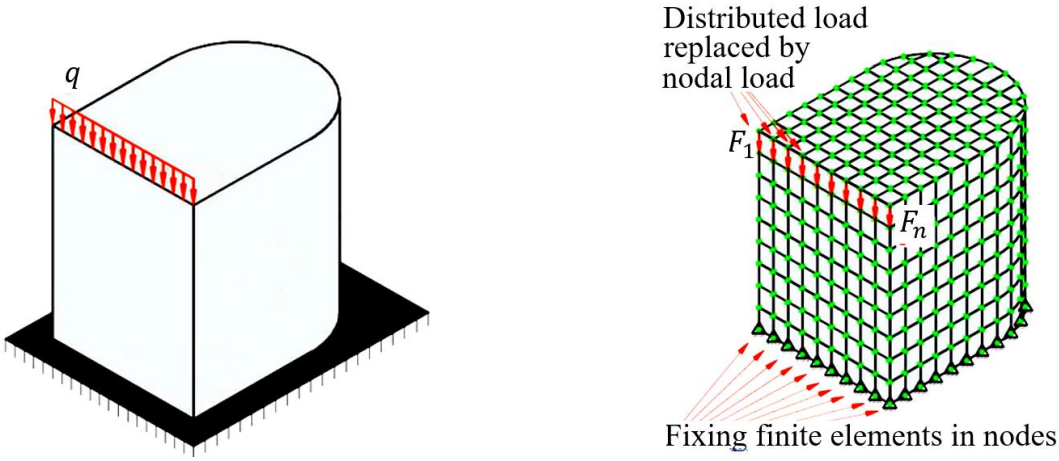
The finite elements themselves are characterized by certain information, both the topological and the information about which nodes belong to the finite element, what is the elasticity of the finite element, what is the cross–section shape, the thickness, the mass, etc.

When performing calculations, the information associated with the finite elements usually has a vector form, and the information about combination of finite elements in a mechanical system, presented in a discrete model, has a matrix form consisting of vectors. The elements of these vectors and matrices can be not only digital, but also the functional.

As part of the finite element method, which is intended for calculating physical systems, in particular mechanical ones, the algorithmic functions are separated.

The form of data for use in calculations of mechanical systems by finite element method includes one or another representation of the mechanical system: a drawing, a 3–D model, the actual physical structure, etc.

The finite element method is sometimes considered as a form of numerical integration when replacing continuous mechanical models with a finite element network, which is at discretizing a continuous region of a body with elements of finite dimensions (Fig. 4.5.1), for which theoretical solutions are known, formally (algebraically) combined in a mechanical system using the discrete model.

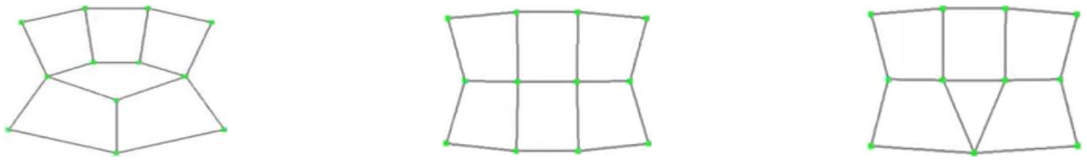


(a) (b)

Fig. 4.5.1

The sizes of finite elements when used for numerical integration correspond to it accuracy. Although, the numerical integration methods are usually associated with specifying the integration step or with determining the integration step, if the magnitude of the error is specified. At the same time, finding the finite elements (which is the determination in vector form of the properties of the finite elements) does not in itself include an indication of the numerical integration step or an indication of the error assigned. If the finite element is a typical repeating object, from the total set of these elements into which the continuous mechanical model is divided, then the definitions of the finite element and the discrete model connecting the finite elements can serve as an algorithm inside the numerical integration cycle. That is, the use of the finite element method in itself is not methodically the numerical integration.

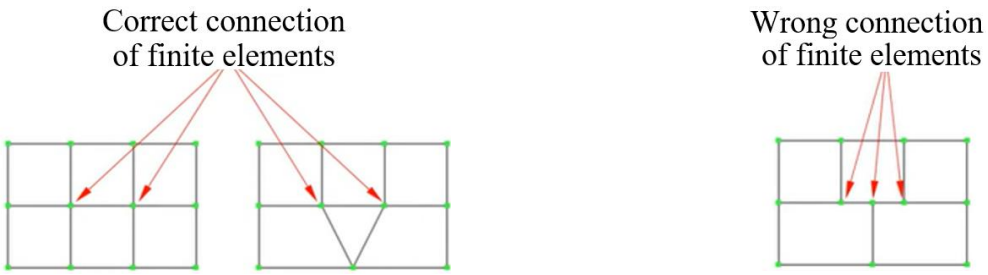
Finite elements in a discrete model of the mechanical system can only be connected at nodes; the examples of finite elements connecting are shown on the schemes (Fig. 4.5.2 a, b and c).



(a) (b) (c)

Fig. 4.5.2

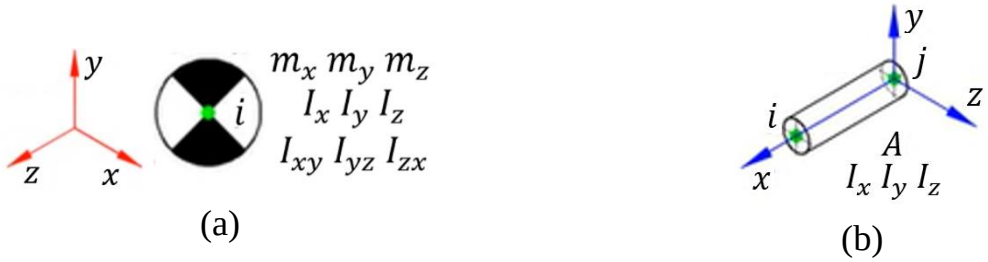
The diagram below shows in comparison of the examples of correct (Fig. 4.5.3 a) and incorrect combination of finite elements in a discrete model (Fig. 4.5.3 b).



(a) (b)

Fig. 4.5.3

Frequently used types of finite elements are shown on schemes. Zero–dimensional single–node finite element in the global coordinate system (with axes: x, y and z) – the object of concentrated mass (Fig. 4.5.4 a), more accurately, the concentrated inertia along the axes (m_x, m_y and m_z), as well as the angular inertia (I_x, I_y, I_z, I_{xy}, I_{yz} and I_{zx}). The number of degrees of freedom of a zero–dimensional single–node finite element associated with the structure of discrete model ranges from 3 to 6.

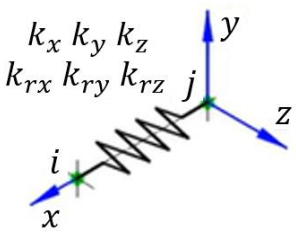


(a) (b)

Fig. 4.5.4

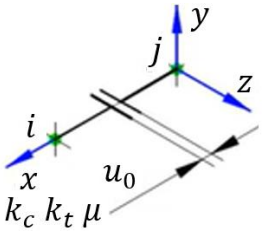
A one–dimensional two–node finite element: the rod, shaft, beam, the combined beam is shown on the diagram (Fig. 4.5.4 b) in the local coordinate system, associated with this finite element. The number of degrees of freedom of a one–dimensional finite element ranges from 2 to 12.

Pseudo–one–dimensional 2–node finite elements: the spring (Fig. 4.5.5 a) and contact element (Fig. 4.5.5 b), in contrast to the one–dimensional two–node finite element, have additional degrees of freedom, for a spring: the linear characteristics (k_x, k_y and k_z) and angular (k_{rx}, k_{ry} and k_{rz}) stiffness, for the contact element: the characteristics of compressive stiffness k_c, the tensile stiffness k_t, the friction μ and the initial gap u_0. The number of degrees of freedom of a pseudo–one–dimensional finite element ranges from 2 to 12.



Spring

(a)

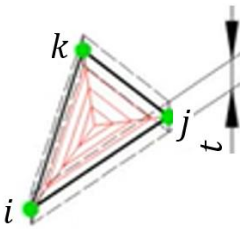


Contact element

(b)

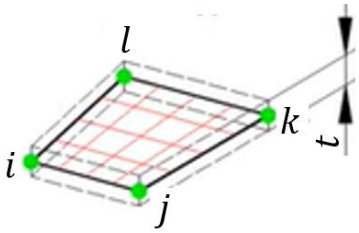
Fig. 4.5.5

Two-dimensional three– and four–node finite elements (Fig. 4.5.6 a and b), for example, a plate and a membrane (at zero bending stiffness), are characterized by a number of degrees of freedom from 6 to 24. Two–dimensional finite elements are specified in their local coordinate systems.



Plate

(a)

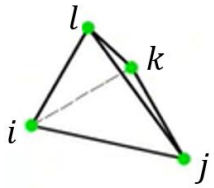


Membrane

(b)

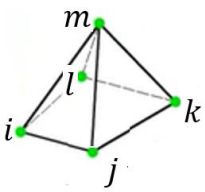
Fig. 4.5.6

Three–dimensional 4–th, 5–th, 6–th and 8–node finite elements, for example: tetrahedron, pyramid, bar and hexahedron (Fig. 4.5.7), have from 12 to 24 degrees of freedom. The three–dimensional finite elements in mechanics usually do not have rotational degrees of freedom.



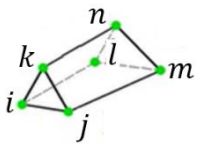
Tetrahedron

(a)



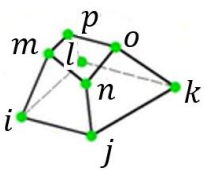
Pyramid

(b)



Bar

(c)

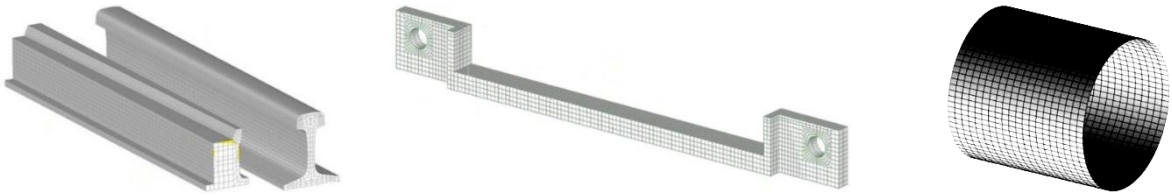


Hexahedron

(d)

Fig. 4.5.7

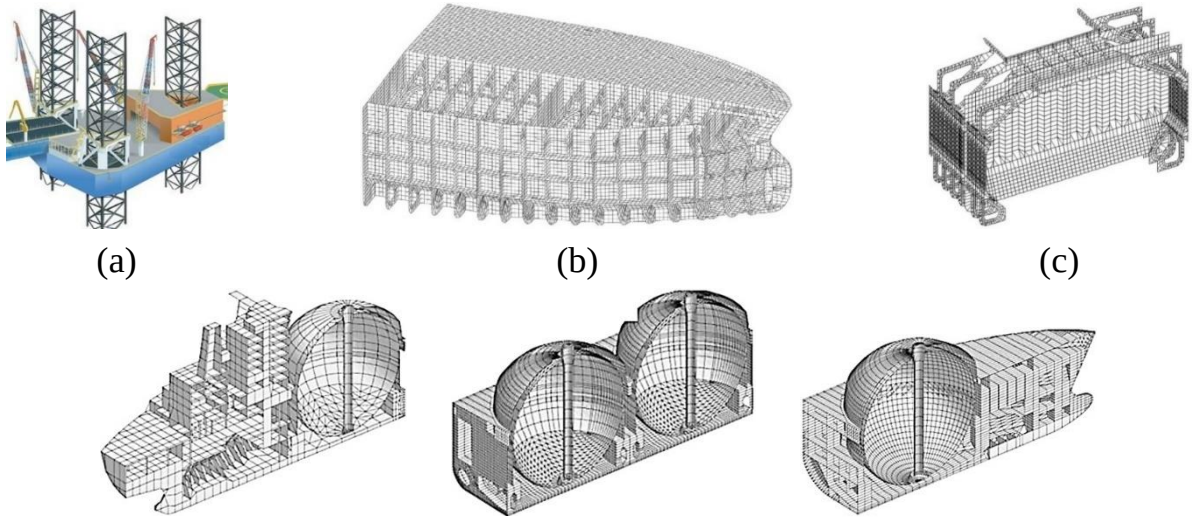
The examples of three–dimensional and surface structures for the calculation of stresses, strains and the displacements, for which the finite element method can be used (Fig. 4.5.8 a, b and c).



(a) (b) (c)

Fig. 4.5.8

An example of rod and beam structures, loads on elements that are applied at the ends (Fig. 4.5.9 a), which is the rods, for which the finite element method can be used, which is convenient in calculations, especially if the rigidity of the nodes of their connection is one can neglect, as well as examples of combined discrete models of marine structures are shown (Fig. 4.5.9 b, c and d, https://slideplayer.com/slide/754595/, DNV) which include one–dimensional finite elements and two–dimensional, that is, surface elements.



(a) (b) (c)

(d)

Fig. 4.5.9

Questions and comments:

4.5.1. The finite element method for calculating of mechanical systems and determining the forces, displacements, stresses and deformations as a mathematical form of combining into a discrete model of the standard solutions available for finite elements – the parts of mechanical system.

4.5.2. The use of local coordinate systems to describe the mathematical properties of individual finite elements and global coordinate system to find the properties of a discrete model of mechanical system.

4.5.3. Replacing of distributed load acting on a mechanical system and its finite elements with a nodal load acting on a discrete model of a mechanical system.

4.5.4. The vector form of information, associated with each finite element (about which nodes belong to the finite element, what is the elasticity of the finite element, what is the cross–section shape, thickness, mass, etc.), and a matrix form of information about the combination of finite elements in a mechanical system, presented in discrete model.

4.5.5. The numerical and functional elements of finite element vectors and matrices of discrete model.

4.5.6. The finite element method as a form of numerical integration when replacing the entire mechanical models with a finite element network.

4.5.7. The accuracy of numerical integration, depending on the assigned integration step, and the error of the finite element method, associated with the finite element size.

4.5.8. Examples of connecting finite elements in nodes. Correct and incorrect structures of finite elements in a discrete model.

4.5.9. Commonly used types of the finite elements. Dimension and number of the finite element nodes.

4.5.10. Examples of three–dimensional and surface structures which can be used by the finite element method for calculating stresses, deformations and displacements.

4.5.11. Examples of rod and beam structures, the loads to elements that are applied at the ends, for which the finite element method can be used, which are rods.

4.5.12. The examples of combined structures including one–dimensional finite elements and two–dimensional, that is, surface elements.

4.6. Determination of the stiffness matrix of a mechanical system when using the finite element method in the EXCEL

Exercise. Calculate statically definable flat truss using the finite element method in the EXCEL. Find displacements of the truss nodes and forces in truss rod elements.

The initial data are specified on the diagram of flat truss (Fig. 4.6.1) and indicated in the Table on fragment of the EXCEL page (Fig. 4.6.2).

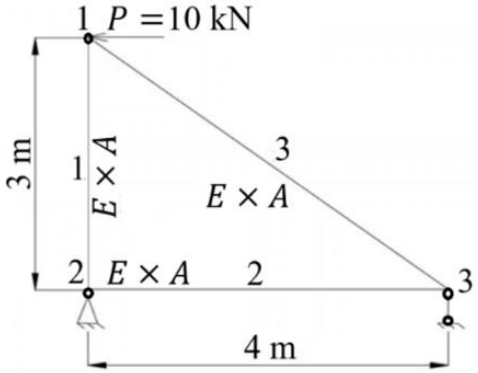


Fig. 4.6.1

	A	B	C	D	E	F	G	H	I	J	K	L
1	Element # i	Nodes coordinates						l, m	$c = \cos\alpha$	$s = \sin\alpha$	E, MPa	A , sq. m
2		First end			Second end							
3		Node #	x_1, m	y_1, m	Node #	x_2, m	y_2, m					
4	1	1	0,0	3,0	2	0,0	0,0	3,0	0,00	-1,00	210 000	0,0025
5	2	2	0,0	0,0	3	4,0	0,0	4,0	1,00	0,00	210 000	0,0025
6	3	1	0,0	3,0	3	4,0	0,0	5,0	0,80	-0,60	210 000	0,0025

Fig. 4.6.2

Node 1 is affected by the force: $P = -10$ kN. The modulus of elasticity E of the material (the carbon steel) is taken equal to 210 000 MPa. The cross–sectional area each of all rod elements is 0.0025 sq. m.

On the right side of the Table in the EXCEL page (Fig. 4.6.2) the values of the length l of the hinged and numbered rod elements of the truss, the direction cosines ($c = \cos\alpha$) and sines ($s = \sin\alpha$) of the sections of truss are determined in the global coordinate system, the characteristics of their linear stiffness ($E \times A$) are determined according to the elastic modulus E of the material and cross–sectional area A.

When filling out the right side of the Table shown on fragment of the EXCEL page (Fig. 4.6.2) to find the lengths of the rod elements l_i and direction cosines (c_i and s_i) taking into account the projections of the rods on the coordinate axes: $(x_{2i} - x_{1i})$ and $(y_{2i} - y_{1i})$ well–known formulas are used: $l_i = \sqrt{(x_{2i} - x_{1i})^2 + (y_{2i} - y_{1i})^2}$, $c_i = (x_{2i} - x_{1i})/l_i$ and $s_i = (y_{2i} - y_{1i})/l_i$.

The rod element 1–2 (the nodes numbers). The stiffness matrix of i–rod finite element of a flat truss in the local coordinate system (the local stiffness matrix) for an element connected by hinges and experiencing a longitudinal load, has the form:

$$r_i = \begin{bmatrix} \frac{E_i \times A_i}{l_i} & -\frac{E_i \times A_i}{l_i} \\ -\frac{E_i \times A_i}{l_i} & \frac{E_i \times A_i}{l_i} \end{bmatrix} = \frac{E_i \times A_i}{l_i} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and the equation relating the acting forces (P_{1i} and P_{2i}) and the resulting deformations (d_{1i} and d_{2i}), one can represent in matrix form:

$$\frac{E_i \times A_i}{l_i} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} d_{1i} \\ d_{2i} \end{bmatrix} = \begin{bmatrix} P_{1i} \\ P_{2i} \end{bmatrix}.$$

Local matrix of element stiffness 1–2 is shown in the Table on fragment of the EXCEL page (Fig. 4.6.3 a).

	M	N
10	175	-175
11	-175	175

(a)

	M	N	O	P
15	0.00	-1.00	0.00	0.00
16	0.00	0.00	0.00	-1.00

(b)

Fig. 4.6.3

Deformation of the i–th bar finite element equals: $\varepsilon_i = (d_{2i} - d_{1i})/l_i$.

The transition from the global coordinate system (u_1, v_1, u_2 and v_2) to the local one, associated with the bar finite element, to find the displacements of the ends of the finite element (d_1 and d_2) and the deformation of the finite element is made by redefining the displacements taking into account the direction cosines of the local coordinate system in the global coordinate system:

since $u_1 \times c + v_1 \times s = d_1 \times \cos\alpha \times \cos\alpha + d_1 \times \sin\alpha \times \sin\alpha =$

$= d_1 \times ((\cos\alpha)^2 + (\sin\alpha)^2) = d_1$, so $d_1 = u_1 \times c + v_1 \times s$ and $d_2 = u_2 \times c + v_2 \times s$.

Transition on a plane from a global coordinate system to a local one for the i–th rod finite element in matrix notation has the form:

$\begin{bmatrix} d_{1i} \\ d_{2i} \end{bmatrix} = \begin{bmatrix} c_i & s_i & 0 & 0 \\ 0 & 0 & c_i & s_i \end{bmatrix} \times \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$, where: $T = \begin{bmatrix} c_i & s_i & 0 & 0 \\ 0 & 0 & c_i & s_i \end{bmatrix}$ – transition matrix (i.e. the cosine matrix, where: $c_i = \cos(\beta_i)$ and $s_i = \sin(\beta_i)$) of i–th bar finite element.

The cosine matrix of element 1–2 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.3 b).

To transpose the cosine matrix for element 1–2 on the EXCEL page, one needs to select the placement area taking into account the dimensions of the transposed matrix, enter a command in the address bar indicating the ranges of cells of the original matrix (Fig. 4.6.3 b): = TRANSPOSE(M15:P16) and use a combination of keystrokes to execute the transpose command: control shift enter. Product of the transposed cosine matrix and the local stiffness matrix is: $\begin{bmatrix} c_i & 0 \\ s_i & 0 \\ 0 & c_i \\ 0 & s_i \end{bmatrix} \times \begin{bmatrix} \frac{E_i \times A_i}{l_i} & -\frac{E_i \times A_i}{l_i} \\ -\frac{E_i \times A_i}{l_i} & \frac{E_i \times A_i}{l_i} \end{bmatrix}$ for the element 1–2 shown in the Table on a fragment of the EXCEL page (Fig. 4.6.4 a).

	M	N
27	0	0
28	-175	175
29	0	0
30	175	-175

(a)

	M	N	O	P
34	0	0	0	0
35	0	175	0	-175
36	0	0	0	0
37	0	-175	0	175

(b)

Fig. 4.6.4

To multiply the transposed cosine matrix by the local stiffness matrix of the rod finite element 1–2 one needs to select the area where the product of matrices is placed, taking into account the dimension of their product, enter a command in the address bar indicating the ranges of cells of the original matrices, observing their sequence in the

product: = MMULT(M20:N23;M10:N11) and use the following keystrokes to execute the matrix multiplication command: control shift enter.

The global stiffness matrix for the i–th bar finite element of the truss corresponds to the multiplication of the transposed cosine matrix by the local stiffness matrix of the element and by the original cosine matrix – in the specified sequence:

$$\begin{bmatrix} c_i & 0 \\ s_i & 0 \\ 0 & c_i \\ 0 & s_i \end{bmatrix} \times \begin{bmatrix} \frac{E_i \times A_i}{l_i} & -\frac{E_i \times A_i}{l_i} \\ -\frac{E_i \times A_i}{l_i} & \frac{E_i \times A_i}{l_i} \end{bmatrix} \times \begin{bmatrix} c_i & s_i & 0 & 0 \\ 0 & 0 & c_i & s_i \end{bmatrix},$$ that is: $T^T \times r \times T$ – the transformation of the stiffness matrix of a finite element of truss from the local coordinate system to a global one, where: T^T – the transposed cosine matrix.

The global stiffness matrix of the rod finite element 1–2 is determined using the action: = MMULT(M27:N30;M15:P16), the result is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.4 b). To execute the matrix multiplication command in EXCEL one needs to use a combination of keystrokes: control shift enter.

Taking into account the number of nodes of a flat truss – 3 and considering the number of degrees of freedom of each node, equal to 2, the dimension of the stiffness matrix of an integral flat truss is equal to: 6 × 6.

The location of the stiffness matrix of the finite element 1–2 in the stiffness matrix of the entire flat truss, which is determined by the node numbers, is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.5).

	U1	V1	U2	V2	U3	V3
U1	0	0	0	0		
V1	0	175	0	-175		
U2	0	0	0	0		
V2	0	-175	0	175		
U3						
V3						

Fig. 4.6.5

Local stiffness matrix of element 2–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.6 a).

	R	S
10	131.25	-131.25
11	-131.25	131.25

(a)

	R	S	T	U
15	1.00	0.00	0.00	0.00
16	0.00	0.00	1.00	0.00

(b)

Fig. 4.6.6

The product of the transposed cosine matrix and the local stiffness matrix for element 2–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.7 a).

	R	S
27	131.25	-131.25
28	0	0
29	-131.25	131.25
30	0	0

(a)

	R	S	T	U
34	131.25	0	-131.25	0
35	0	0	0	0
36	-131.25	0	131.25	0
37	0	0	0	0

(b)

Fig. 4.6.7

The global matrix of stiffness of the finite element 2–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.7 b).

The location of the stiffness matrix of the rod finite element 2–3 in the stiffness matrix of the entire flat truss, which is determined by the node numbers, is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.8).

	U1	V1	U2	V2	U3	V3
U1						
V1						
U2			131.25	0	-131.25	0
V2			0	0	0	0
U3			-131.25	0	131.25	0
V3			0	0	0	0

Fig. 4.6.8

The local stiffness matrix of element 1–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.9 a).

	W	X
10	105	-105
11	-105	105

(a)

	W	X	Y	Z
15	0.80	-0.60	0.00	0.00
16	0.00	0.00	0.80	-0.60

(b)

Fig. 4.6.9

The product of the transposed cosine matrix and the local stiffness matrix for element 1–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.10 a).

	W	X
27	84	-84
28	-63	63
29	-84	84
30	63	-63

(a)

	W	X	Y	Z
34	67.2	-50.4	-67.2	50.4
35	-50.4	37.8	50.4	-37.8
36	-67.2	50.4	67.2	-50.4
37	50.4	-37.8	-50.4	37.8

(b)

Fig. 4.6.10

The global matrix of stiffness of the rod finite element 1–3 is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.10 b).

The location of the stiffness matrix of the rod finite element 1–3 in the stiffness matrix of the entire flat truss, which is determined by the numbers of the nodes, is shown in the Table on a fragment of the EXCEL page (Fig. 4.6.11).

	U1	V1	U2	V2	U3	V3
U1	67.2	-50.4			-67.2	50.4
V1	-50.4	37.8			50.4	-37.8
U2						
V2						
U3	-67.2	50.4			67.2	-50.4
V3	50.4	-37.8			-50.4	37.8

Fig. 4.6.11

The total stiffness matrix of a flat truss is determined by adding the stiffness matrices of the finite element bars in the global coordinates of the flat truss, which is shown in the Table on the EXCEL page fragment (Fig. 4.6.12). To execute the matrix addition command, select on the EXCEL page the area for inserting the resulting stiffness matrix of the flat truss, highlighting the added matrices on the page as terms when typing the sum in the address bar: =M42:R47+R50:W55+W58:AB63, one needs to use keystroke: control shift enter to perform matrix addition.

	U1	V1	U2	V2	U3	V3
U1	67.2	-50.4	0	0	-67.2	50.4
V1	-50.4	212.8	0	-175	50.4	-37.8
U2	0	0	131.25	0	-131.25	0
V2	0	-175	0	175	0	0
U3	-67.2	50.4	-131.25	0	198.45	-50.4
V3	50.4	-37.8	0	0	-50.4	37.8

Fig. 4.6.12

When determining the displacements of nodes of a bar plane truss (Fig. 4.6.1) one needs to take into account that two nodes of the truss are fixed: the second node is fixed in two directions (u and v) of the global coordinate system, and the third node is fixed in one direction v. To take into account the fixation of the specified nodes, the values of the diagonal stiffness matrix are set equal to one for displacements: u_2, v_2 and v_3, and the remaining values of the elements of the rigidity matrix of the flat truss in the rows and columns: u_2, v_2 and v_3 are assigned equal to zero. With the elements in the rows and columns corresponding to the fixed nodes adjusted the stiffness matrix r of the flat frame is shown in the Table on the page fragment (Fig. 4.6.13).

	U1	V1	U2	V2	U3	V3
U1	67.2	-50.4	0	0	-67.2	0
V1	-50.4	212.8	0	0	50.4	0
U2	0	0	1	0	0	0
V2	0	0	0	1	0	0
U3	-67.2	50.4	0	0	198.45	0
V3	0	0	0	0	0	1

Fig. 4.6.13

Since in the global coordinate system the product of the stiffness matrix of the mechanical system by the vector of displacements is equal to the vector of effective loads, then to find the displacements it is required the matrix r^{-1}– the inverse of the stiffness matrix r to multiply by specified vector of loads.

In the EXCEL program, the inverse matrix with respect to the rigidity matrix of a flat frame is searched using the function: = MINVERSE(…), to execute the command to find the inverse matrix in EXCEL one needs to use a combination of keystrokes: control shift enter. The inverse stiffness matrix is shown in the Table on fragment of the EXCEL page (Fig. 4.6.14).

	U1	V1	U2	V2	U3	V3
U1	0.02571	0.00429	0	0	0.00762	0
V1	0.00429	0.00571	0	0	0	0
U2	0	0	1	0	0	0
V2	0	0	0	1	0	0
U3	0.00762	0	0	0	0.00762	0
V3	0	0	0	0	0	1

Fig. 4.6.14

The load vector (Fig. 4.6.15 a) and vector of displacements (Fig. 4.6.15 b) are shown in the Table on fragment of the EXCEL page.

U1	-10
V1	0
U2	0
V2	0
U3	0
V3	0

(a)

U1	-0.25714
V1	-0.04286
U2	0
V2	0
U3	-0.07619
V3	0

(b)

Fig. 4.6.15

To move from a global system to a local one to find displacements in coordinate systems associated with finite bar elements, the above matrix formula is used:

$$\begin{bmatrix} d_{1i} \\ d_{2i} \end{bmatrix} = \begin{bmatrix} c_i & s_i & 0 & 0 \\ 0 & 0 & c_i & s_i \end{bmatrix} \times \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}.$$

Finding local displacements from global ones is shown in the Table on fragment of the EXCEL page (Fig. 4.6.17). An area is selected on the working page for the result of the product of matrices, the matrix of cosines for the corresponding rod finite element is multiplied by the matrix of displacements of the finite element nodes found in the global coordinate system, the action is applied: = MMULT(N15:Q16;P107:P110), to execute the command to multiply matrix by vector in EXCEL one needs to use a combination of key-strokes: control shift enter.

	N	O	P	Q	R
106	Finite element ends		1-2	2-3	1-3
107	First end	U	-0.2571	0	-0.2571
108		V	-0.0429	0	-0.0429
109	Second	U	0	-0.0762	-0.0762
110	end	V	0	0	0
111	Local	d1	0.04286	0	-0.18
112	displac.	d2	0	-0.0762	-0.061

(a)

	N	O	P	Q
115	Deformations		Stresses	Forces
116	1-2	-0,0143	-3000	-7,5
117	2-3	-0,019	-4000	-10
118	1-3	0,02381	5000	12,5

(b)

Fig. 4.6.17

Deformation of the i–th bar finite element equals: $\varepsilon_i = (d_{2i} - d_{1i})/l_i$.

Finding strains (m) and stresses (kPa) is shown in the Table on fragment of the EXCEL page (Fig. 4.6.17 b).

Questions and comments:

4.6.1. The required initial data for determining the displacements of truss nodes and forces in the rod elements of the truss using the finite element method in the EXCEL program.

4.6.2. Determination in the EXCEL table of the length values of hinged and numbered truss rod elements, the direction cosines and sines of truss sections in the global coordinate system, their linear stiffness characteristics.

4.6.3. Stiffness matrix of a finite element rod in a local coordinate system, the local stiffness matrix in an EXCEL table.

4.6.4. Matrix form of the equation relating the acting forces and deformations.

4.6.5. Transition from the global coordinate system to local one, associated with the rod finite element, to find the displacements of ends of the finite element and its

deformation, taking into account the direction cosines of local coordinate system in the global coordinate system.

4.6.6. Transposing a cosine matrix in EXCEL using the TRANSPOSE function.

4.6.7. Multiplying the transposed cosine matrix by the local stiffness matrix of a rod finite element in EXCEL using the MMULT function.

4.6.8. Finding the stiffness matrix of a rod finite element by multiplying the transposed cosine matrix by the local stiffness matrix of the element and by the original cosine matrix.

4.6.9. The location of the stiffness matrix of rod finite element in the stiffness matrix of entire flat truss on a fragment of an EXCEL page, determined by node numbers.

4.6.10. Determination in the EXCEL program of the general stiffness matrix of a flat truss by adding the stiffness matrices of rod finite elements in global coordinates.

4.6.11. Adjustment of stiffness matrix elements considering the fixation of truss nodes.

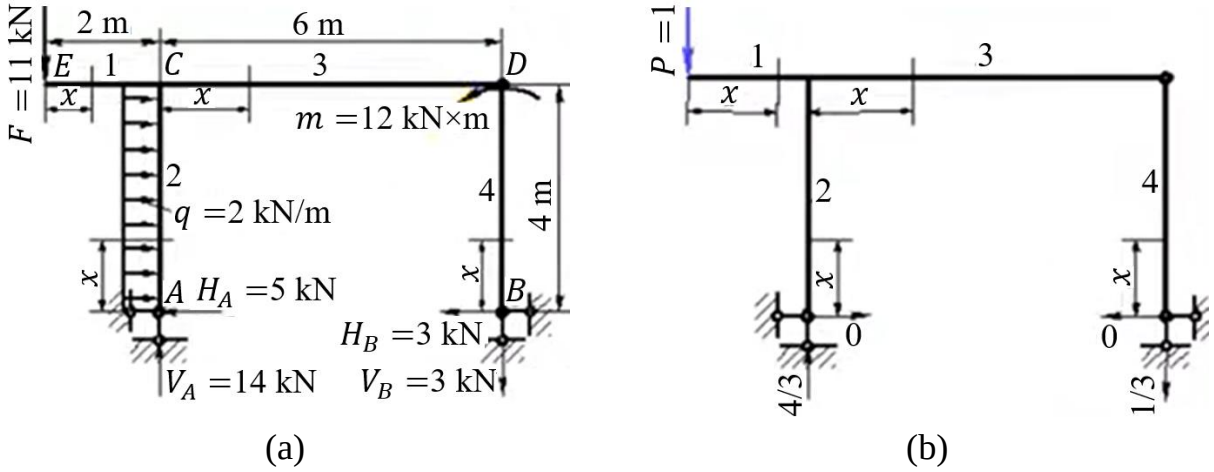
4.6.12. Finding the inverse matrix of a mechanical system in the EXCEL program using the MINVERSE function.

4.6.13. Determination of local displacements of mechanical system nodes based on the found global displacements, transition from the global system to local one to find displacements in coordinate systems associated with finite rod elements.

4.6.14. Finding the deformations of the rod finite elements of a mechanical system and the stresses acting in the finite elements in the EXCEL program.

4.7. Calculation of the Mohr's integral and displacements of frame nodes in the Mathcad

Exercise. The exercise assigns one to calculate the vertical displacement of the point E in a flat statically definable frame (Fig. 4.7.1 a) using the Mohr's integral in the computer algebra system Mathcad, using the direct integration of products of diagrams without the use of the numerical methods for finding integrals of the product of diagrams (such as the methods of Vereshchagin, Simpson, trapezoids, etc.).



(a) (b)

Fig. 4.7.1

Statically definable frame is loaded with concentrated force: F =11 kN at the point E, with the distributed load: q =2 kN/m at section AC and with the concentrated moment: m =12 kN×m. In the auxiliary diagram (Fig. 4.7.1 b) a unit load: P =1 is applied in the direction of the displacement sought at the point E. Also, on the auxiliary diagram along the rod elements of the frame the directions of reference of the integration variables are shown x.

Since the frame (Fig. 4.7.1 a) is statically definable, then by assumptions of the balance the support reactions are determined: H_A =5 kN, V_A =14 kN, H_B =3 kN and V_B =3 kN. The support reactions under the influence of a unit load are equal: at the point A – 4/3 and 0, at the point B – 1/3 and 0.

It is recommended to assign the previously found reactions of supports of a statically definable frame and reactions of unit load in the form of vectors in the Mathcad page (a fragment of the page is shown in the diagram, Fig. 4.7.2). When arranging vectors along a line, it is recommended to use transpose in the Mathcad. It is necessary in the form of vectors based on the assigned data to determine in the Mathcad page the geometric characteristics of the frame: the values of the lengths of the sections, the number of sections as elements of the vector and to assign an index of variation of the section number.

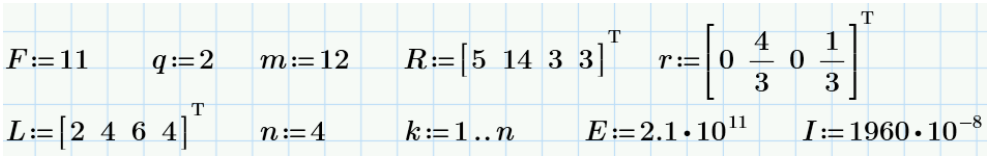


Fig. 4.7.2

In addition, to solve the problem one needs assign stiffness characteristics on the Mathcad page: $E = 2.1 \times 10^{11}$ Pa (210 GPa) – the modulus of elasticity of the material, which is typical for carbon steel, and indicate the moment of inertia of the cross–section frame rod elements: $I = 1960 \times 10^{-8}$ m^4 (that is 1960 cm^4). In the exercise the stiffness characteristics of the frame sections are assumed to be the same.

To find the vertical displacement of the point E the Mohr's integral is determined using the formula: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx + \sum \int \mu \times \frac{Q \times Q_i}{G \times A} dx + \sum \int \frac{N \times N_i}{E \times A} dx$, where M, Q and N – the diagrams of bending moments, shear forces and longitudinal forces of the specified load acting on a mechanical system; M_i, Q_i and N_i – the diagrams of bending moments, shear forces and longitudinal forces of unit load (of the unit force or the moment), in the direction of which the displacement under assigned load is sought; A – the cross–sectional area of a mechanical system element; μ – the section shape factor (the dimensionless coefficient) taken into account when finding the shear displacement.

When considering only bending moments as the factors of displacements of nodes and sections of the frame, neglecting the displacements of frame points under the influence of longitudinal and transverse (the shear) forces, the Mohr's integral formula can be applied with the simplification: $\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx$.

Under the assumed condition of linearity of the diagrams, the displacements of a given point of the mechanical system in one direction or another by the Mohr's integral could be calculated using the Vereshchagin's rule as:

$\Delta_i = \sum \int \frac{M \times M_i}{E \times I} dx = \sum \frac{l}{6 \times E \times I} \times (M_1 \times M_{i,1} + 4 \times M_{1/2} \times M_{i,1/2} + M_2 \times M_{i,2})$,

where the moments: M_1 and M_2 are the values at the ends of the linear diagram.

In the form of a vector in the Mathcad computer algebra system one needs to specify a diagram of moments of forces acting on the frame (the fragment of the page in the Mathcad, Fig. 4.7.3 a) and a diagram of moments of unit load (Fig. 4.7.3 b) applied to specified point in direction of the displacement sought. Multiplying the vector of moments of acting forces by 10^3 corresponds to converting the vector dimension into newtons.

$$M_F(x) := \begin{bmatrix} -F \cdot x \\ R_1 \cdot x - \frac{q \cdot x^2}{2} \\ -F \cdot \left(x + L_1\right) + R_2 \cdot x + R_1 \cdot L_2 - \frac{g \cdot L_2^{\,2}}{2} \\ R_3 \cdot x \end{bmatrix} \cdot 10^3 \qquad M_1(x) := \begin{bmatrix} -1 \cdot x \\ 0 \\ -1 \cdot \left(L_1 + x\right) + \frac{4}{3} \cdot x \\ 0 \end{bmatrix}$$

(a) (b)

Fig. 4.7.3

One can perform the summation operation of the total by the sections of the integrals of diagrams product by pressing the keys together: ctrl shift 4 and set summation limits and assign a summable indicator (the elements of a vector or matrix) in the Mathcad computer algebra system necessary for finding the vertical displacement of the point E (the fragment of page is shown in the diagram, Fig. 4.7.4). One can call the integration operation in the Mathcad on the Mathematics tab in the Operators panel.

$$\Delta := \sum_{k=1}^{n} \int_0^{L_k} \frac{M_F(x)_k \cdot M_1(x)_k}{E \cdot I} \, dx \qquad \Delta = 0.025$$

Fig. 4.7.4

In the case if one needs to find other frame displacements, for example, the section turning at the point B and the horizontal displacement of the point E (Fig. 4.7.1 a) in the Mathcad computer algebra system it is recommended to set diagrams of moments of unit forces applied to points B and E in the directions of the required displacements, not in the form of a vector (Fig. 4.7.5 a and b), but in the form of a matrix (the fragment of the page in the Mathcad, Fig. 4.7.6).

Each column in the matrix of diagrams of moments of unit forces in given directions of the required displacements is a function of the moment under corresponding unit impact, and each row of the matrix corresponds to a specific section of the frame.

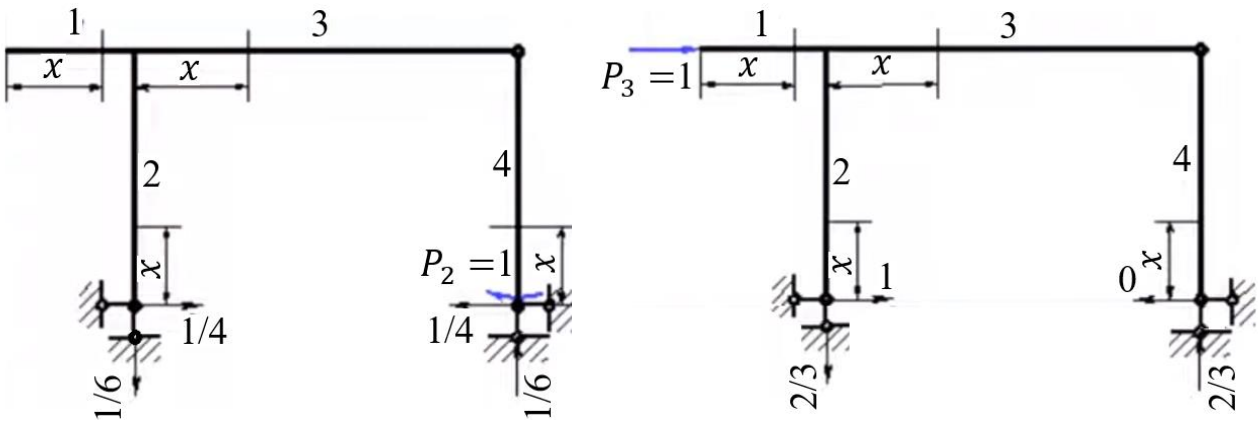


Fig. 4.7.5

$$M_1(x) := \begin{bmatrix} -1 \cdot x & 0 & 0 \\ 0 & \frac{-1}{L_4} \cdot x & 1 \cdot x \\ -1 \cdot (L_1 + x) + r_2 \cdot x & \frac{1}{L_3} \cdot x - 1 & 1 \cdot L_4 - \frac{L_4}{L_3} \cdot x \\ 0 & -1 + \frac{1}{L_4} \cdot x & 0 \end{bmatrix}$$

Fig. 4.7.6

Wherein, the values of the Mohr's integral in the Mathcad will be found as elements of a vector consisting of three elements corresponding to the displacements of the point of the frame in considered directions (the fragment of the page in the Mathcad, Fig. 4.7.7).

$$i := 1 \ldots 3 \qquad \Delta_i := \sum_{k=1}^{n} \int_0^{L_k} \frac{M_F(x)_k \cdot M_1(x)_{k,i}}{E \cdot I} \, dx$$

$$\Delta = \begin{bmatrix} 0.025 \\ 0.004 \\ -0.025 \end{bmatrix} \begin{matrix} m \\ radian \\ m \end{matrix}$$

Fig. 4.7.7

Questions and comments:

4.7.1. Direct integration of products of diagrams without using numerical methods for finding integrals of the product of diagrams (such as the Vereshchagin's, the Simpson's, method of trapezoids, etc.).

4.7.2. Determining the diagrams of bending moments in frame elements under the assigned load and the diagrams of bending moments under the unit forces in the directions of displacements sought.

4.7.3. Specifying the geometric characteristics of the frame in the form of vectors in the Mathcad, as well as the stiffness characteristics of the frame elements and the geometric characteristics of the sections of the frame elements.

4.7.4. The general form of the Mohr's integral for determining the displacement of a given section of a mechanical system (the frame).

4.7.5. Simplification of the form of the Mohr's integral with possible neglect of the influence of longitudinal and transverse (the shear) forces when determining the displacements of nodes and sections of a frame subject to bending.

4.7.6. Summation of a finite series in the Mathcad by pressing the keys together: ctrl shift 4 and assigning summation limits, as well as the summed indicator (the elements of a vector or matrix).

4.7.7. Calling the integration operation in the Mathcad on the Mathematics tab in the Operators panel, the parameters of the integration operation in the Mathcad.

4.7.8. The values of the Mohr's integral in the Mathcad as components of a vector consisting of elements corresponding to mechanical system sections displacements which are sought the in assigned directions.

4.8. The use of the Mathcad computer algebra system to calculate forces in the nodes of a flat truss using the method of cutting nodes

Exercise. The flat statically determined truss (Fig. 4.8.1 a) is subject to three forces: P=3 kN, Q=4 kN and G=5 kN. Using the Mathcad computer algebra system determine the forces in the truss rods (Fig. 4.8.1 b): S_1, S_2, S_3, S_4, S_5, S_6, S_7 and S_8. Specified truss includes five rod elements and three supporting elements.

To find the forces in the rod elements of a flat truss after determining the reactions of the supports, the method of cutting nodes or the method of sections are typically used. When using computer systems, the node cutting method is usually used.

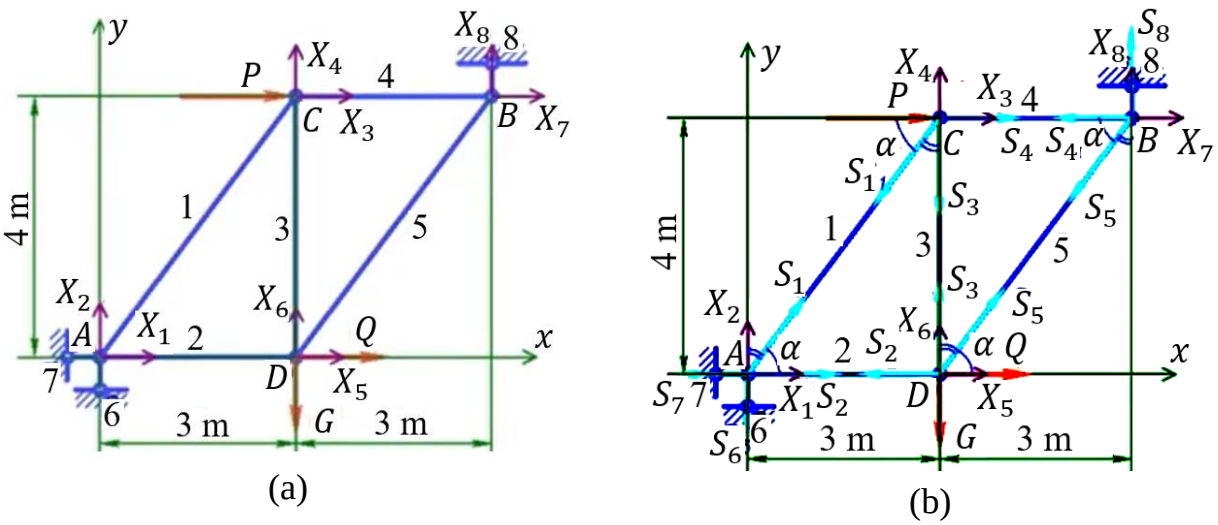


Fig. 4.8.1

On the diagram, in addition to the axes of the coordinate system: x and y, the additional axes are also used – two in each of nodes of the flat truss: X_1, X_2, X_3, X_4, X_5, X_6, X_7 and X_8. By introducing eight additional coordinate axes – two in each node of the flat truss, one can draw up eight balance equations of nodes. These eight equations will include five forces in the truss bars and three forces in the support bars of the truss.

The equations of balance of the cut node A of the flat truss:

$$\sum F_{X_1} = 0,\ S_1 \times \cos\alpha + S_2 \times 1 + S_6 \times 0 + S_7 \times (-1) = 0,$$

$$\sum F_{X_2} = 0,\ S_1 \times \sin\alpha + S_2 \times 0 - S_6 \times 1 + S_7 \times 0 = 0.$$

The equations of balance of the node B:

$$\sum F_{X_7} = 0,\ -S_4 - S_5 \times \cos\alpha = 0,$$

$$\sum F_{X_8} = 0,\ S_8 - S_5 \times \sin\alpha = 0.$$

The equations of balance of the node C:

$$\sum F_{X_3} = 0,\ P + S_4 - S_1 \times \cos\alpha = 0,$$

$$\sum F_{X_4} = 0,\ -S_1 \times \sin\alpha + S_3 = 0.$$

The equations of balance of the node D:

$$\sum F_{X_5} = 0,\ Q + S_5 \times \cos\alpha - S_2 = 0,$$

$$\sum F_{X_6} = 0,\ S_3 + S_5 \times \sin\alpha - G = 0.$$

The considered equations of the balance of nodes of flat truss correspond to the method of cutting nodes typical for calculating a truss.

When using the Mathcad computer algebra system, a geometric description of the truss is required as data, for example, in the form of a table of cosines of angles α_{ij} between each force in each of the truss rod elements: S_1, S_2, S_3, S_4, S_5, S_6, S_7 and S_8 and each of the additional coordinate axes: X_1, X_2, X_3, X_4, X_5, X_6, X_7 and X_8, entered two in each of the truss nodes. It is recommended to fill out the specified table in the EXCEL program (shown on fragment of the EXCEL page, Fig. 4.8.2).

	A	B	C	D	E	F	G	H	I
1		S1	S2	S3	S4	S5	S6	S7	S8
2	x1	0.6	1	0	0	0	0	-1	0
3	x2	0.8	0	0	0	0	-1	0	0
4	x3	-0.6	0	0	1	0	0	0	0
5	x4	-0.8	0	-1	0	0	0	0	0
6	x5	0	-1	0	0	0.6	0	0	0
7	x6	0	0	1	0	0.8	0	0	0
8	x7	0	0	0	-1	-0.6	0	0	0
9	x8	0	0	0	0	-0.8	0	0	1

Fig. 4.8.2

After filling out the table, one needs to copy the data from the EXCEL program into the Mathcad file and redefine it as a matrix of direction cosines of the angles α_{ij} between the forces in each of the truss rod elements and each of the additional coordinate axes (the fragment of the page in the Mathcad, Fig. 4.8.3).

$$C := \begin{bmatrix} 0.6 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -0.6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 4.8.3

Setting external load in projections onto eight additional coordinate axes: X_1, X_2, X_3, X_4, X_5, X_6, X_7 and X_8 (the fragment of the page in the Mathcad, Fig. 4.8.4).

$$F := [0 \ 0 \ P \ 0 \ Q \ -G \ 0 \ 0]^T$$

$$F = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 4 \\ -5 \\ 0 \\ 0 \end{bmatrix}$$

Fig. 4.8.4

Because direction cosines with respect to each of the additional coordinate axes: X_1, X_2, X_3, X_4, X_5, X_6, X_7 and X_8, represented as matrix $[C]$ on pages fragments: the

EXCEL and the Mathcad, serve as coefficients in the system of the balance equations of the truss nodes, in each of which the sum of the projections of forces and external load ($[F]$ – in vector notation) is equal to zero, then the solution in matrix form has the form: $[S] = -[C]^{-1} \times [F]$.

The solution to the issue is the vector $[S]$, composed of the found eight efforts: S_1, S_2, S_3, S_4, S_5, S_6, S_7 and S_8 – five in the bar elements of the flat truss and three in the support bars. These efforts at the nodes of the flat truss are shown in vector form (the fragment of the page in the Mathcad, Fig. 4.8.5).

$$S := \left(-C^{-1} \cdot F\right)^{\mathrm{T}} \qquad S = [-0.625 \;\; 7.375 \;\; 0.5 \;\; -3.375 \;\; 5.625 \;\; -0.5 \;\; 7 \;\; 4.5]$$

Fig. 4.8.5

Questions and comments:

4.8.1. Methods for finding forces in rod systems after determining the support reactions: the method of cutting nodes and the method of sections.

4.8.2. The coordinate systems and the additional coordinate axes in each of the nodes of the mechanical system (truss), the forces in which are determined by the method of cutting nodes.

4.8.3. The balance equations for cut nodes of the flat truss, included sums of projections onto additional axes of forces acting in the nodes.

4.8.4. The geometric description of the truss when performing calculations in the Mathcad computer algebra system of the forces acting in the nodes: the determination of cosines of the direction angles between the forces in each of the rod elements of the truss and each of the additional coordinate axes, two introduced in each of the nodes of the flat truss, using the EXCEL program for geometric description of the truss and exporting data from the EXCEL program to the Mathcad.

4.8.5. Specifying an external load in the Mathcad in projections onto additional coordinate axes in the nodes of mechanical system (the truss).

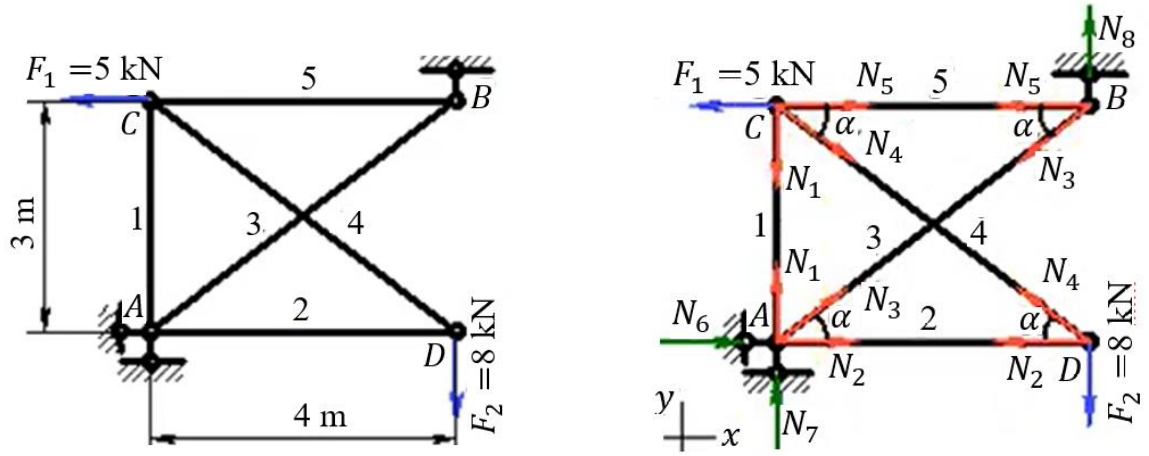
4.8.6. Matrix of direction cosines between the forces in each of the rod elements and each of the additional coordinate axes, used in the Mathcad computer algebra system when calculating forces in the elements of mechanical system (the truss), serving as a matrix of coefficients for the balance equations of nodes.

4.8.7. Solving a system of equilibrium equations when calculating a mechanical system (the truss) by cutting out nodes and finding the forces in the rod elements.

4.8.8. Solution in the Mathcad of balance equations specified in matrix and vector form of factors.

4.9. Calculation of displacements of nodes of a statically definable truss in the computer algebra system Mathcad

Exercise. Find the displacement of nodes of statically definable truss under the load acting on it. Data is shown on the diagram of the truss (Fig. 4.9.1). In the nodes: C and D the concentrated forces are applied: F_1 =5 kN and F_2 =8 kN. Length of the inclined rods is 5 m. The truss material is carbon steel, the elastic modulus of which is: $E = 2.1 \times 10^{11}$ Pa. The cross–sectional areas of all rod elements of the truss are set to be the same: $A = 1.2 \times 10^{-4}$ sq. m.



(a) (b)

Fig. 4.9.1

On the Mathcad page (fragment of the page is shown in the diagram, Fig. 4.9.2) one needs to determine the data of the flat truss: the elastic modulus E and cross-sectional area A, characterizing the linear rigidity of the rod elements of the form, as well as the length of the rod elements l – in the form of vector.

$$E := 2.1 \cdot 10^{11} \qquad A := 1.2 \cdot 10^{-4} \qquad l := \begin{bmatrix} 3 & 4 & 5 & 5 & 4 & 0 & 0 & 0 \end{bmatrix}^{T}$$

Fig. 4.9.2

The displacements of the nodes of truss with hinged joints, which rod elements experience tensile and compressive forces uniform along the length of each rod element, one can find using Mohr's integral in the form: $\Delta_i = \sum \int \frac{N_F \times N_i}{E \times A} dx = \sum N_F \times N_i \times \frac{l}{E \times A}$, or in the matrix one: $[\Delta] = \{N_i\}^T \times [L] \times \{N_F\}$, where $L = \frac{l}{E \times A}$ – the pliance index of the corresponding element of the rod system, which is specified in the form of diagonal pliance matrix $[L]$, N_i – the force of unit load in the direction of i–th displacement, N_F – the

force of external load – the vector of longitudinal forces in the rod elements of the truss from the action of the unit force.

Vector $\{N_i\}$ (the matrix–column) of forces in the rods of the flat truss from the action of the unit force in the direction of each i–th displacement of n displacements sought and vector of the longitudinal forces $\{N_F\}$ in the sections of the assigned load action, as well as the diagonal pliance matrix $[L]$ of the rod system have the form:

$$\{N_i\} = \begin{Bmatrix} N_{i1} \\ N_{i2} \\ \dots \\ N_{in} \end{Bmatrix}, \{N_F\} = \begin{Bmatrix} N_{F1} \\ N_{F2} \\ \dots \\ N_{Fn} \end{Bmatrix} \text{ and } [L] = \begin{bmatrix} \left(\frac{l}{E \times A}\right)_1 & 0 & \dots \\ 0 & \left(\frac{l}{E \times A}\right)_2 & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

On the diagram (Fig. 4.9.3) the actions of unit forces are shown in the hinge joints of the flat truss in the directions in which the displacements are sought.

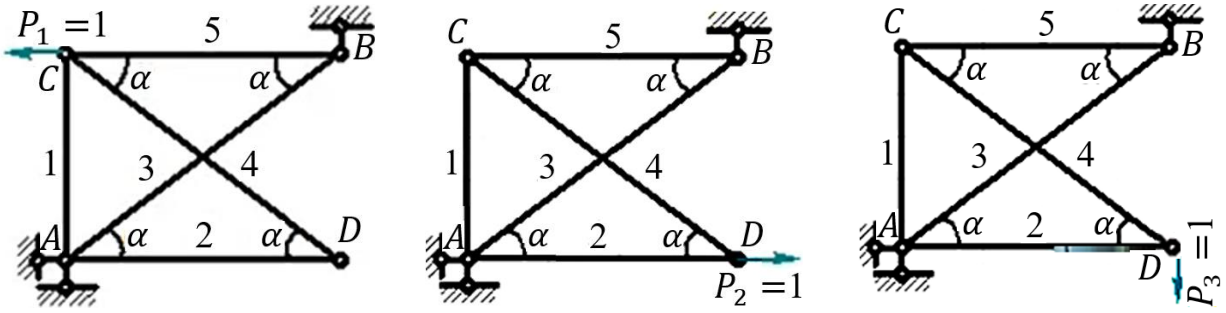


Fig. 4.9.3

The forces in the eight bars of the flat truss one can find taking into account the balance equations in the directions of two axes (x and y) for each of the four nodes of the truss. To compile the balance equations, one can recommend a sequence of nodes: A, C, B and D.

Cosines of direction angles α_{ij} for given external forces (F_1 and F_2) and acting ones in the truss rods, including the support rods (having zero length): N_1, N_2, N_3, N_4, N_5, N_6, N_7 and N_8 with respect to the coordinate axes (x and y) are sought as the ratio of the lengths of horizontal, vertical and diagonal rod elements, and also when determining the sign of the direction cosine, the directions of the coordinate axes and the directions of projections of forces on these axes are taken into account.

The balance equation of the cut node A of the flat truss has the form:

$\sum X = 0, N_1 \times 0 + N_2 \times 1 + N_3 \times 0.8 + N_4 \times 0 + N_5 \times 0 + N_6 \times 1 + N_7 \times 0 +$
$+ N_8 \times 0 + 0 = 0,$

$\sum Y = 0, N_1 \times 1 + N_2 \times 0 + N_3 \times 0.6 + N_4 \times 0 + N_5 \times 0 + N_6 \times 0 + N_7 \times 1 +$

$+N_8 \times 0 + 0 = 0$, the specified external forces do not act on the node, the reactions: N_6, N_7 and N_8 in the supports correspond to the forces in the supporting rod elements (of zero length).

Equations of balance of the node C are: $\sum X = 0$, $N_1 \times 0 + N_2 \times 0 + N_3 \times 0 +$
$+N_4 \times 0.8 + N_5 \times 1 + N_6 \times 0 + N_7 \times 0 + N_8 \times 0 - F_1 = 0$,
$\sum Y = 0$, $-N_1 \times 1 + N_2 \times 0 + N_3 \times 0 - N_4 \times 0.6 + N_5 \times 0 + N_6 \times 0 + N_7 \times 0 +$
$+N_8 \times 0 + 0 = 0$.

The equilibrium equations of the node B are: $\sum X = 0$, $N_1 \times 0 + N_2 \times 0 -$
$-N_3 \times 0.8 + N_4 \times 0 - N_5 \times 1 + N_6 \times 0 + N_7 \times 0 + N_8 \times 0 + 0 = 0$,
$\sum Y = 0$, $N_1 \times 0 + N_2 \times 0 - N_3 \times 0.6 + N_4 \times 0 + N_5 \times 0 + N_6 \times 0 +$
$+N_7 \times 0 + N_8 \times 1 + 0 = 0$.

Equations of balance of the node D are: $\sum X = 0$, $N_1 \times 0 - N_2 \times 1 + N_3 \times 0 -$
$-N_4 \times 0.8 + N_5 \times 0 + N_6 \times 0 + N_7 \times 0 + N_8 \times 0 + 0 = 0$,
$\sum Y = 0$, $N_1 \times 0 + N_2 \times 0 + N_3 \times 0 + N_4 \times 0.6 + N_5 \times 0 + N_6 \times 0 + N_7 \times 0 +$
$+N_8 \times 0 - F_2 = 0$.

The above equations of balance of nodes are given for clarity in studying the matrix method in the Mathcad computer algebra system for finding the displacements of nodes of a rod flat truss.

For practical purposes to find the displacements of truss nodes it is necessary to draw up matrix of direction cosines between the acting force in each rod (there are eight of them in exercise, including the support rods of zero length) or the external force and each of the axes (two for flat rod trusses) in all nodes of the truss (in exercise the truss has four nodes), and the equations of balance of nodes are given above to illustrate this issue.

When using the Mathcad computer algebra system as data, the geometric description of the truss is required, for example, in the form of a table of cosines of the angles α_{ij} between the forces and of each of the coordinate axes. It is recommended to fill out the specified table of cosines of angles α_{ij} in the EXCEL program (shown on a fragment of the EXCEL page, Fig.4.9.4).

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		N1	N2	N3	N4	N5	N6	N7	N8	F	P1	P2	P3
2	A, c	0	1	0.8	0	0	1	0	0	0	0	0	0
3	A, s	1	0	0.6	0	0	0	1	0	0	0	0	0
4	C, c	0	0	0	0.8	1	0	0	0	-5	-1	0	0
5	C, s	-1	0	0	-0.6	0	0	0	0	0	0	0	0
6	B, c	0	0	-0.8	0	-1	0	0	0	0	0	0	0
7	B, s	0	0	-0.6	0	0	0	0	1	0	0	0	0
8	D, c	0	-1	0	-0.8	0	0	0	0	0	0	1	0
9	D, s	0	0	0	0.6	0	0	0	0	-8	0	0	-1

Fig. 4.9.4

The last four columns in the Table correspond to the cosines (three columns of the Table) between the unit forces applied in the directions of the initial displacements of the nodes of the bar truss, which is required to determine the Mohr's integral, and the projections of the given external forces on the axis (in relation to the nodes of the truss, one column), which corresponds to the free (that is independent on displacements) terms of considered balance equations of the rod truss nodes.

The Table can be conveniently transferred from the EXCEL to the Mathcad and redefined as a matrix. To do this, the READEXCEL function can be used, located in the Mathcad panel on the Input/Output tab, wherein the EXCEL file used must be located in the same folder as the Mathcad file, the READEXCEL function line has the form:
T :=READEXCEL(">\Data in EXCEL Determining of truss nodes displacements
By matrix way,xlsx","Sheet1!B2:M9")

Table characterizing the geometric elements of the truss, with direction cosines between the forces in the truss rods applied in nodes: A, C, B and D, and coordinate axes, with projections of assigned external forces, as well as with direction cosines of unit forces specified in the directions of displacements sought in the truss nodes, which displacements are kinematically possible, is shown in the form of a matrix $[T]$ on the Mathcad page (fragment of the page is shown in the diagram, Fig. 4.9.5).

$$T=\begin{bmatrix} 0 & 1 & 0.8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 1 & 0 & 0 & 0 & -5 & -1 & 0 & 0 \\ -1 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & -1 \end{bmatrix}$$

Fig. 4.9.5

The matrix C of direction cosines between the acting force in each rod (in the exercise there are eight of them, including the support rods of zero length) or external force and each of the axes (two ones – for flat bar trusses) in all nodes of the truss (in considered exercise the truss has four nodes) one can select in the original matrix using the operation: submatrix (fragment of the page in the Mathcad, Fig. 4.9.6), which in the Mathcad is located on the Matrices/Tables tab in the Functions with vectors/matrices panel. To perform the submatrix operation, one needs to specify parameters – the intervals of row and column indices in the source matrix corresponding to the selected matrix.

$$C := \text{submatrix}(T, 1, 8, 1, 8)$$

$$C = \begin{bmatrix} 0 & 1 & 0.8 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0.6 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.8 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.9.6

Selecting in the matrix $[T]$ the ninth column corresponding to the vector of given external forces $\{F\}$, one can do in the Mathcad using the appropriate command on the Matrices/Tables tab in the Operators with Vectors/Matrixes panel: $M^{\langle\rangle}$ (column selection is shown on a fragment of a Mathcad page, Fig. 4.9.7 a).

$$F := T^{\langle 9 \rangle}$$

$$F^{\mathrm{T}} = [0 \;\; 0 \;\; -5 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -8]$$

(a)

$$PI := \text{submatrix}(T, 1, 8, 10, 12)$$

$$PI^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(b)

Fig. 4.9.7

The matrix $[P_i]$ with direction cosines of the unit forces specified in the directions of truss nodes displacements sought, which are kinematically possible, one can select in the original matrix using the operation: submatrix (fragment of the page in the Mathcad, Fig. 4.9.7 b).

Since the equations of the balance of the nodes of the bar truss have a matrix form: $\lfloor C \rfloor \times \{N_F\} + \{F\} = 0$, then finding the force vector in the rods of the flat truss one can represent in matrix form as: $\{N_F\} = -\lfloor C \rfloor^{-1} \times \{F\}$, which is shown in a fragment of the Mathcad page (Fig. 4.9.8), in which the force vector in the rods of the flat truss is found.

$$NF := -C^{-1} \cdot F \qquad NF^{\mathrm{T}} = [-8 \;\; -10.667 \;\; 7.083 \;\; 13.333 \;\; -5.667 \;\; 5 \;\; 3.75 \;\; 4.25]$$

Fig. 4.9.8

The matrix of forces in the rod elements of the flat truss under unit load $[P_i]$, acting in the direction of dislacements sought, is determined as: $\{N_i\} = -\lfloor C \rfloor^{-1} \times [P_i]$, this is shown in the Mathcad (Fig. 4.9.9).

$$NI := -C^{-1} \cdot PI$$

$$NI^{T} = \begin{bmatrix} 0 & 0 & -1.25 & 0 & 1 & 1 & 0.75 & -0.75 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1.333 & 1.667 & 1.667 & -1.333 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 4.9.9

The determining of diagonal matrix of rod flat truss pliance is shown in the Mathcad (Fig. 4.9.10).

$$L := \frac{1}{E \cdot A} \cdot \mathrm{diag}(l)$$

$$L = \begin{bmatrix} 1.19 \cdot 10^{-7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.587 \cdot 10^{-7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.984 \cdot 10^{-7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.984 \cdot 10^{-7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.587 \cdot 10^{-7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.9.10

The vector of kinematically possible displacements of nodes of the flat truss in the matrix form is: $[\Delta] = \{N_i\}^T \times [L] \times \{N_F\}$ is defined in the computer algebra system Mathcad (fragment of the page is shown in the diagram, Fig. 4.9.10).

$$\Delta := NI^{T} \cdot L \cdot NF \cdot 10^{3} \qquad \Delta^{T} = \begin{bmatrix} -0.003 & -0.002 & 0.011 \end{bmatrix}$$

Fig. 4.9.11

Questions and comments:

4.9.1. Data for finding of displacements of nodes in a statically definable truss, rod elements which experience tensile and compressive forces uniform along the length of the rod elements using Mohr's integral in the Mathcad computer algebra system.

4.9.2. Drawing up of the Mohr's integral for mechanical system (the truss) of rod elements that are subject to tension or compression and do not experience bending moments and transverse (the shear) forces.

4.9.3. The matrix form with longitudinal forces uniformly distributed along the length in the truss rods.

4.9.4. Vector (the matrix–column) of longitudinal forces acting on the rod elements of truss under external load. The required force vector in the rod elements of the mechanical system (the truss).

4.9.5. Equations for the balance of force projections in the nodes of mechanical system. The use of table of direction cosines of angles between coordinate axes and forces applied in the nodes to the elements of mechanical system under external forces and the unit forces in the direction of sought displacements.

4.9.6. Forming table in the EXCEL and transferring the data from table in the EXCEL to a matrix in the Mathcad using the READEXCEL function, the parameters of this function. Selecting a matrix of direction cosines between the acting force in each rod or external force and each of the axes, using the submatrix operation in the Mathcad.

4.9.7. Selecting a matrix column in the Mathcad using the appropriate command on the Matrices/Tables tab in the Vector/Matrix Operators panel.

4.9.8. The matrix form of balance equations of mechanical system nodes and finding the force vector in the truss rods by solution in matrix form.

4.9.9. Finding in the Mathcad the matrix of forces in the rod elements of the truss from under unit load in the direction of displacements sought.

4.9.10. The determining of a diagonal pliance matrix in the Mathcad.

4.9.11. Determination in the Mathcad of kinematically possible displacements vector of the rod truss nodes.

4.10. Calculation of statically indefinable frame using the finite element method in the Mathcad. Setting the geometric characteristics of the frame in matrix and vector forms

Exercise. Calculate a flat statically indefinable frame using the finite element method and the Mathcad computer algebra system. The data set is shown on the scheme of frame (Fig. 4.10.1 a).

The frame consists of two racks: (1–2) and (3–5), the inclined section (3–4) and the horizontal crossbar (2–3). Frame supports – the rigid embeddings in the nodes: 1 and 4, the hinged support in node 5. The node 2 of the frame – rigid, the node 3 – hinged (three rod elements of the frame are hinged).

The concentrated force: $P_1 = 50$ kN is applied to node 2 of the frame. The distributed load of constant intensity: $q_1 = 20$ kN/m acts on the horizontal crossbar.

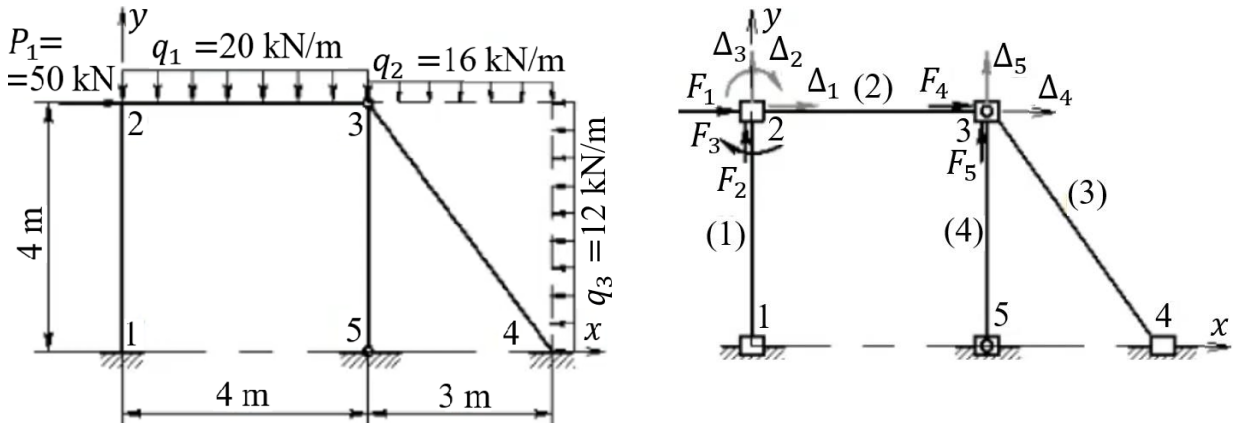


Fig. 4.10.1

The inclined element of the frame is subject to a distributed load of constant intensity, including the vertical and horizontal components: q_2 =16 kN/m and q_3 =12 kN/m.

Rod finite elements have a beginning and an end, the beginning corresponds to the lower number of the node in which the rod finite element is connected to another finite element or to a support. It is assumed that beam finite elements can be classified into 4 types depending on how the finite elements are connected at their ends to other finite elements or to supports.

The discrete frame model is shown on the scheme (Fig. 4.10.1 b). The discrete model contains the finite elements of mechanical system (the frame):

– the rod finite element (1–2) of the 1–st type – the ends of the rod finite element are rigidly connected to a fixed base and to another finite element (2–3);

– the rod finite element (2–3) of the 2–nd type – at the beginning of the rod finite element there is a rigid embedding (the node 2), and at the end there is a hinged connection with other finite elements (the node 3);

– the rod finite element (3–4) of the 3–rd type – at the beginning of the rod finite element there is a hinged connection with other finite elements (the node 3), and at the end there is a rigid embedding (fixed support 4);

– the rod finite element (3–5) of the 4–th type – both ends of which are hingedly connected, both with other rod finite elements – in node 3, and with a support in node 5.

In addition to the rod finite elements of the discrete model, the required displacements of nodes (the kinematic characteristics of frame deformation) are shown:

– the fixed nodes: 1, 5 and 4 have no displacements;

– the nodes 2 and 3 can displace when the flat frame is deformed, and the displacements of nodes correspond to the end displacements of those rod finite elements that are connected at these nodes;

– rigid node 2 experiences both the linear displacement and the angular one, and hinge node 3 experiences the linear displacement.

In the displacing nodes of mechanical system during deformation of the discrete model, which shows the displacements sought (Fig. 4.10.1 b): Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5, the acting external forces are also indicated: F_1, F_2, F_3, F_4 and F_5.

It is recommended to enter the initial data associated with the geometric description of the discrete model in the EXCEL program (see fragment of the page on the diagram, Fig. 4.10.2), and then to transfer the data to the Mathcad file.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Element # i	First-second ends	FEM type	Nodes coordinates				l, m	$c = \cos\alpha$	$s = \sin\alpha$	f, N/m	g, N×m
2				First end		Second end						
3				x_i, m	y_i, m	x_j, m	y_j, m					
4	1	1-2	1	0.0	0.0	0.0	4.0	4.0	0.00	1.00	25	0.25
5	2	2-3	2	0.0	4.0	4.0	4.0	4.0	1.00	0.00	25	0.25
6	3	3-4	3	4.0	4.0	7.0	0.0	5.0	0.60	-0.80	20	0.2
7	4	3-5	4	4.0	4.0	4.0	0.0	4.0	0.00	-1.00	25	0.25

Fig. 4.10.2

The denominations used in the Data Table EXCEL: x_i, y_i, x_j and y_j – the coordinates of the beginning and end of rod finite element; l – the length of rod finite element: $l = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$; $(x_j - x_i)$ and $(y_j - y_i)$ – the projections of rods on coordinate axes; $\cos\alpha = (x_j - x_i)/l$ and $\sin\alpha = (y_j - y_i)/l$ – the cosine and sine of the angle α, that forms the rod with positive axis direction x; $f = \frac{E \times A}{l}$ – the longitudinal linear stiffness of rod finite element; $g = \frac{E \times I}{l}$ – the bending linear stiffness of rod finite element.

Matrix with node coordinates is copied to the Mathcad from Microsoft EXCEL (fragment of the page in the Mathcad, Fig. 4.10.3). The EXCEL and Mathcad files have to be located in the same folder to use the READEXCEL command in the Mathcad, which is located on the Input/Output tab.

$xy :=$ READEXCEL(".\Data in EXCEL for the FEM calculation of a statically indeterminate frame in Mathcad.xlsx", "Sheet1!D4:G7")

$$xy = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 4 & 4 & 4 \\ 4 & 4 & 7 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

Fig. 4.10.3

In addition to use of the READEXCEL command in the Mathcad one can copy a Table selected in the EXCEL («copy paste», fragment of the page in the Mathcad, Fig. 4.10.4). After the copying one needs to designate columns that become vectors (in the

considered exercise – the vectors of the coordinates of the beginning and end of each finite element) into the Mathcad.

FE	$Type$	x_i	y_i	x_j	y_j
“1–2”	1	0.0	0.0	0.0	4.0
“2–3”	2	0.0	4.0	4.0	4.0
“3–4”	3	4.0	4.0	7.0	0.0
“3–5”	4	4.0	4.0	4.0	0.0

$$x_i=\begin{bmatrix}0\\0\\4\\4\end{bmatrix}\quad y_i=\begin{bmatrix}0\\4\\4\\4\end{bmatrix}\quad x_j=\begin{bmatrix}0\\4\\7\\4\end{bmatrix}\quad y_j=\begin{bmatrix}4\\4\\0\\0\end{bmatrix}$$

Fig. 4.10.4

After specifying the coordinates of the ends of the rods in the Mathcad file by copying the table from the EXCEL, one needs to transform the coordinates into the form of projections of vectors of the coordinates of the finite elements on the axes: $(x_j - x_i)$ and $(y_j - y_i)$, and to determine the length l of rod finite elements, cosines and sines of the direction angles with respect to the positive direction of the abscissa axis x: $\cos\alpha$ and $\sin\alpha$ (fragment of the page in the Mathcad is shown on scheme, Fig. 4.10.5).

$$x := x_j - x_i \qquad y := y_j - y_i \qquad l := \sqrt{x^2 + y^2}$$

$$x^{\mathrm{T}} = [0\ \ 4\ \ 3\ \ 0] \qquad y^{\mathrm{T}} = [4\ \ 0\ \ -4\ \ -4] \qquad l^{\mathrm{T}} = [4\ \ 4\ \ 5\ \ 4]$$

$$C := \frac{x}{l} \qquad C^{\mathrm{T}} = [0\ \ 1\ \ 0.6\ \ 0] \qquad S := \frac{y}{l} \qquad S^{\mathrm{T}} = [1\ \ 0\ \ -0.8\ \ -1]$$

Fig. 4.10.5

To find the reactions of the supports of statically indefinable frame and to find the displacements of nodes loose in the supports when the frame is deformed, which is assigned in the exercise, one needs to use the values of the stiffness of the rod finite elements, which depends on the structural material used, on the length of the finite element, on the shape and dimensions of its cross–section.

For general information, the modulus of elasticity of carbon steel is about 220 GPa (that is 220×10^9 N/m^2), of the aluminum–based alloys – about 70 GPa, the elastic modulus of copper is usually 130 GPa, the bronze elastic modulus is about 100 GPa, the brass elastic modulus is from 80 up to 100 GPa, the elastic modulus of cast iron – about 80 GPa, of the titanium – about 110 Gpa, of glass – from 50 up to 80 GPa. One needs to note that in determining the reactions of the supports of a statically indefinable frame, the

magnitude of the elastic modulus does not affect the reactions of the supports if all the rod elements are made of the same linear elastic material.

The length of each rod does not affect the ratio of its linear bending stiffness and the linear tensile/compressive stiffness.

It needs also be clarified that for indicators in different measurement systems, for example, in the International System of Units SI, that is, regardless of the measurement system, the flexibility (deformability, the opposite indicator of rigidity) of the rod elements of frames for bending usually has a nominal value that is several orders of magnitude greater than the flexibility in tension and compression. For example, for a square solid cross–section with a size of 0.1 m, the moment of inertia of the section affecting the bending stiffness is: 8.3×10^{-6}, m^4, and the cross–sectional area is: 1.0×10^{-2}, m^2. The difference in these values corresponds to approximately a thousand–fold increase in the cross-section stiffness in tension and compression of the rod element over its bending stiffness. This circumstance can be taken into account when finding the displacements of nodes of rod frames or reactions of supports of statically indeterminate frames, when the displacements of tension and compression can be neglected, which does not apply to the calculation of rod trusses, all elements of which are hinged and do not experience bending.

Since the decision to determine the reactions of the supports of a statically indefinable frame and of the displacements of it free nodes does not depend on the stiffness of the rod elements, but also on the ratio of the stiffness of rod elements having different lengths, for the exercise it is assigned to assume that all rod elements have the same transverse section and set the linear stiffness of the rod elements, adhering to the conventional values of the rod element rigidity in tension and compression: $E \times A$=100 and the bending rigidity: $E \times I$=1. This ratio of nominal values of stiffness is possible if the rod elements of the frame are, for example, pipes, channels, the posts of which are in the plane of the frame, I-beams, etc. Vectors of linear values of longitudinal and bending stiffness are specified in the Mathcad (fragment of the page in the Mathcad, Fig. 4.10.6).

$$EA := 100 \qquad EI := 1$$

$$f := \frac{EA}{l} \qquad f^{T} = [25 \;\; 25 \;\; 20 \;\; 25] \qquad g := \frac{EI}{l} \qquad g^{T} = [0.25 \;\; 0.25 \;\; 0.2 \;\; 0.25]$$

Fig. 4.10.6

Questions and comments:

4.10.1. Designation of frame nodes and directions of its rod elements.

4.10.2. Four types of rod finite elements of the frame, differing in the form of connection with adjacent elements and with supports: the first type, both ends of which are

rigidly connected, the second type, the beginning of which is rigidly connected and the end is hingedly supported, the third type, which has a hinged connection at the beginning and at the end – a rigid embedding, and a fourth type of rod element, both ends of which are hingedly supported.

4.10.3. Indication on a discrete model of a mechanical system of possible displacements of nodes displaceable during its deformation and external forces acting in these nodes or transformed to points of application in these nodes.

4.10.4. Entering initial data related to the geometric description of a discrete model in the EXCEL. Determination in the EXCEL of the lengths of rod finite elements, the lengths of their projections on the axes and direction cosines. Indication of linear longitudinal and bending stiffness of frame elements. Importing data from the EXCEL to the Mathcad.

4.10.5. Stiffness values of linearly elastic structural materials necessary to find reactions in the supports of a statically indefinable frame and displacements of units not secured in the supports when the frame is deformed.

4.10.6. Independence of the choice of measurement system for the ratio of the linear bending stiffness of frame elements and their linear stiffness during their longitudinal deformation in tension and compression.

4.10.7. A multiple excess of the nominal values of compliance (i.e., the inverse indicator to rigidity) during bending of rod frames and their elements compared to compliance during their longitudinal deformation of tension or compression, which serves as a prerequisite for neglecting tension and compression when finding the displacements of nodes and sections during bending of frames, in contrast to determining the displacements of hinged truss nodes, rod elements that are hingedly connected do not experience bending, but experience tension and compression.

4.11. Determination of external forces in finite element nodes in the Mathcad

The determining of geometric characteristics of the frame in matrix and vector forms is made in the previous section (item 4.10).

Formation of vector of the external load. The external forces: F_1, F_2, F_3, F_4 and F_5 are applied in the connection nodes of the rod finite elements – two racks (Fig. 4.10.1 a and b): (1–2) and (3–5), inclined section (3–4) and horizontal crossbar (2–3), and the components of specified forces correspond to the variables sought – the displacements of free (the loose) nodes: Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5.

At the node 2 one horizontal concentrated force is applied: P_1 =50 kN. In addition, the horizontal crossbar (2–3) is subject to a distributed load of constant intensity: q_1 =20

kN/m, and the inclined section (3–4) is subject to distributed load of constant intensity, including vertical and horizontal components: q_2 =16 kN/m and q_3 =12 kN/m.

The vector of external load F, reduced to the nodes of the discrete model, is determined as the sum of the vectors of external load from concentrated forces F', applied to the nodes, and concentrated forces Q, to which the distributed load acting on the rod finite elements is reduced: $F = F' + Q$,

To find the concentrated forces Q, equivalent to the distributed external load when acting on the rod finite elements, one needs to specify the vectors of the distributed load in projections in the Mathcad: q_x and q_y along all the rod elements of the flat frame (fragment of the page in the Mathcad, Fig. 4.11.1).

$P := 50$ $F'_1 := P$ $kk := 2..5$ $F'_{kk} := 0$ $F'^{T} = [50\ 0\ 0\ 0\ 0]$

$q_x := [0\ 0\ -12\ 0]^T$ $q_y := [0\ -20\ -16\ 0]^T$

Fig. 4.11.1

To apply the finite element method it is required the distributed load of the distributed load: q_x and q_y, acting on each element $(i-j)$, to the global coordinate system (Fig. 4.10.1 a) to the nodal load Q, which corresponds to opposite directions of the reactions: P_1, P_2, P_3, P_4, P_5 and P_6 (Fig. 4.11.2), which act in nodes 2 and 3, i.e. to transform the distributed load when it acts on the rod finite elements to concentrated forces equivalent to this distributed external load.

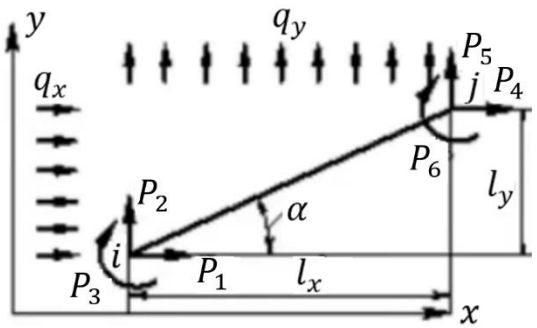


Fig. 4.11.2

In addition, the number of components of the vector of concentrated forces Q, equivalent to a distributed external load when acting on the rod finite elements, in the considered exercise being solved is five (Fig. 4.11.3), as is the number of components of the vector of applied concentrated forces, since in the considered exercise there are five variables sought – the number of degrees of displaceability of loose nodes, and not six – as the number of displacements on the plane of the two ends of each rod finite element.

The concentrated forces Q, equivalent to a distributed external load when acting on the rod finite elements, correspond to the reactions arising from this load in the nodes of the finite elements, but have opposite signs.

(a) (b)

Fig. 4.11.3

The vector of acting forces F when calculating the frame using the finite element method is equal to the sum of external concentrated force F' and the concentrated forces Q, equivalent to a distributed external load when acting on rod finite elements, that is the vector of acting forces: $F = F' + Q$ opposites the reactions on the rod finite element.

In the global coordinate system one needs to find vectors of end forces from the action of a distributed load for all rod finite elements included in a flat frame, taking into account the types of fastening the ends of these elements.

For a finite element of the 1–st type, the ends of which are rigidly connected to a fixed base and/or to another finite element (in exercise this is finite element number 1, that is 1–2), in the global coordinate system the vector of end forces $\{p\}$ under the influence of a distributed load has the form:

$$\{p\} = \begin{Bmatrix} -\frac{q_x \times l_y}{2} \\ -\frac{q_y \times l_x}{2} \\ \frac{-q_x \times l_y \times l \times \sin\alpha + q_y \times l_x \times l \times \cos\alpha}{12} \\ -\frac{q_x \times l_y}{2} \\ -\frac{q_y \times l_x}{2} \\ \frac{q_x \times l_y \times l \times \sin\alpha - q_y \times l_x \times l \times \cos\alpha}{12} \end{Bmatrix}.$$

The symbols correspond to the scheme of the rod element (Fig. 4.11.2).

For a finite element of the 2–nd type, which is the horizontal crossbar number 2, that is 2–3, at the beginning of which there is a rigid embedding, and at the end a hinged connection with other finite elements, in the global coordinate system the vector of end forces $\{p\}$ under the influence of a distributed load is:

$$\{p\} = \begin{Bmatrix} -\frac{5\times q_x\times l_y}{8} \\ -\frac{5\times q_y\times l_x}{8} \\ \frac{-q_x\times l_y\times l\times \sin\alpha + q_y\times l_x\times l\times\cos\alpha}{8} \\ -\frac{3\times q_x\times l_y}{2} \\ -\frac{3\times q_y\times l_x}{2} \\ 0 \end{Bmatrix}.$$

For a finite element of the 3–rd type, which is the inclined section number 3, that is 3–4, at the beginning of which there is a hinged connection, and at the end there is a rigid embedding, in the global coordinate system the vector of end forces $\{p\}$ under the influence of a distributed load one can determine as:

$$\{p\} = \begin{Bmatrix} -\frac{3\times q_x\times l_y}{8} \\ -\frac{3\times q_y\times l_x}{8} \\ 0 \\ -\frac{5\times q_x\times l_y}{8} \\ -\frac{5\times q_y\times l_x}{8} \\ \frac{q_x\times l_y\times l\times \sin\alpha - q_y\times l_x\times l\times\cos\alpha}{8} \end{Bmatrix}.$$

For a finite element of the 4–th type, that is for vertical column number 4, that is 3–5, both ends of which are hinged, in the global coordinate system the vector of end forces $\{p\}$ under the influence of a distributed load has the form:

$$\{p\} = \begin{Bmatrix} -\frac{q_x\times l_y}{2} \\ -\frac{q_y\times l_x}{2} \\ 0 \\ -\frac{q_x\times l_y}{2} \\ -\frac{q_y\times l_x}{2} \\ 0 \end{Bmatrix}.$$

The vector components of the end forces under the influence of a distributed load for different rod elements as part of a flat frame in the global coordinate system have a certain mathematical similarity, which one can use in the Mathcad to assign the auxiliary vectors, and before that it is required to specify the index of the number of rod finite elements in the frame (the sections, fragment of the page in the Mathcad, Fig. 4.11.4).

$i_u := 4 \quad i := 1 .. i_u \quad v := \overrightarrow{q_x \cdot y} \quad w := \overrightarrow{q_y \cdot x} \quad u := \overrightarrow{-v \cdot l \cdot S + w \cdot l \cdot C}$

Fig. 4.11.4

These auxiliary vectors allow one to represent compactly end force vectors for four types of rod finite elements in a global coordinate system: 1–st type, the ends of which are rigidly connected to a fixed support and/or to another finite element (in considered exercise this is the finite element number 1: $i = 1$, Fig. 4.10.1 a and b), 2–nd type, at the beginning of which there is a rigid embedding, and at the end there is a hinged connection with other finite elements ($i = 2$), 3–rd type, at the beginning of which there is a hinged connection, and at the end there is a rigid embedding ($i = 3$) and 4–th type, both ends of which are hingedly connected ($i = 4$). End force vectors using auxiliary vectors for the listed four finite elements are determined in the Mathcad (fragment of the page in the Mathcad, Fig. 4.11.5).

$$p_1(i) := \frac{1}{2} \cdot \left[-v_i \;\; w_i \;\; \frac{1}{6} \cdot u_i \;\; -v_i \;\; -w_i \;\; -\left(\frac{1}{6} \cdot u_i \right) \right]^{\mathrm{T}}$$

$$p_1(1)^{\mathrm{T}} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$p_2(i) := \frac{1}{8} \cdot \left[-5 \cdot v_i \;\; -5 \cdot w_i \;\; u_i \;\; -3 \cdot v_i \;\; -3 \cdot w_i \;\; 0 \right]^{\mathrm{T}}$$

$$p_2(2)^{\mathrm{T}} = [0 \;\; 50 \;\; -40 \;\; 0 \;\; 30 \;\; 0]$$

$$p_3(i) := \frac{1}{8} \cdot \left[-3 \cdot v_i \;\; -3 \cdot w_i \;\; 0 \;\; -5 \cdot v_i \;\; -5 \cdot w_i \;\; -u_i \right]^{\mathrm{T}}$$

$$p_3(3)^{\mathrm{T}} = [-18 \;\; 18 \;\; 0 \;\; -30 \;\; 30 \;\; -6]$$

$$p_4(i) := \frac{1}{2} \cdot \left[-v_i \;\; -w_i \;\; 0 \;\; -v_i \;\; -w_i \;\; 0 \right]^{\mathrm{T}}$$

$$p_4(4)^{\mathrm{T}} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

Fig. 4.11.5

When finding the concentrated forces equivalent to a distributed external load acting on rod finite elements, the found reactions are not of interest: –30, 30 and –6 in the rigid embedding of the rod inclined section (3–4, the support 4) under the action of a distributed load, because the forces are sought: F_1, F_2, F_3, F_4 and F_5 , which are applied in the nodes connecting the rod finite elements – two racks (Fig. 4.10.1 a and b): (1–2) and (3–5), the inclined section (3–4) and horizontal crossbar (2–3), corresponding to the main variables sought – the displacements of free (the loose) nodes: Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5.

The number of components of the vector of concentrated forces Q, equivalent to the distributed external load when acting on the rod finite elements, is equal to five, as is the number of components of the vector of applied concentrated forces, since in the issue there

are five unknown degrees of freedom of loose nodes. And the inclined rod in the support is subject to forces that are opposite to the reactions found at its loose end.

The determining in the global coordinate system of vector components of concentrated forces Q, equivalent to a distributed external load acting on the rod finite elements, the summation of the found vector with assigned concentrated forces F', acting on the nodes of the finite elements and determination of the external load vector F, acting on the nodes of statically indefinable flat frame, is performed in the Mathcad (fragment of page is in the diagram, Fig. 4.11.6).

$$Q_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad Q_2 := \begin{bmatrix} 0 & -50 & 40 & 0 & -30 \end{bmatrix}^T \quad Q_3 := \begin{bmatrix} 0 & 0 & 0 & 18 & -18 \end{bmatrix}^T$$

$$Q_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$Q := Q_1 + Q_2 + Q_3 + Q_4 \qquad F := F' + Q \qquad F^T = \begin{bmatrix} 50 & -50 & 40 & 18 & -48 \end{bmatrix}$$

Fig. 4.11.6

One needs to note that the preparing of initial data for their use in the calculation of mechanical systems using the finite element method, in particular, replacing the load, including distributed one applied to a mechanical system (to frame or other structure), with a load concentrated in nodes, can not always be completely formalized using a universal algorithm. For the individual elements of the mechanical system that have certain features, after determining the concentrated forces applied to each element, when this element is considered separately from the mechanical system, the formation of vectors or matrices of data is required, taking into account the methodological features related to each finite element or to all.

For example, after finding the expected reactions that occur when loading individual finite elements the issue may arise to redefine the external load, and the reference of the external load to the nodes of the mechanical system may change. At the same time, the number of methodological features, in particular, when finding forces on the side of fixed supports, is greater with a smaller number of finite elements, when the number of reactions from the supports is proportional to the number of elements in the mechanical system.

On the other hand, the need to redefine the external load after finding the support reactions for individual finite elements loaded with given external forces is greater with a variety of types of finite elements and forms of their connection in a mechanical system.

For these reasons, it is relevant to consider the preparing of initial data as a separate issue that precedes the use of the finite element method to determine forces in finite elements, the displacements of nodes in which finite elements are connected, and to find the support reactions for statically indefinable mechanical systems.

Questions and comments:

4.11.1. The action of reactions on each finite element at the joints with other finite elements or with supports, which are equivalent in magnitude to the total distributed and concentrated external load acting on this element and are opposite to this load.

4.11.2. Setting the distributed load acting on the frame using the finite element method and the Mathcad and converting distributed external forces to nodal forces, the number of which, like the number of concentrated external forces, corresponds to the number of degrees of freedom of loose nodes.

4.11.3. Determination of the vector of end forces from the distributed load acting on the finite element. The formal similarity in the global coordinate system used for a compact representation in the Mathcad and for determining the end forces from the influence of a distributed load on rod elements of different types (of four ones).

4.11.4. Forming the set of initial data (referring the distributed and concentrated loads to finite elements connection nodes) as the separate issue prior to the considered method use.

4.11.5. Methodological features of referring distributed load to the connection nodes of finite elements, associated with the ratio of the number of finite elements and supports of the mechanical system, as well as with the variety of types of finite elements and forms of their connection in the mechanical system.

4.12. Stiffness matrix of rod finite elements in the global coordinate system, determined in the Mathcad

The specification of the geometric characteristics of the frame in matrix and vector forms, as well as the determination in the Mathcad of external forces in the finite element nodes is carried out in the previous sections (the items 4.10 and 4.11).

Since the rod finite elements of four types and the local coordinate systems associated with them are multidirectional, then in order to find the total interactions of the finite elements at the nodes of the discrete model, the problem arises of converting the end reactions of each finite element into the global coordinate system of the discrete model.

In the global coordinate system there is a relationship between the end reactions $\{r\}_e$ in the rod finite element and its end displacements $\{\delta\}_e$, that cause the reactions, which one can represent in matrix form: $\{r\}_e = [k]_e \times \{\delta\}_e$, where $[k]_e$ – the stiffness matrix of the rod finite element in the global coordinate system, and the directions of forces in the global coordinate system corresponding to each element k_{ij} of the stiffness matrix are shown on the schemes of vector decomposition of the load acting on the ends

of the rod element (Fig. 4.12.1 a–f), resulting from displacements in each direction j: Δ_1, Δ_2, Δ_3, Δ_4, Δ_5 and Δ_6.

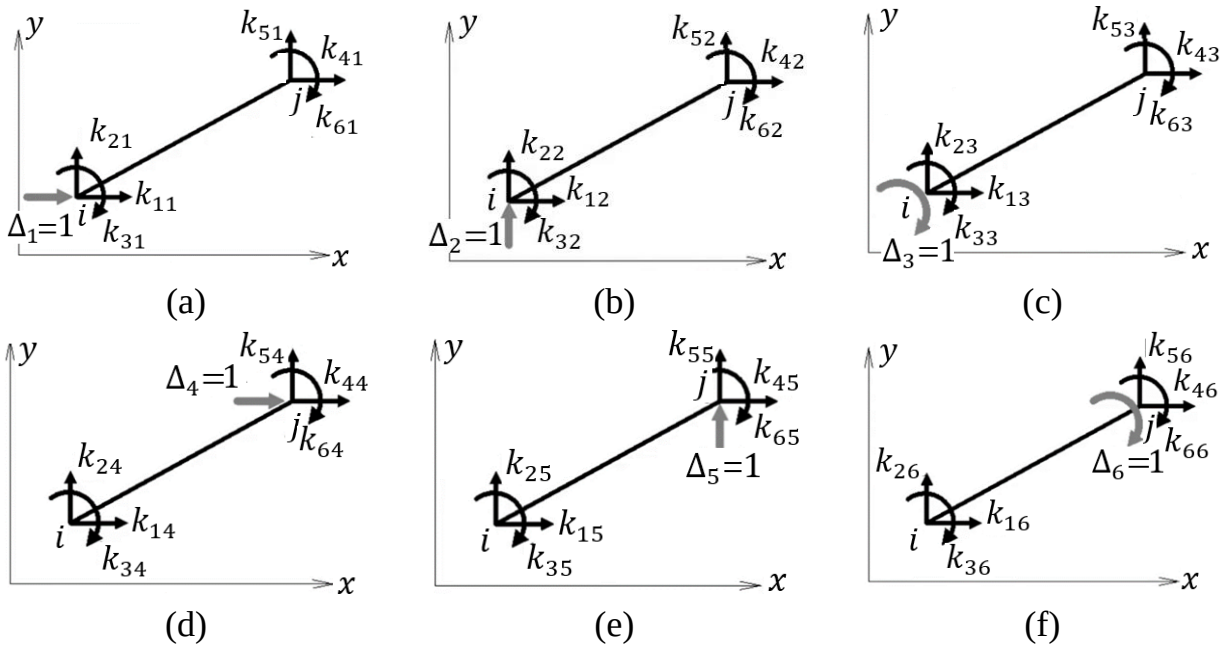


Fig. 4.12.1

Stiffness matrix of a finite element rod in the global coordinate system has the form:

$$[k]_e = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}.$$

Matrix elements in global coordinate system for the option of rigid nodes at the beginning and at the end of the rod finite element, which ends are rigidly connected to the fixed base and to other elements (in considered exercise the finite element of the 1–st type is rod 1–2) are:

$$k_{11} = k_{44} = -k_{14} = -k_{41} = f \times (\cos\alpha)^2 + \frac{12\times g}{l^2} \times (\sin\alpha)^2,$$

$$k_{22} = k_{55} = -k_{25} = -k_{52} = f \times (\sin\alpha)^2 + \frac{12\times g}{l^2} \times (\cos\alpha)^2,$$

$$k_{33} = k_{66} = 4 \times g,$$

$$k_{12} = k_{21} = k_{45} = k_{54} = -k_{15} = -k_{51} = -k_{24} = -k_{42} =$$

$$= \left(f - \frac{12\times g}{l^2}\right) \times \sin\alpha \times \cos\alpha,$$

$$k_{13} = k_{31} = k_{16} = k_{61} = -k_{46} = -k_{64} = -k_{34} = -k_{43} = \frac{6\times g}{l} \times \sin\alpha,$$

$$k_{35} = k_{53} = k_{56} = k_{65} = -k_{23} = -k_{32} = -k_{26} = -k_{62} = \frac{6\times g}{l} \times \cos\alpha,$$

$$k_{36} = k_{63} = 2 \times g.$$

Definition in the Mathcad of elements of the stiffness matrix in global coordinate system for the 1–st type rode finite element (fragment of the page in the Mathcad, Fig. 4.12.2).

$$a1_i := f_i \cdot C_i^2 + \frac{12 \cdot g_i}{l_i^2} \cdot S_i^2 \qquad a2_i := \left(f_i - \frac{12 \cdot g_i}{l_i^2} \right) \cdot S_i \cdot C_i \qquad a3_i := \frac{6 \cdot g_i}{l_i} \cdot S_i$$

$$a4_i := f_i \cdot S_i^2 + \frac{12 \cdot g_i}{l_i^2} \cdot C_i^2 \qquad a5_i := \frac{6 \cdot g_i}{l_i} \cdot C_i \qquad a6_i := 4 \cdot g_i \qquad a7_i := 2 \cdot g_i$$

Fig. 4.12.2

Combination of stiffness matrix elements in global coordinate system for the 1–st type rode finite element in the Mathcad (fragment of the page in the Mathcad, Fig. 4.12.3).

$$k1(i) := \begin{bmatrix} a1_i & a2_i & a3_i & -a1_i & -a2_i & a3_i \\ a2_i & a4_i & -a5_i & -a2_i & -a4_i & -a5_i \\ a3_i & -a5_i & a6_i & -a3_i & a5_i & a7_i \\ -a1_i & -a2_i & -a3_i & a1_i & a2_i & -a3_i \\ -a2_i & -a4_i & a5_i & a2_i & a4_i & a5_i \\ a3_i & -a5_i & a7_i & -a3_i & a5_i & a6_i \end{bmatrix}$$

Fig. 4.12.3

Elements of the stiffness matrix in the global coordinate system for rod finite elements of the 2–nd type, for which there is a rigid embedding of beginning of the rod element, and hinged connection of the end with other finite elements (in considered exercise, the finite element of the 2–nd type is rod 2–3):

$k_{11} = k_{44} = -k_{14} = -k_{41} = f \times (\cos\alpha)^2 + \frac{3 \times g}{l^2} \times (\sin\alpha)^2,$

$k_{22} = k_{55} = -k_{25} = -k_{52} = f \times (\sin\alpha)^2 + \frac{3 \times g}{l^2} \times (\cos\alpha)^2,$

$k_{33} = 3 \times g,$

$k_{12} = k_{21} = k_{45} = k_{54} = -k_{15} = -k_{51} = -k_{24} = -k_{42} =$

$= \left(f - \frac{3 \times g}{l^2} \right) \times \sin\alpha \times \cos\alpha,$

$k_{13} = k_{31} = -k_{34} = -k_{43} = \frac{6 \times g}{l} \times \sin\alpha,$

$k_{35} = k_{53} = -k_{23} = -k_{32} = \frac{3 \times g}{l} \times \cos\alpha,$

$k_{16} = k_{26} = k_{36} = k_{46} = k_{56} = k_{66} = k_{61} = k_{62} = k_{63} = k_{64} = k_{65} = 0.$

Determination in the Mathcad of the elements of stiffness matrix in global coordinate system for a rod finite element of 2–nd type (fragment of the page in the Mathcad, Fig. 4.12.4).

$$b1_i := f_i \cdot C_i^2 + \frac{3 \cdot g_i}{l_i^2} \cdot S_i^2 \qquad b2_i := \left(f_i - \frac{3 \cdot g_i}{l_i^2} \right) \cdot S_i \cdot C_i \qquad b3_i := \frac{6 \cdot g_i}{l_i} \cdot S_i$$

$$b4_i := f_i \cdot S_i^2 + \frac{3 \cdot g_i}{l_i^2} \cdot C_i^2 \qquad b5_i := \frac{3 \cdot g_i}{l_i} \cdot C_i \qquad b6_i := 3 \cdot g_i$$

Fig. 4.12.4

Combination of stiffness matrix elements in global coordinate system for rod finite element of 2–nd type (fragment of the page in the Mathcad, Fig. 4.12.5).

$$k2(i) := \begin{bmatrix} b1_i & b2_i & b3_i & -b1_i & -b2_i & 0 \\ b2_i & b4_i & -b5_i & -b2_i & -b4_i & 0 \\ b3_i & -b5_i & b6_i & -b3_i & b5_i & 0 \\ -b1_i & -b2_i & -b3_i & b1_i & b2_i & 0 \\ -b2_i & -b4_i & b5_i & b2_i & b4_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.12.5

The stiffness matrix in global coordinate system of 3–rd type rod finite element, at the beginning of which there is a hinge connection with other finite elements, and at the end there is a rigid embedding (in assigned issue the finite element of 3–rd type is rod 3–4, fragment of the page in the Mathcad, Fig. 4.12.6).

$$k3(i) := \begin{bmatrix} b1_i & b2_i & 0 & -b1_i & -b2_i & b3_i \\ b2_i & b4_i & 0 & -b2_i & -b4_i & -b5_i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b1_i & -b2_i & 0 & b1_i & b2_i & -b3_i \\ -b2_i & -b4_i & 0 & b2_i & b4_i & b5_i \\ b3_i & -b5_i & 0 & -b3_i & b5_i & b6_i \end{bmatrix}$$

Fig. 4.12.6

The stiffness matrix in global coordinate system of 4–th type rod finite element, both ends of which are hingedly connected to other rod finite elements or to a support (in assigned issue the finite element 4–th type is rod 3–5, fragment of the page in the Mathcad, Fig. 4.12.7).

$$c1_i := f_i \cdot C_i^2 \qquad c2_i := f_i \cdot S_i \cdot C_i \qquad c3_i := f_i \cdot S_i^2$$

$$k4(i) := \begin{bmatrix} c1_i & c2_i & 0 & -c1_i & -c2_i & 0 \\ c2_i & c3_i & 0 & -c2_i & -c3_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c1_i & -c2_i & 0 & c1_i & c2_i & 0 \\ -c2_i & -c3_i & 0 & c2_i & c3_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.12.7

The vector of four considered types of rod finite elements is introduced. To generalize the stiffness matrices of finite element rods in the global coordinate system into a general array in the Mathcad, the function can be used: if (the Mathcad page fragment, Fig. 4.12.8).

$$t := [1 \;\; 2 \;\; 3 \;\; 4]^T \qquad k(i) := \text{if}\left(t_i > 3, k4(i), \text{if}\left(t_i > 2, k3(i), \text{if}\left(t_i > 1, k2(i), k1(i)\right)\right)\right)$$

Fig. 4.12.8

Values of elements of stiffness matrices of rod finite elements: $k(1)$ for rod element 1–2, $k(2)$ for horizontal crossbar 2–3, $k(3)$ for a sloping section 3–4 and $k(4)$ for vertical stand 3–5 (fragment of the page in the Mathcad is on the diagram, Fig. 4.12.9).

$$k(1)=\begin{bmatrix} 0.188 & 0 & 0.375 & -0.188 & 0 & 0.375 \\ 0 & 25 & 0 & 0 & -25 & 0 \\ 0.375 & 0 & 1 & -0.375 & 0 & 0.5 \\ -0.188 & 0 & -0.375 & 0.188 & 0 & -0.375 \\ 0 & -25 & 0 & 0 & 25 & 0 \\ 0.375 & 0 & 0.5 & -0.375 & 0 & 1 \end{bmatrix} \quad k(2)=\begin{bmatrix} 25 & 0 & 0 & -25 & 0 & 0 \\ 0 & 0.047 & -0.188 & 0 & -0.047 & 0 \\ 0 & -0.188 & 0.75 & 0 & 0.188 & 0 \\ -25 & 0 & 0 & 25 & 0 & 0 \\ 0 & -0.047 & 0.188 & 0 & 0.047 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k(3)=\begin{bmatrix} 7.215 & -9.588 & 0 & -7.215 & 9.588 & -0.192 \\ -9.588 & 12.809 & 0 & 9.588 & -12.809 & -0.072 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -7.215 & 9.588 & 0 & 7.215 & -9.588 & 0.192 \\ 9.588 & -12.809 & 0 & -9.588 & 12.809 & 0.072 \\ -0.192 & -0.072 & 0 & 0.192 & 0.072 & 0.6 \end{bmatrix} \quad k(4)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.12.9

Questions and comments:

4.12.1. Transformation of stiffness matrices of rod finite elements from multidirectional local coordinate systems into a common global coordinate system to find the total interactions of finite elements at the nodes of a discrete model.

4.12.2. Dependencies between the end reactions in the rod finite element and its end displacements that cause them, as well as the stiffness matrix of the rod finite element in the global coordinate system.

4.12.3. Form of the stiffness matrix of a rod finite element in the global coordinate system. Determining the stiffness matrix of rod finite element in the Mathcad.

4.12.4. The elements of the stiffness matrix of a rod finite element of the first type, both ends of which are in rigid embeddings when connected to a support or to other finite elements.

4.12.5. The elements of the stiffness matrix of a rod finite element of the second type, at the beginning of which there is a rigid embedding with other finite elements, and at the end there is a hinged connection.

4.12.6. Elements of the stiffness matrix of a rod finite element of the third type, at the beginning of which there is a hinged connection with other finite elements, and at the end there is rigid embedding.

4.12.7. The elements of the stiffness matrix of a rod finite element of fourth type, both ends of which are hingedly connected to other rod finite elements or to a support.

4.12.8. Generalization in the Mathcad of stiffness matrices of finite element bars in the global coordinate system into a common array using the function: if.

4.13. Determination in the Mathcad of the incidence matrix and stiffness matrix of a discrete model when calculating frame using the finite element method

Determining of the geometric characteristics of the frame in matrix and vector forms, as well as the determination in the Mathcad of external forces in the finite element nodes is performed above (items 4.10–4.12). To find the support reactions of a statically indefinable flat frame consisting of rod finite elements and to determine the displacements of frame nodes not fixed in supports one needs to form a stiffness matrix of discrete model (Fig. 4.10.1 b) with use of the individual finite elements matrices, which combines these finite elements in addition to the stiffness matrices of individual finite elements.

The condition of balance of nodes of discrete model has a matrix form: $\{R\} - \{F\} = 0$, where $\{F\}$ – the vector of external forces; $\{R\}$ – the vector of nodal reactions. The vector of nodal reactions $\{R\}$ of a linearly deformable mechanical system is associated with the vector of displacements of nodes $\{\Delta\}$ in system of canonical equations as a function of displacements, which can be represented in matrix form as: $\{R\} = [K] \times \{\Delta\}$.

Discrete model stiffness matrix $[K]$ has the form:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \dots & \dots & \dots & \dots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix}.$$

Each element K_{ij} of the stiffness matrix $[K]$ of discrete model serves as a proportionality coefficient of the nodal reaction R_i, caused by the nodal displacement: $\Delta_j = 1$ with zero displacements of the remaining nodes.

Initially, all the stiffness matrices of individual elements are combined into a quasi–diagonal matrix, the dimension of which corresponds to the sum of the dimensions of all finite element stiffness matrices, and these finite element matrices are located diagonally of indicated quasi–diagonal matrix:

$$\left[\overline{K}\right] = \begin{bmatrix} [K]_1 & 0 & \dots & 0 \\ 0 & [K]_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & [K]_m \end{bmatrix}.$$

To create quasi–diagonal matrix, a matrix of zero elements is preliminarily formed, the dimension to which corresponds to the quasi–diagonal matrix (fragment of the page in the Mathcad, Fig. 4.13.1).

To form a quasi–diagonal matrix, the stiffness matrices of individual finite elements are placed on diagonal of the matrix of zero elements. To form a quasi–diagonal matrix the «putregion» command can be used in the Mathcad, which is located on the Functions tab in the All functions panel.

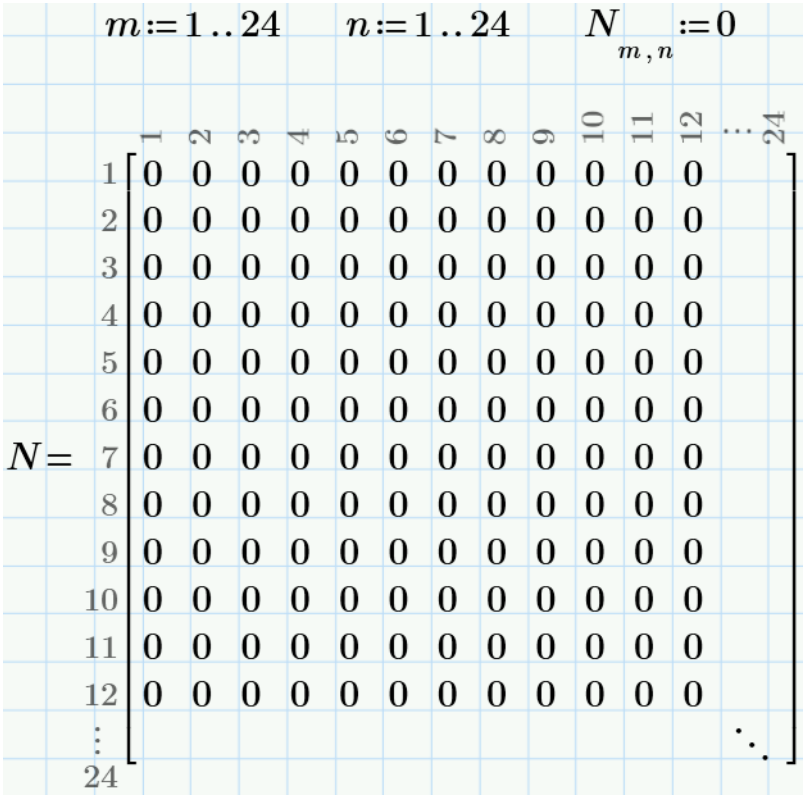


Fig. 4.13.1

The coordinates of insertions of finite element stiffness matrices on the diagonal one can determine using an algorithm in the Mathcad. This is shown (fragment of the page in the Mathcad, Fig. 4.13.2) and the use of such an algorithm in the Mathcad, which allows one to find the parameters for inserting the stiffness matrix of each finite element (the indices: 1, 7, 13 and 16 on the diagonal of the zero matrix correspond to the first elements of the placed finite element stiffness matrices in a given sequence). The use of this algorithm is convenient for large dimension of discrete model, which is for a large number of finite elements in its composition.

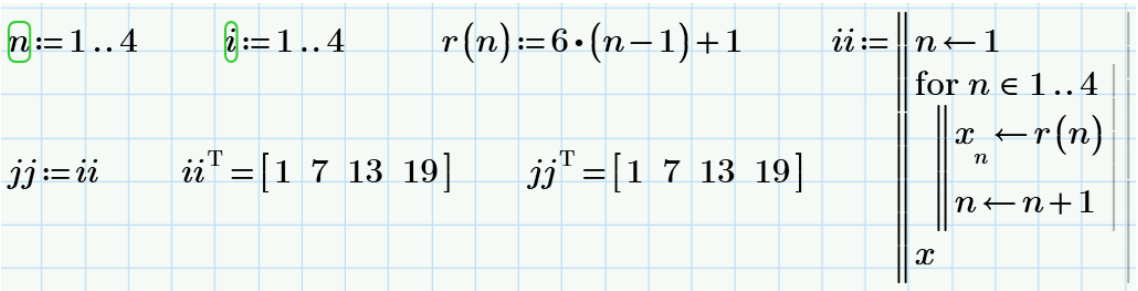


Fig. 4.13.2

The quasi–diagonal matrix is drawing up by including the finite element stiffness matrices in the zero element matrix using the «putregion» function located on the Functions tab in the All Functions panel (fragment of the page in the Mathcad, Fig. 4.13.3).

$KD(i) := \text{putregion}\left(N, k(i), ii_i, jj_i\right)$ $\quad KD := \sum_i^n KD(i)$

$KD =$

	1	2	3	4	5	6	7	8	9	10	11	12	…
1	0.188	0	0.375	−0.188	0	0.375	0	0	0	0	0	0	
2	0	25	0	0	−25	0	0	0	0	0	0	0	
3	0.375	0	1	−0.375	0	0.5	0	0	0	0	0	0	
4	−0.188	0	−0.375	0.188	0	−0.375	0	0	0	0	0	0	
5	0	−25	0	0	25	0	0	0	0	0	0	0	
6	0.375	0	0.5	−0.375	0	1	0	0	0	0	0	0	
7	0	0	0	0	0	0	25	0	0	−25	0	0	
8	0	0	0	0	0	0	0	0.047	−0.188	0	−0.047	0	
9	0	0	0	0	0	0	0	−0.188	0.75	0	0.188	0	
10	0	0	0	0	0	0	−25	0	0	25	0	0	
11	0	0	0	0	0	0	0	−0.047	0.188	0	0.047	0	
12	0	0	0	0	0	0	0	0	0	0	0	0	
⋮													⋱
24													

Fig. 4.13.3

The quasi–diagonal matrix describes the elastic properties of finite elements that are not interconnected. Since the finite elements have common connection nodes, they need to be combined into a common discrete model. For this, a structural matrix $[I]$, is formed, which is an incidence matrix of correspondences of displacements of the ends of each finite element and of displacements of nodes of the discrete model.

Each column of the incidence matrix $[I]$ corresponds to one of the generalized displacements of discrete model, and each row corresponds to one end displacement of the finite element. Wherein, if the displacement of the end of the rod corresponds to the displacement of the node of discrete model, the element of the structural matrix $[I]$ is equal to one, and if the displacement of the end of the rod does not mean the displacement of the node of discrete model – the element of the structural matrix $[I]$ is equal to zero.

A unit in the Table at the intersection of a certain column and a certain row corresponds to the equality of the nodal displacement to the end one; otherwise, zeros are indicated in the Table. Single and zero values of the elements of the incidence matrix correspond to the use of stiffness matrices of individual finite elements in global coordinates.

And if the stiffness matrices of individual finite elements were formed in local coordinate systems associated with each finite element separately, then the values of the elements of the incidence matrix would correspond to the direction cosines of the displacements of the ends of the rod finite elements in the global system of discrete model.

A non–zero value of an element of the incidence matrix indicates the relationship between the displacements of the ends of the rod finite element and the displacements of the nodes of the discrete model (their incidence, in other words, compatibility). It is more convenient to type the incidence matrix in the form of a table in EXCEL (see fragment of

the page, Fig. 4.13.4). In the EXCEL table, the designations of the rod finite elements (1–2, 2–3, 2–4 and 3–4), are entered, along the lines – the designations of displacements: δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6 of the ends of each rod finite element and incidence values of displacements of the ends of each of rod elements and along the columns – the node displacements of discrete model: Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 in global coordinates.

	A	B	C	D	E	F	G	H
1				Displacements of Discrete Model				
2		Finite Element	Displacement	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
3	1	1 – 2	δ_1	0	0	0	0	0
4	2		δ_2	0	0	0	0	0
5	3		δ_3	0	0	0	0	0
6	4		δ_4	1	0	0	0	0
7	5		δ_5	0	1	0	0	0
8	6		δ_6	0	0	1	0	0
9	7	2 – 3	δ_1	1	0	0	0	0
10	8		δ_2	0	1	0	0	0
11	9		δ_3	0	0	1	0	0
12	10		δ_4	0	0	0	1	0
13	11		δ_5	0	0	0	0	1
14	12		δ_6	0	0	0	0	0
15	13	3 – 4	δ_1	0	0	0	1	0
16	14		δ_2	0	0	0	0	1
17	15		δ_3	0	0	0	0	0
18	16		δ_4	0	0	0	0	0
19	17		δ_5	0	0	0	0	0
20	18		δ_6	0	0	0	0	0
21	19	3 – 5	δ_1	0	0	0	1	0
22	20		δ_2	0	0	0	0	1
23	21		δ_3	0	0	0	0	0
24	22		δ_4	0	0	0	0	0
25	23		δ_5	0	0	0	0	0
26	24		δ_6	0	0	0	0	0

Fig. 4.13.4

After filling out the EXCEL table one needs to copy it from the EXCEL program into the Mathcad computer algebra system and redefine it as a matrix. In addition, if the files: the EXCEL and the Mathcad are located in a shared folder in the directory, then one can use the READEXCEL function, which is located on the Input/Output tab, to transfer an array of data from EXCEL to Mathcad (the fragment of the page in the Mathcad, Fig. 4.13.5). The incidence matrix in considered exercise contains 5 columns and 24 rows (6×4).

The stiffness matrix $[K]$ of discrete model is searched as the product of three matrices in succession: the transposed incidence matrix $[I]^T$, the quasi–diagonal stiffness matrix $\left[\overline{K}\right]$ and the original incidence matrix $[I]$: $[K] = [I]^T \times \left[\overline{K}\right] \times [I]$. The stiffness matrix of discrete model is determined in the Mathcad (fragment of the page, Fig. 4.13.6).

$$I := \text{READEXCEL}(\text{".\Incidence matrix FEM calculation of a statically indeterminate frame in Mathcad.xlsx"}, \text{"Sheet1!D3:H26"})$$

$$I^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.13.5

$$K := I^{\mathrm{T}} \cdot KD \cdot I$$

$$K = \begin{bmatrix} 25.188 & 0 & -0.375 & -25 & 0 \\ 0 & 25.047 & -0.188 & 0 & -0.047 \\ -0.375 & -0.188 & 1.75 & 0 & 0.188 \\ -25 & 0 & 0 & 32.215 & -9.588 \\ 0 & -0.047 & 0.188 & -9.588 & 37.856 \end{bmatrix}$$

Fig. 4.13.6

Questions and comments:

4.13.1. Formation of a stiffness matrix of a discrete model from the stiffness matrices of finite elements of a statically indeterminate mechanical system to determine the reactions of its supports and the movements of free nodes.

4.13.2. Matrix form of the condition of the nodes balance of discrete model of a linearly deformable mechanical system as the equality to zero of the difference between the vectors of nodal reactions and external forces.

4.13.3. Matrix form of system of the canonical equations in the displacement method when determining the vector of nodal reactions of a linearly deformable mechanical system.

4.13.4. Drawing up the quasi–diagonal matrix, the dimension of which corresponds to the sum of the dimensions of the stiffness matrices of all finite elements of the mechanical system. The drawing up of a zero matrix in the Mathcad used to determine the quasi–diagonal matrix. The use of algorithm in the Mathcad to find the insertion coordinates of finite element stiffness matrices in a quasi–diagonal matrix.

4.13.5. The need to create and use the stiffness matrix of discrete model when calculating a mechanical system using the finite element method, since the quasi–diagonal matrix, like the stiffness matrices of individual finite elements, describes the elastic properties of finite elements which are not interconnected.

4.13.6. The use of incidence matrix (the structural), showing the relationship between the displacements of the ends of the rod finite element and the displacements of the nodes of the discrete model.

4.13.7. Finding in the Mathcad the stiffness matrix of a discrete model as a sequential product of three matrices: the transposed incidence matrix on the quasi–diagonal matrix and on the incidence matrix.

4.13.8. Drawing up the incidence matrix in the form of table in the EXCEL program, copying the incidence table from the EXCEL into the Mathcad and redefining the table in the Mathcad as the incidence matrix. The use of the READEXCEL function when copying a table to the Mathcad file.

4.14. Finding nodal displacements when calculating a frame in the Mathcad using the finite element method, converting the displacement matrix to local coordinates

The vector of nodal displacements is given by the equation: $\{\Delta\} = [K]^{-1} \times \{F\}$, where $[K]^{-1}$ – the matrix inverse to the stiffness matrix $[K]$.

Element K_{ij}^{-1} of the matrix $[K]^{-1}$, inverse of the stiffness matrix $[K]$, is determined as: $K_{ij}^{-1} = (-1)^{i+j} \times \frac{M_{ji}}{Det[K]}$, where $Det[K]$ – the determinant of the stiffness matrix $[K]$, M_{ij} – the minor of the element K_{ij} in the matrix determinant.

The vector of nodal displacements $\{\Delta\}$ and the matrix $[K]^{-1}$, the inverse of the stiffness matrix are determined in the Mathcad computer algebra system (fragment of the page in the Mathcad, Fig. 4.14.1).

$$K^{-1} = \begin{bmatrix} 0.242 & 4.424 \cdot 10^{-4} & 0.046 & 0.203 & 0.051 \\ 4.424 \cdot 10^{-4} & 0.04 & 0.004 & 3.803 \cdot 10^{-4} & 1.242 \cdot 10^{-4} \\ 0.046 & 0.004 & 0.581 & 0.038 & 0.007 \\ 0.203 & 3.803 \cdot 10^{-4} & 0.038 & 0.204 & 0.051 \\ 0.051 & 1.242 \cdot 10^{-4} & 0.007 & 0.051 & 0.039 \end{bmatrix}$$

$$\Delta := K^{-1} \cdot F \qquad \Delta^{T} = [15.101 \quad -1.8 \quad 25.702 \quad 12.829 \quad 1.852]$$

Fig. 4.14.1

After finding the nodal displacements in the global coordinate system, one can find the force vectors acting in the local coordinate systems in each of the finite elements. The transition to local coordinate systems in considered issue arises due to the fact that the forces arising in the rod finite elements depend on the relative displacements of the ends of the rod elements. The stiffness matrices of the finite elements of the frame in local

coordinate systems defined below do not depend on those shown in the diagram (item 4.10, fragment of the page in the Mathcad, Fig. 4.10.5) vectors of sines and cosines of direction angles for each finite element, which were taken into account when determining the stiffness matrices of rod finite elements in the global system coordinates (item 4.12, fragment of the page in the Mathcad, Fig. 4.12.2 and 4.12.4 and resulting finite element stiffness matrices on fragments of the pages in the Mathcad, Fig. 4.12.3 and 4.12.5–4.12.9).

The stiffness matrices in local coordinate systems for finite elements of four types are determined in the Mathcad according to the assigned order of numbers of these finite elements (it would be mentioned that in the considered issue, the number of the finite element i is equal to the number of its type): 1–st type, the ends of which are rigidly connected to a fixed base and/or to another finite element (in exercise this is finite element number 1, that is 1–2: $[k_1']$, fragment of the page in the Mathcad, Fig. 4.14.2); 2–nd type, for a horizontal crossbar 2–3, at the beginning of which there is a rigid embedding, and at the end a hinged connection with other finite elements ($[k_2']$, fragment of the page in the Mathcad, Fig. 4.14.3); 3–rd type, for inclined section 3–4, at the beginning of which there is a hinged connection, and at the end there is a rigid embedding ($[k_3']$, fragment of the page in the Mathcad, Fig. 4.14.4); 4–th type, for vertical column 3–5, both ends of which are hinged ($[k_4']$, fragment of the page in the Mathcad, Fig. 4.14.5).

$$k1'(i) := \begin{bmatrix} f_i & 0 & 0 & -f_i & 0 & 0 \\ 0 & \frac{12 \cdot g_i}{l_i^2} & -\frac{6 \cdot g_i}{l_i} & 0 & -\frac{12 \cdot g_i}{l_i^2} & -\frac{6 \cdot g_i}{l_i} \\ 0 & -\frac{6 \cdot g_i}{l_i} & 4 \cdot g_i & 0 & \frac{6 \cdot g_i}{l_i} & 2 \cdot g_i \\ -f_i & 0 & 0 & f_i & 0 & 0 \\ 0 & -\frac{12 \cdot g_i}{l_i^2} & \frac{6 \cdot g_i}{l_i} & 0 & \frac{12 \cdot g_i}{l_i^2} & \frac{6 \cdot g_i}{l_i} \\ 0 & -\frac{6 \cdot g_i}{l_i} & 2 \cdot g_i & 0 & \frac{6 \cdot g_i}{l_i} & 4 \cdot g_i \end{bmatrix}$$

Fig. 4.14.2

$$k2'(i) := \begin{bmatrix} f_i & 0 & 0 & -f_i & 0 & 0 \\ 0 & \frac{3 \cdot g_i}{l_i^2} & -\frac{3 \cdot g_i}{l_i} & 0 & -\frac{3 \cdot g_i}{l_i^2} & 0 \\ 0 & -\frac{3 \cdot g_i}{l_i} & 3 \cdot g_i & 0 & \frac{3 \cdot g_i}{l_i} & 0 \\ -f_i & 0 & 0 & f_i & 0 & 0 \\ 0 & -\frac{3 \cdot g_i}{l_i^2} & \frac{3 \cdot g_i}{l_i} & 0 & \frac{3 \cdot g_i}{l_i^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.14.3

$$k3'(i) := \begin{bmatrix} f_i & 0 & 0 & -f_i & 0 & 0 \\ 0 & \frac{3 \cdot g_i}{l_i^2} & 0 & 0 & -\frac{3 \cdot g_i}{l_i^2} & -\frac{3 \cdot g_i}{l_i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -f_i & 0 & 0 & f_i & 0 & 0 \\ 0 & -\frac{3 \cdot g_i}{l_i^2} & 0 & 0 & \frac{3 \cdot g_i}{l_i^2} & \frac{3 \cdot g_i}{l_i} \\ 0 & -\frac{3 \cdot g_i}{l_i} & 0 & 0 & \frac{3 \cdot g_i}{l_i} & 3 \cdot g_i \end{bmatrix}$$

Fig. 4.14.4

$$k4'(i) := \begin{bmatrix} f_i & 0 & 0 & -f_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -f_i & 0 & 0 & f_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4.14.5

To generalize the stiffness matrices in local coordinate systems of four types of bar finite elements into a common array in the Mathcad, the function can be used: if (fragment of the page in the Mathcad, Fig. 4.14.6).

$$k'(i) := \text{if}\left(t_i > 3, k4'(i), \text{if}\left(t_i > 2, k3'(i), \text{if}\left(t_i > 1, k2'(i), k1'(i)\right)\right)\right)$$

Fig. 4.14.6

Questions and comments:

4.14.1. Matrix form of canonical equations for determining the displacements of free nodes of discrete model of a mechanical system. Determining the inverse matrix with respect to the stiffness matrix of the discrete model and multiplying it by the vector of the acting external load when finding the displacement vector of free nodes.

4.14.2. The determination in the Mathcad of the inverse matrix with respect to the stiffness matrix of a discrete model and the displacements of free (the loose) nodes of a mechanical system (a given statically indefinable frame) in the global coordinate system.

4.14.3. Finding the stiffness matrices of rod finite elements in local coordinate systems after determining the displacements of loose nodes in the global coordinate system to find the forces in rod finite elements corresponded with their deformations (relative movements of the ends of rod elements in local coordinate systems associated with these elements).

4.14.4. Determination in the Mathcad of the stiffness matrices of rod finite elements in local coordinate systems without the use of vectors of sines and cosines of direction angles for each finite element, which were taken into account when determining the stiffness matrices of rod finite elements in the global coordinate system.

4.14.5. Combining stiffness matrices in local coordinate systems of rod finite elements into a common array in the Mathcad using the function: if.

4.15. Determination in the Mathcad of forces in finite elements in a local coordinate system under end reactions and under external load, transformation of the displacement matrix to local coordinates

Vector R (Fig. 4.15.1) in the global coordinate system (ixy) is the sum of projections: r_1 and r_2, and in the local coordinate system (ix_1y_1), which is rotated by angle α, with respect to the global one, the vector R is the sum of projections: r_1' and r_2' on the axes. The magnitude of the moment on the plane when rotating the coordinate axes does not change.

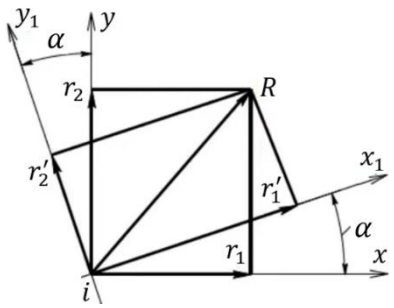


Fig. 4.15.1

The converting the end characteristics of a finite element, specified in the global coordinate system, into the corresponding characteristics in the local system, which is turned in the global coordinate system by an angle α, for the initial $\{r'\}_i$ and for the end $\{r'\}_j$ rod sections for each finite element has the matrix form:

$$\begin{Bmatrix} r_1' \\ r_2' \\ r_3' \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_i \times \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}_i \text{ and } \begin{Bmatrix} r_4' \\ r_5' \\ r_6' \end{Bmatrix}_j = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_j \times \begin{Bmatrix} r_4 \\ r_5 \\ r_6 \end{Bmatrix}_j$$

In collapsed form, the transformation of the end characteristics of a finite element, specified in the global coordinate system, into the corresponding characteristics in the local system has the form: $\{r'\}_i = [T]_i \times \{r\}_i$.

To convert the end forces specified in the global coordinate system into the corresponding forces in the local coordinate system of the finite element, an auxiliary matrix (of cosines and sines) is used:

$$[T]_e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(fragment of the page in the Mathcad, Fig. 4.15.2).

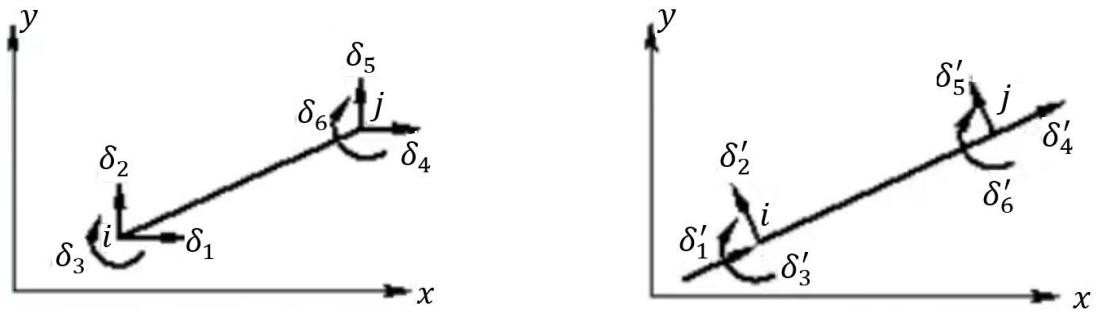
$$T(i) := \begin{bmatrix} C_i & S_i & 0 & 0 & 0 & 0 \\ -S_i & C_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_i & S_i & 0 \\ 0 & 0 & 0 & -S_i & C_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 4.15.2

It is required to find the end forces $\{s'\}_e$ of the rod finite elements in local coordinate systems, that is the internal forces corresponding to the longitudinal force, the shear force and the bending moment. End forces are equal to the sum of reactions $\{r'\}_e$ from other finite elements and from the external load $\{p'\}_e$ (focused and distributed), applied to each finite element and reduced to the ends of the element (to the nodes of the discrete model): $\{s'\}_e = \{r'\}_e + \{p'\}_e = [k']_e \times [T]_e \times \{\delta\}_e + \{p'\}_e$.

Vector of the end displacements of the finite element in the local coordinate system: $\{\delta'\}_e = [T]_e \times \{\delta\}_e$ is determined using an auxiliary matrix (cosines and sines) $[T]_e$ to move from the global coordinate system of the discrete model, in which the displacements of nodes are sought, to the local coordinate system of the finite element.

The relationship between the end displacements of rod finite element in the global coordinate system $\{\delta\}_e$ and in the local coordinate system $\{\delta'\}_e$ is shown on the finite element diagram (Fig. 4.15.3 a and b).



(a) (b)

Fig. 4.15.3

When drawing up vectors of end displacements of rod finite elements in the global coordinate system, one can use the scheme of discrete model for clarity (Fig. 4.15.4).

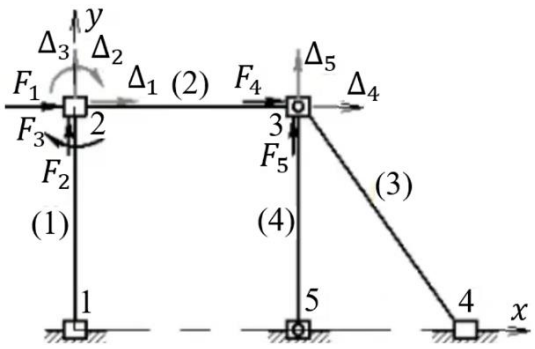


Fig. 4.15.4

One needs to note that the sizes of matrices and vectors (that is the number of components of the vectors, rows and columns of matrices) when using the finite element method in calculation of mechanical systems are associated both with the vector geometry

of mechanical systems and with matrix mathematical operations on data arrays whose size corresponds to the number of finite elements. The number of components of the displacement vector of a node on the plane is three, and in space – six, the number of components of displacements of the two ends of the rod finite element on the plane is six, etc.

The end forces $\{s'\}_e$ for each end of the rod finite element have three components (the projections on the local axes, two linear projections and one angular projection), and the total number of end force components for each rod finite element on the plane is six: $\{s'\}_e = \{\{s'\}_i, \{s'\}_j\} = \{s'_1, s'_2, s'_3, s'_4, s'_5, s'_6\}$.

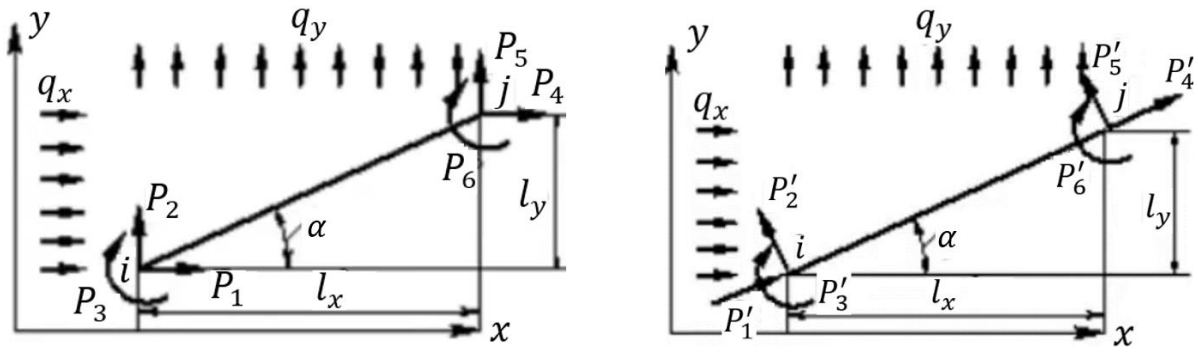
Vectors of end displacements $\{\delta\}_e$ of finite elements in the local coordinate system are determined in the Mathcad by multiplying the auxiliary matrix by the vector of displacements of discrete model nodes in the global coordinate system, and each of these components, in turn, is a four–component vector, to be more exact – the multicomponent one according to the number of finite elements (fragment of the page in the Mathcad, Fig. 4.15.5).

$$\delta 1 := \begin{bmatrix} 0 \\ \Delta_1 \\ \Delta_4 \\ \Delta_4 \end{bmatrix} \quad \delta 2 := \begin{bmatrix} 0 \\ \Delta_2 \\ \Delta_5 \\ \Delta_5 \end{bmatrix} \quad \delta 3 := \begin{bmatrix} 0 \\ \Delta_3 \\ 0 \\ 0 \end{bmatrix} \quad \delta 4 := \begin{bmatrix} \Delta_1 \\ \Delta_4 \\ 0 \\ 0 \end{bmatrix} \quad \delta 5 := \begin{bmatrix} \Delta_2 \\ \Delta_5 \\ 0 \\ 0 \end{bmatrix} \quad \delta 6 := \begin{bmatrix} \Delta_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\delta(i) := \begin{bmatrix} \delta 1_i & \delta 2_i & \delta 3_i & \delta 4_i & \delta 5_i & \delta 6_i \end{bmatrix}^{\mathrm{T}}$$

Fig. 4.15.5

Vectors of end forces from a distributed load in the global coordinate system (Fig. 4.15.6 a) are found above (section 4.11, Fig. 4.11.2).



(a) (b)

Fig. 4.15.6

For a finite element of the 1–st type, the ends of which are rigidly connected to a fixed base and/or to another finite element (in exercise this is finite element number 1, that

is 1–2), in the local coordinate system the vector of end forces $\{p'\}$ under the influence of a distributed load has the form:

$$\{p'\} = \begin{Bmatrix} -\frac{q_x \times l_y \times \cos\alpha + q_y \times l_x \times \sin\alpha}{2} \\ -\frac{q_y \times l_x \times \cos\alpha - q_x \times l_y \times \sin\alpha}{2} \\ \frac{-q_x \times l_y \times l \times \sin\alpha + q_y \times l_x \times l \times \cos\alpha}{12} \\ -\frac{q_x \times l_y \times \cos\alpha + q_y \times l_x \times \sin\alpha}{2} \\ -\frac{q_y \times l_x \times \cos\alpha + q_x \times l_y \times \sin\alpha}{2} \\ \frac{q_x \times l_y \times l \times \sin\alpha - q_y \times l_x \times l \times \cos\alpha}{12} \end{Bmatrix}.$$

The symbols correspond to the scheme of the rod element (Fig. 4.15.6 b).

For a finite element of the 2–nd type, which is the horizontal crossbar number 2, that is 2–3, at the beginning of which there is a rigid embedding, and at the end a hinged connection with other finite elements, in the local coordinate system the vector of end forces $\{p'\}$ under the influence of a distributed load is:

$$\{p'\} = \begin{Bmatrix} -\frac{5 \times q_x \times l_y \times \cos\alpha + 5 \times q_y \times l_x \times \sin\alpha}{8} \\ -\frac{5 \times q_x \times l_y \times \cos\alpha - 5 \times q_y \times l_x \times \sin\alpha}{8} \\ \frac{-q_x \times l_y \times l \times \sin\alpha + q_y \times l_x \times l \times \cos\alpha}{8} \\ -\frac{3 \times q_x \times l_y \times \cos\alpha + 3 \times q_y \times l_x \times \sin\alpha}{8} \\ -\frac{3 \times q_y \times l_x \times \cos\alpha - 3 \times q_x \times l_y \times \sin\alpha}{8} \\ 0 \end{Bmatrix}.$$

For a finite element of the 3–rd type, which is the inclined section number 3, that is 3–4, at the beginning of which there is a hinged connection, and at the end there is a rigid embedding, in the local coordinate system the vector of end forces $\{p'\}$ under the influence of a distributed load one can determine as:

$$\{p'\} = \begin{Bmatrix} -\frac{3 \times q_x \times l_y \times \cos\alpha + 3 \times q_y \times l_x \times \sin\alpha}{8} \\ -\frac{3 \times q_y \times l_x \times \cos\alpha - 3 \times q_x \times l_y \times \sin\alpha}{8} \\ 0 \\ -\frac{5 \times q_x \times l_y \times \cos\alpha + 5 \times q_y \times l_x \times \sin\alpha}{8} \\ -\frac{5 \times q_y \times l_x \times \cos\alpha - 5 \times q_x \times l_y \times \sin\alpha}{8} \\ \frac{q_x \times l_y \times l \times \sin\alpha - q_y \times l_x \times l \times \cos\alpha}{8} \end{Bmatrix}.$$

For a finite element of the 4–th type, that is for vertical column number 4, that is 3–5, both ends of which are hinged, in the local coordinate system the vector of end forces $\{p'\}$ under the influence of a distributed load has the form:

$$\{p'\} = \begin{Bmatrix} -\frac{q_x \times l_y \times \cos\alpha + q_y \times l_x \times \sin\alpha}{2} \\ -\frac{q_y \times l_x \times \cos\alpha - q_x \times l_y \times \sin\alpha}{2} \\ 0 \\ -\frac{q_x \times l_y \times \cos\alpha + q_y \times l_x \times \sin\alpha}{2} \\ -\frac{q_y \times l_x \times \cos\alpha - q_x \times l_y \times \sin\alpha}{2} \\ 0 \end{Bmatrix}.$$

It is required to find vectors of end forces under this load in local coordinate systems (Fig. 4.15.6 b) for all rod finite elements included in the flat frame, taking into account the types of fastening of these elements. Since mathematical operators in the Mathcad are quite clear, one can generate the Mathcad calculation file without first writing down the formulas in text with sufficient experience in computer algebra. The symbols correspond to the diagram of the rod element (Fig. 4.15.6 b). The components of the end force vector under the distributed load within the finite rod element for different elements in the flat frame, as previously noted (item 4.12) have certain mathematical similarities in local co-ordinate systems, which one can use in the Mathcad to determine auxiliary vectors, and above in the considered issue the index of the number of rod finite elements in the frame was specified in the Mathcad (fragment of the page in the Mathcad, Fig. 4.15.7).

$a := \overrightarrow{q_x \cdot y \cdot C + q_y \cdot x \cdot S}$ $b := \overrightarrow{q_y \cdot x \cdot C + q_x \cdot y \cdot S}$ $d := \overrightarrow{q_x \cdot y \cdot l \cdot S + q_y \cdot x \cdot l \cdot C}$

Fig. 4.15.7

These auxiliary vectors allow compact representation of end force vectors for four types of rod finite elements in local coordinate systems: 1–st type, the ends of which are rigidly connected to a fixed base and/or to another finite element (in the task this is the finite element number 1: $i = 1$), 2–nd type, at the beginning of which there is rigid embedding, and at the end there is a hinged connection with other finite elements ($i = 2$), 3–rd type, at the beginning of which there is a hinged connection, and at the end there is a rigid embedding ($i = 3$) and 4–th type, both ends of which are hingedly supported ($i = 4$). End force vectors using auxiliary vectors for the listed four finite elements are defined in the Mathcad (fragment of the page in the Mathcad, Fig. 4.15.8):

$$p1'(i) := \frac{1}{2} \cdot \left[-a_i \;\; b_i \;\; \frac{1}{6} \cdot d_i \;\; -a_i \;\; b_i \;\; -\left(\frac{1}{6} \cdot d_i \right) \right]^T$$

$$p1'(1)^T = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$p2'(i) := \frac{1}{8} \cdot \left[-5 \cdot a_i \;\; 5 \cdot b_i \;\; d_i \;\; -3 \cdot a_i \;\; 3 \cdot b_i \;\; 0 \right]^T$$

$$p2'(2)^T = [0 \;\; -50 \;\; -40 \;\; 0 \;\; -30 \;\; 0]$$

$$p3'(i) := \frac{1}{8} \cdot \left[-3 \cdot a_i \;\; 3 \cdot b_i \;\; 0 \;\; -5 \cdot a_i \;\; 5 \cdot b_i \;\; -d_i \right]^T$$

$$p3'(3)^T = [-25.2 \;\; -25.2 \;\; 0 \;\; -42 \;\; -42 \;\; 42]$$

$$p4'(i) := \frac{1}{2} \cdot \left[-a_i \;\; b_i \;\; 0 \;\; -a_i \;\; b_i \;\; 0 \right]^T$$

$$p4'(4)^T = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

Fig. 4.15.8

To generalize vectors of end forces in local coordinate systems of rod finite elements into a common array in the Mathcad, the function can be used: if (fragment of the page in the Mathcad, Fig. 4.15.9).

$$p(i) := \text{if}\left(t_i > 3, p4'(i), \text{if}\left(t_i > 2, p3'(i), \text{if}\left(t_i > 1, p2'(i), p1'(i)\right)\right)\right)$$

Fig. 4.15.9

The end forces found in local coordinate systems in the finite elements of a given statically indefinable flat frame are determined in the Mathcad (fragment of page is in the diagram, Fig. 4.15.10).

$$s'(i) := k'(i) \cdot T(i) \cdot \delta(i) + p'(i)$$

$$s'(1) = \begin{bmatrix} 45.01 \\ -6.807 \\ 7.188 \\ -45.01 \\ 6.807 \\ 20.039 \end{bmatrix} \quad s'(2) = \begin{bmatrix} 56.807 \\ -54.99 \\ -20.039 \\ -56.807 \\ -25.01 \\ 0 \end{bmatrix} \quad s'(3) = \begin{bmatrix} 99.114 \\ -24.927 \\ 0 \\ -166.314 \\ -42.273 \\ 40.635 \end{bmatrix} \quad s'(4) = \begin{bmatrix} -46.297 \\ 0 \\ 0 \\ 46.297 \\ 0 \\ 0 \end{bmatrix}$$

Fig. 4.15.10

Questions and comments:

4.15.1. Matrix transformation from the local coordinate system to the global system (rotating the axes) and from the global coordinate system to the local system (cancelling the rotation of the axes) by multiplying the matrix of direction cosines and sines by the coordinate vector of the section of the rod finite element, for the initial and final sections of the element.

4.15.2. Determining the vector of end displacements in the local coordinate system by multiplying the auxiliary matrix (the cosines and sines) by the vectors of end displacements in the global coordinate system.

4.15.3. End forces in the rod finite elements of a mechanical system, i.e. the internal forces corresponding to the longitudinal force, transverse force and bending moment, equal to the sum of the forces acting on the rod finite element from other finite elements, and the external load acting on the finite element.

4.15.4. The number of end force components for a finite element on a plane (and in space), associated both with the vector geometry of mechanical systems and with data arrays whose size corresponds to the number of finite elements.

4.15.5. Determination of end displacement vectors of finite elements in the Mathcad in a local coordinate system, i.e. finding the components of these vectors, which, in turn, are multicomponent vectors according to the number of finite elements.

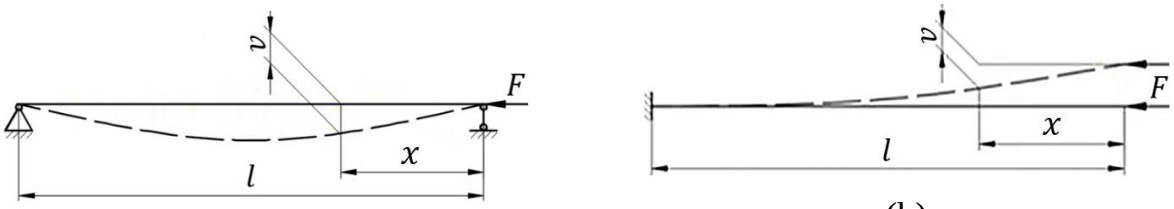
4.15.6. Finding auxiliary vectors in the Mathcad, taking into account existing similarities and mathematical analogies when determining the vector of end forces from the influence of a load applied within the finite rod element for other elements in the flat frame. The generalization of end force vectors in local coordinate systems of rod finite elements into a common array in the Mathcad.

4.15.8. Finding the end forces in local coordinate systems in the Mathcad of the finite elements of a given statically indefinable flat frame.

Chapter 5. Mechanical sustainability of marine structures elastic elements, the influence of material plasticity on sustainability

5.1. Sustainability of a rod compressed along the axis, finding the critical force using the Euler's formula

Euler's formula for finding the compression force along the axis of a simply supported elastic rod with a cross–section constant along the length, at which the rod loses mechanical sustainability. Before compression begins, a rod with hinged supports may have a slight deflection (the initial bend, Fig. 5.1.1 a) caused by various or any reasons (physical or technological).



(a) (b)

Fig. 5.1.1

For example, a rod may have sagging or certain slight deviations of the cross–section from its geometric shape, have a curved axis, assuming, due to uneven heating of the sides of the rod, or may experience a temporary deviation in the shape of the axis from straightness due to impact or vibration. The acting force may be applied not at the point of the geometric center of the cross–sectional area, or this force can act non–parallel to the axis of the rod as part of a frame with rigid joints of the connecting rods, or a rod as part of a truss, if there is friction in the hinged connection of the truss, there is no bushing and the bolt is pinched, or the truss assembly is susceptible to corrosion, etc.

Regardless of the initial cause of loss of mechanical sustainability, for example, taken from those considered or others, the magnitude of the greatest load that a rod, compressed along the axis, is able to withstand without loss of mechanical sustainability is its own characteristic of its rigidity (the rod dimensions, cross–sectional shape, length and the structural material) and conditions of connection of the rod with the support or with other rods.

Deflection in the section located at a distance z from the point of the external axial of the force F application is $v(x) = v$, and the bending moment in the section from the axial compressive of the force is equal: $M = -F \times v$. The deflection v, which is the action arm of the force F, is measured from the line of it action.

The equation of the curved axis of a beam has the form: $v'' = \frac{M}{E \times I}$.

By substituting the value of the moment into the formula, one can obtain an equation in the form: $v'' + \frac{F}{E \times I} \times v = 0$.

When using the symbol: $k^2 = \frac{F}{E \times I}$ the equation of the curved axis of rod of the application axial force F takes the form: $v'' + k^2 \times v = 0$.

The solution to the resulting linear homogeneous equation of the second order for the curved axis of the beam has the form: $v(x) = A \times \sin(k \times x) + B \times \cos(k \times x)$.

The solution check one can do by finding the second derivative v'' and substituting the second derivative v'' and the function $v(x)$ into the initial equation:
$v'' = -k^2 \times (A \times \sin(k \times x) + B \times \cos(k \times x))$
and $-k^2 \times (A \times \sin(k \times x) + B \times \cos(k \times x)) +$
$+k^2 \times (A \times \sin(k \times x) + B \times \cos(k \times x)) \equiv 0$.

The boundary conditions for the solution $v(x)$ are: at $x = 0$, $v(0) = 0$
and at $x = l$, $v(l) = 0$.

In the case when the coefficient found for the given boundary conditions is equal to zero: $A = 0$, there is no deflection v.

And when: $A \neq 0$ the deflection v has a sinusoidal shape.

The sine wave takes zero values: $\sin(k \times l) = 0$, if the value in the sine argument is a multiple of π, that is equal: $\pi \times n$, where n – the integer, then: $k \times l = \pi \times n$,
$k^2 \times l^2 = \pi^2 \times n^2$ and $\frac{F}{E \times I} \times l^2 = \pi^2 \times n^2$.

Therefore, the critical force (the Euler's force) equals: $F_E = \frac{\pi^2 \times n^2 \times E \times I}{l^2}$.

The minimum axial force ($F_E = F_{Emin}$), at which the loss of mechanical sustainability of the rod occurs, corresponds to the following data: $n = 1$ and $I = I_{min}$, that is the buckling mode is the form of the least stiffness of the compressed rod, wherein the critical force equals: $F_{CR} = \frac{\pi^2 \times E \times I_{min}}{l^2}$.

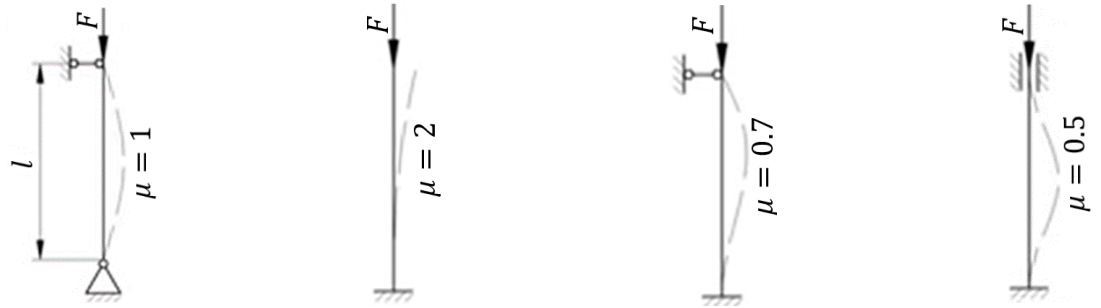
Finding the critical force for cantilever fastening of a rod (Fig. 5.1.1 b). Boundary conditions at the x dimension from the point of the application external axial force F are: $v(0) = 0$ at $x = 0$ and $\theta(l) = v'(l) = 0$ at $x = l$, where θ – the angle of the section turn at bending of a rod compressed by an axial load during loss of mechanical sustainability.

By substituting boundary conditions into the equation, one can find:
$v(0) = A \times \sin 0 + B \times \cos 0 = B = 0$,
$\theta(l) = v'(l) = k \times (A \times \cos(k \times l) - B \times \sin(k \times l)) = k \times A \times \cos(k \times l) = 0$,
$k \times l = \frac{\pi}{2} + \pi \times n$, $k^2 \times l^2 = \left(\frac{\pi}{2} + \pi \times n\right)^2$ and $\frac{F}{E \times I} \times l^2 = \left(\frac{\pi}{2} + \pi \times n\right)^2$.

The minimum non–zero value of the critical force F_{Emin} corresponds to following conditions: $n = \frac{1}{2}$ and $I = I_{min}$, wherein, the critical force at which a cantilevered rod loses mechanical sustainability, equals: $F_{CR} = \frac{\pi^2 \times E \times I_{min}}{4 \times l^2}$ or $F_{CR} = \frac{\pi^2 \times E \times I_{min}}{(2 \times l)^2}$.

The factor of the conditions for fastening a rod compressed by an axial force, when finding the critical force at which the rod loses mechanical sustainability, can be characterized by the coefficient μ of the influence of length l of the rod, wherein the critical force determined according to the Euler's formula, equals: $F_{CR} = \frac{\pi^2 \times E \times I_{min}}{(\mu \times l)^2}$.

The influence coefficient μ (the force reduction) of length l of the rod for different fastening conditions has the values indicated in the diagrams (Fig. 5.1.2 a–d).



(a) (b) (c) (d)

Fig. 5.1.2

Questions and comments:

5.1.1. The force under which a rod, which dimensions and shape of the section can be constant along its length, loses mechanical sustainability when compressed along a line passing through the hinges of the supports.

5.1.2. Factors of initial geometric deviations from the straightness of the axes of elements of mechanical systems and the symmetry of section shapes: the initial deformations due to uneven heating, initial deformations due to their own weight, technological errors, etc., which are part of the primary causes of loss of mechanical sustainability.

5.1.3. The magnitude of the greatest compressive load on an element of a mechanical system (rod, plate, shell, etc.), which this element is able to withstand without loss of mechanical sustainability, which is own characteristic of the rigidity of the system element and the conditions of its connection with the support or with other elements of the mechanical system.

5.1.4. The correlation of the bending moment arm with the resulting deflection when compressing an element of a mechanical system along a line passing through the hinges of the supports, in particular, when compressing a rod along its axis, which causes the form of the curved axis equation, including the deflection of the axis – the proportional

bending moment arm and the second derivative of the deflection, corresponding to the diagram of the bending moment on the scale of the rigidity of the element of the mechanical system.

5.1.5. The shape of the curved axis of a rod subject to axial compression and solutions to a linear homogeneous second–order differential equation.

5.1.6. The minimum axial force (the Eulerian force) at which the loss of mechanical sustainability of the rod on hinged supports at both ends occurs.

5.1.7. Minimum axial force at which loss of mechanical sustainability occurs when the rod is cantilevered. The influence of boundary conditions (the conditions of rod ends supporting or pinching) on mechanical sustainability (on the magnitude of critical force).

5.1.8. The coefficient of influence of the rod length on the critical force at which it loses mechanical sustainability, depending on the conditions of the rod fastening.

5.2. The influence of material plasticity on the sustainability of compressed elements of mechanical systems, the limits of applicability of Euler's formula for finding critical forces

The formula for finding critical force includes the elastic modulus E, which corresponds to the proportionality of stress σ and strain ε in an elastic material according to the Hooke's law, if the stress value does not exceed the yield strength σ_y of the material, as this is indicated in the diagram of stress–strain state (Fig. 5.2.1). The modulus of elasticity corresponds to the tangent of the line in the stress–strain state diagram – in its initial section: $tg\ \alpha = E = \frac{\sigma}{\varepsilon}$.

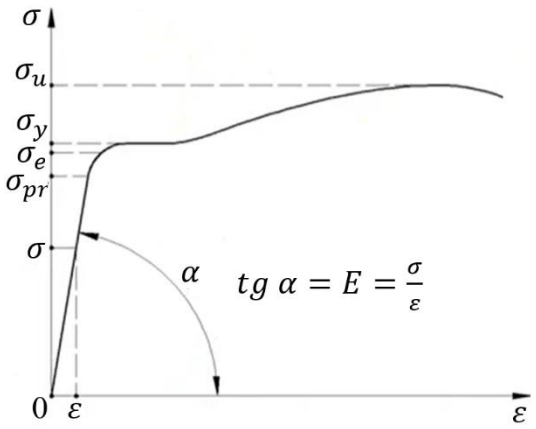


Fig. 5.2.1

The critical stress (item 5.1, the mechanical compressive stress in the cross–section of an elastic rod, corresponding to the critical load F_{CR}), in which an axially compressed rod element of a mechanical system loses sustainability, equals:

$\sigma_{CR} = \frac{F_{CR}}{A} = \frac{\pi^2 \times E \times \frac{I_{min}}{A}}{(\mu \times l)^2} = \pi^2 \times E \times \frac{i_{min}^2}{(\mu \times l)^2} = \pi^2 \times E \times \left(\frac{i_{min}}{\mu \times l}\right)^2$,

where i_{min} – the minimum radius of inertia of the section corresponding to the minimum moment of inertia I_{min} of the section: $i_{min} = \sqrt{\frac{I_{min}}{A}}$.

The flexibility index λ_{max} of a rod characterizes the ratio of its longitudinal and transverse dimensions and the tendency to lose sustainability under compression: $\lambda_{max} = \frac{\mu \times l}{i_{min}}$.

Loss of sustainability in compression of a rod occurs in the form of its greatest flexibility, and the maximum value of the flexibility index can be designated λ. Taking into account the accepted designation, the critical strain is determined as: $\sigma_{CR} = \frac{\pi^2 \times E}{\lambda^2}$.

The maximum stress at which the Hooke's law of linear dependence of stress and strain is valid corresponds to the yield strength: $\sigma_{CR} = \sigma_y$. For example, for carbon structural steel the Young's modulus is usually equal: E=200 000 MPa, and the yield stress is: σ_y= 240 MPa.

The limiting value of the flexibility index of a structural steel rod, at which Euler's formula is applicable to find the critical stresses beyond which the rod loses sustainability, equals: $\lambda_{max} = \pi \times \sqrt{\frac{E}{\sigma_y}} = 3.1415 \times \sqrt{\frac{200\,000}{240}} = 91$.

Provided that a rod made of steel with the specified characteristics has greater flexibility than: $\lambda_{max} = 91$, then before buckling the rod has elastic characteristics in compression, and the buckling load one can find using the Euler's formula.

If the rod has less flexibility, then the loss of sustainability occurs beyond the proportionality of stresses and strains, then the Euler's formula gives an overestimated value of the critical force. For many materials it is known that the critical stress exceeding the yield strength depends linearly on the flexibility of the sample (the element of the mechanical system, rod).

In the plane of greatest flexibility of the rod by estimating the minimum radius of inertia of its cross-section: $i = i_{min} = \frac{I_{min}}{A}$ and $\lambda = \mu \times l/i$ to find the critical stress when compressing the rod element of a mechanical system until it loses the sustainability at which the yield strength of the elastic material may be exceeded σ_y, one can use the recommended formula of F.S. Yasinsky: $\sigma_{CR} = a - b \times \lambda$, where a and b – the empirical coefficients depending on the elasticity and the yield of the material.

In this case, the critical force under compression of the rod element is: $F_{CR} = (a - b \times \lambda) \times A$.

If the flexibility index of the rod is less than certain initial value λ_0, then with an increase of acting load the rod element can go into a plastic state (with the residual

deformations and the exceeding of yield stress σ_y) before losing sustainability, wherein a sustainability calculation is not required, but the calculation of the strength of the compressed element is required or the calculation under limiting deformations.

The example, for the characteristic values of structural carbon steel, which equal: a =310 MPa, σ_y=240 MPa and b = 1.14 MPa, one can determine the initial flexibility index as: $\lambda_0 = \frac{a-\sigma_y}{b} = \frac{310-240}{1.14} = 60$.

To find the critical stresses that arise during loss of mechanical sustainability due to compression in the plane of greatest flexibility of a rod element the choice of approach can be made taking into account the characteristics and sizes of this element, and considering the selection criterion of compliance with the range of flexibility values on the generalized diagram of critical stresses during buckling (Fig. 5.2.2).

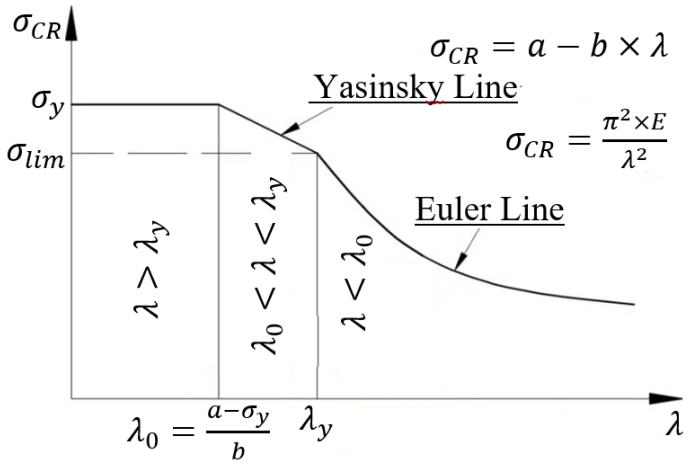


Fig. 5.2.2

With great flexibility of the compressible structural element ($\lambda > \lambda_y$) the calculation of the critical stress and critical force would be done using the Euler's formula. With average flexibility of the axially compressed rod element of the structure ($\lambda_0 < \lambda < \lambda_y$), at which plastic deformation of the mechanical element may occur due to loss of mechanical sustainability, the calculation of the critical force and critical stress would be done taking into account plastic deformation, for example, according to the recommendations of F.S. Yasinsky. With low flexibility ($\lambda < \lambda_0$) of elements of mechanical systems, a calculation of their sustainability is not required, but the calculation of compressive strength or the calculation based on limiting deformations are required.

Questions and comments:

5.2.1. Linear proportionality of stresses and strains according to the Hooke's law under compression of an element of a mechanical system made of an elastic material until the yield point of the material occurs.

5.2.2. Formula for finding the critical stress at which an axially compressed rod element of a mechanical system loses sustainability, depending on the radius of inertia of the section, the length of the rod, the elastic modulus of the material and the coefficient of influence of its length on the critical force, which depends on the conditions of fastening the rod.

5.2.3. An indicator of the flexibility of a rod, depending on its length, on the radius of inertia of the section, on the coefficient of influence of its length and on the critical force, which depends on the conditions of fastening of the rod. Relationship between the critical stress at which the rod becomes unsustainable and the flexibility index.

5.2.4. The minimal magnitude of flexibility indicator of a structural steel rod, at which the Euler's formula is applicable to find the critical stresses beyond which the rod loses sustainability.

5.2.5. Linear dependence of the critical stress exceeding the yield strength of the material on the flexibility index of the sample (the mechanical system element, rod).

5.2.6. Formula of F.S. Yasinsky to find the critical stress when a rod element of a mechanical system is compressed until it loses its sustainability, at which the yield strength can be exceeded, and a deviation from the proportionality according to the Hooke's law of the resulting stresses and deformations will appear.

5.2.7. The plastic state of a compressed rod at lower values of the flexibility index, which occurs before the loss of mechanical sustainability at which sustainability calculation is not required, but calculation of the strength of the compressed element or the calculation based on limiting deformations are required.

5.2.8. Choosing an approach to find the critical stresses that arise during the loss of mechanical sustainability according to the criterion of compliance with range of flexibility values of an element of mechanical system on the generalized critical stress diagram.

5.3. Assessing the sustainability of a compressed rod using stress reduction factors

Formally, when an element of the mechanical system is compressed the condition of sustainability is similar to the condition for the strength under compression of this element: $\sigma = F/A \leq \varphi \times \sigma_y$, where φ – the coefficient of reduction of permissible stress (in considered case – the yield stress σ_y) due to possibility of the sustainability loss.

Typically, for metallic materials the permissible stress reduction factor φ characterizes the ratio of permissible mechanical stresses in compression and in tension.

The values of the permissible stress reduction factor φ are the subject of experimental studies of structural elements differing in rigidity (the material flexibility, the element size, etc.), that is: $\varphi = \varphi(\lambda)$.

To find the reduction factor of the permissible stress φ in such studies, the critical stresses are determined by the studied samples as: $\sigma_{CR}(\lambda) = F_{CR}(\lambda)/A$ and the values of the permissible stress reduction coefficient are found: $\varphi(\lambda) = \sigma_{CR}(\lambda)/\sigma_y$.

The schematic graphs of the dependence of stress under compression of the mechanical system element on the element deformation for various stiffnesses of otherwise similar elements are shown in the diagram (Fig. 5.3.2).

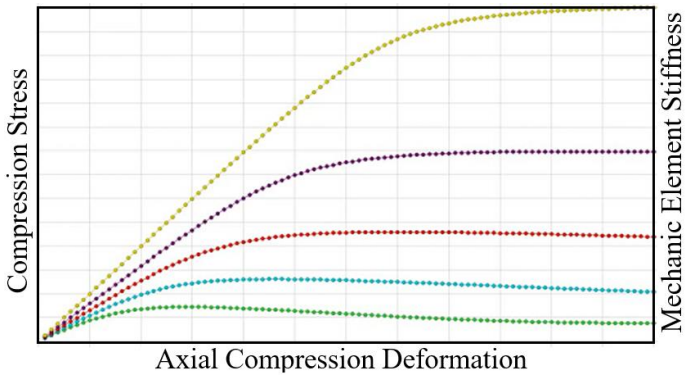


Fig. 5.3.1

Types of standard calculations for assessing the sustainability of elements of mechanical systems are:

1. The verification calculation. Data: F, A, φ and σ_y. The compression sustainability of an element of a mechanical system is assessed, for this purpose, the critical force F_{CR} and critical stress σ_{CR} are sought.

2. The design calculation. Data: F and σ_y. The cross–sectional area A of an element of the mechanical system is determined for the required sustainability: $A \geq F/(\varphi \times \sigma_y)$.

In the design calculation, the reduction factor for the permissible stress φ depends on the flexibility λ, which is unknown, because in design, the flexibility depends on characteristics of section of the mechanical system element, that is the design calculation is non–linear and is usually performed by successive approximations.

As a first approximation, the value of the permissible stress reduction coefficient φ_1 is taken in the range: from 0 to 1, for example: φ_1=0.5. A calculation is performed and the actual value of the permissible stress reduction factor is found φ. If the actual value of the permissible stress reduction factor φ differs significantly from the initially accepted value, then the second approximation is performed, for which: $\varphi_2 = \varphi$.

3. Finding the maximum load $[F]$ (the load capacity of the mechanical system) taking into account the reduction factor φ of the permissible stress σ_y at buckling of the element of the mechanical system. Data: A, φ and σ_y. Taking into account the yield strength σ_y and the reduction factor of the permissible stress φ the ultimate load is sought: $[F] \leq \varphi \times \sigma_y \times A$.

The ratio of the ultimate load corresponding to the critical compression force that an element of a mechanical system can withstand without loss of sustainability to the effective force is the sustainability factor of the element of mechanical system.

Questions and comments:

5.3.1. Formal similarity between the strength conditions of an element of a mechanical system and its sustainability under compression.

5.3.2. The coefficient of reduction in the permissible stress of loss of sustainability of mechanical system element, which is the ratio of the critical stress dependent on the flexibility of the element of the mechanical system to the yield stress of the material.

5.3.3. The form of the dependence of stress in an element of a mechanical system on deformation under compression for various values of stiffness of elements that differ in rigidity and are otherwise similar.

5.3.4. Types of standard calculations when assessing the sustainability of elements of mechanical systems: the verification, in which the sustainability under compression is sought, and the characteristics of the element of the mechanical system are specified, and the design calculation, in which the cross–sectional area is determined taking into account it necessary sustainability.

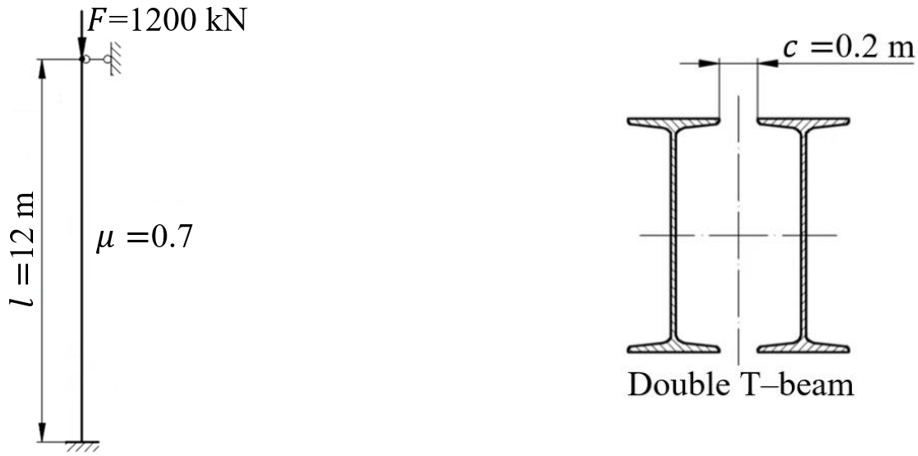
5.3.5. The nonlinearity of the design calculation, in which the area or other characteristics are determined which influence the stiffness sufficient for compression sustainability of an element of a mechanical system, associated with the existing feedback, because the critical force after which the mechanical sustainability is lost depends on required rigidity of the system element (the data depends on the results).

5.3.6. Using the method of successive approximations to determine the coefficient of permissible stress reduction, depending on the rigidity of the element of the mechanical system: initially the coefficient is taken in the range from 0 to 1, and in subsequent approximations the critical forces and buckling stresses are determined, as well as the actual values of the coefficient of permissible stress reduction.

5.3.7. The sustainability of element of the mechanical system factor is the ratio of the maximum load corresponding to the critical force under compression, which the element can withstand without loss of sustainability, to actual compression force.

5.4. Mechanical sustainability of an axially compressed rod made of standard profile parts

Exercise. The standard section rod (double I–beam, Fig. 5.4.1) material is structural carbon steel. Effective force equals: F=1200 kN. The length of compressed rod, rigidly fixed at the lower end is: l =12 m. Coefficient of the influence μ (the reduction) of the length l of the rod for given fastening conditions is equal: μ =0.7. The gap between the double I–sections of compressed rod is equal: c =0.2 m. The yield stress is: σ_y= 240 MPa, the elasticity modulus: E= 2.1×10^5 MPa. Determine the dimensions of standard and non–standard cross–sections by calculating sustainability taking into account the flexibility index of the rod, find the value of the critical force and sustainability safety factor for assigned section dimensions.



(a) (b)

Fig. 5.4.1

Condition of sustainability of the rod at compression is used: $\sigma = F/A \leq \varphi \times \sigma_y$, and the cross–sectional area of a double I–beam corresponding to the condition of its sustainability at compression is sought: $A \geq F/(\varphi \times \sigma_y)$.

Finding the critical force F_{CR} and the cross–sectional dimensions of a double I–beam corresponding to the condition of its sustainability in compression. If, as a first approximation, to take the value of the permissible stress reduction factor: φ_1=0.5, then the required cross–sectional area is equals: $A \geq \frac{1200\times10^3}{0.5\times240\times10^6} = 10^{-2}$, m^2 (100 cm^2). The required cross–sectional area A of each of the double I–beams in the beam must be at least 50 cm^2.

As a first approximation, the nomenclature assumes a standard I–beam profile # 33, which has the geometric characteristics of the section: the cross–sectional area of the I-beam: A =53.8 cm^2, moment of inertia of the section in the plane of greatest rigidity is:

I_x =9 840 cm^4, the moment of inertia of the section in the plane of the minimal rigidity is: I_y =419 cm^4, the flange width is: b =140 mm.

The geometric characteristics of double I–beam profile # 33: the cross–sectional area of double I–beam: A_2 =2×53.8=107.6, cm^2, moment of inertia of the double I–beam section in the plane of greatest rigidity is: I_{x2} =2×9 840=19 680, cm^4, moment of inertia of a double I–section in the plane of lesser rigidity is:

$$I_{y2} = 2 \times \left(I_y + \left(\frac{c}{2} + \frac{b}{2}\right)^2 \times A\right) = 2 \times \left(419 + \left(\frac{20}{2} + \frac{14}{2}\right)^2 \times 53.8\right) = 31\,930, \text{ cm}^4.$$

Moment of inertia of the section in the plane of least rigidity of a double I–beam profile # 33 equals: $I_{min} = I_{x2}$ =19 680 cm^4.

Radius of inertia of the section of a double I–beam in the plane of its least rigidity is equal: $i_{min} = \sqrt{\frac{I_{min}}{A_2}} = \sqrt{\frac{19\,680}{107.6}} = 13.52$, mm.

Greatest flexibility of the rod is: $\lambda_{max} = \frac{\mu \times l}{i_{min}} = \frac{0.7 \times 12}{13.52 \times 10^{-2}} = 62.13$.

The stress permissible reduction coefficient φ one can find by the method of successive approximations using reference values of this coefficient depending on the flexibility index λ of the beam (the data for structural carbon steel are indicated in the Table 5.4.1).

Table 5.4.1

λ	0	10	20	30	40	50	60	70	80	90	100
φ	1.00	0.99	0.97	0.95	0.92	0.89	0.86	0.81	0.75	0.69	0.60
λ	110	120	130	140	150	160	170	180	190	200	210
φ	0.52	0.45	0.40	0.36	0.32	0.29	0.26	0.23	0.21	0.19	0.17

At beam flexibility: $\lambda = 60$, the permissible stress reduction factor for carbon structural steel is equal: φ_{60} =0.86, and at $\lambda = 70$, φ_{70} =0.81. For structural steel, one can find the permissible stress reduction factor by linear interpolation from these data:

$$\varphi_1' = 0.86 + \frac{0.81 - 0.86}{70 - 60} \times (62.13 - 60) = 0.849.$$

The maximum load ($[F] \le \varphi \times \sigma_y \times A_2$, the load capacity of the mechanical system) is: $[F] \le 0.849 \times 240 \times 10^6 \times 107.6 \times 10^{-4} = 2\,192 \times 10^3$, N. The found safe permissible force is significantly greater than the specified load capacity of the mechanical system: F=1200 kN.

The error of the initial approximation is:

$$\Delta_{\varphi 1} = \left|\frac{\varphi_1 - \varphi_1'}{\varphi_1'}\right| \times 100\% = \left|\frac{0.5 - 0.849}{0.849}\right| \times 100\% = 41.1\%.$$

The second approximation required.

The reduction factor for the permissible stress is taken within the range of previous values equals: $\varphi_2 = \frac{0.5+0.849}{2} = 0.675$, $A \geq \frac{1200\times10^3}{0.675\times240\times10^6} = 0.74 \times 10^{-2}$, m^2 (74.1 cm^2).

The required cross–sectional area A of each of the double I–beams in the beam must be at least 37 cm^2.

As a second approximation, the nomenclature assumes a standard I–beam profile # 24a, which has the geometric characteristics of the section: the cross–sectional area of the I–beam: A =37.5 cm^2, moment of inertia of the section in the plane of greatest rigidity is: I_x =3 800 cm^4, moment of inertia of the section in the plane of the minimal rigidity is: I_y =260 cm^4, the flange width is: b =125 mm.

The geometric characteristics of the double I–beams # 24a are: the cross–sectional area of the double I–beams equals: A_2 =2×37.5=75.0, cm^2, moment of inertia of the section of the double I–beams in the plane of greatest rigidity is: I_{x2} =2×3 800=7 600, cm^4, the moment of inertia of the double I–beam section in the plane of the minimal rigidity is: $I_{y2} = 2 \times \left(260 + \left(\frac{20}{2} + \frac{12.5}{2}\right)^2 \times 37.5\right)$=20 320, cm^4.

Moment of inertia of the section in the plane of the minimal rigidity of the double I–beams # 24a equals: $I_{min} = I_{x2}$ =7 600 cm^4.

Radius of inertia of the section of the double I–beams in the plane of the minimal rigidity is equal: $i_{min} = \sqrt{\frac{I_{min}}{A_2}} = \sqrt{\frac{7\,600}{75.0}}$ =10.07, mm.

The maximal beam flexibility index is: $\lambda_{max} = \frac{\mu\times l}{i_{min}} = \frac{0.7\times12}{10.07\times10^{-2}}$ =83.42.

At beam flexibility of: $\lambda = 80$, the permissible stress reduction coefficient for carbon structural steel is equal: φ_{80} =0.75, and at $\lambda = 90$, φ_{90} =0.69. For structural steel, one can find the reduction factor for the permissible stress by linear interpolation from these data as: $\varphi_2' = 0.75 + \frac{0.69-0.75}{90-80} \times (83.42 - 80)$ =0.729.

The limiting load (the load capacity of the mechanical system) is:

$[F] \leq \varphi \times \sigma_y \times A_2 = 0.729 \times 240 \times 10^6 \times 75.0 \times 10^{-4} = 1\,312 \times 10^3$, N.

The found safe permissible force is significantly greater than the assigned load capacity of the mechanical system: F=1200 kN.

The second approximation error is:

$\Delta_{\varphi 2} = \left|\frac{\varphi_2 - \varphi_2'}{\varphi_2'}\right| \times 100\% = \left|\frac{0.675-0.729}{0.729}\right| \times 100\% = 7.4\%$.

Third approximation is required.

The reduction factor for the permissible stress is taken within the range of previous values: $\varphi_3 = \frac{0.675+0.729}{2} = 0.702$, $A \geq \frac{1200\times10^3}{0.702\times240\times10^6} = 0.712 \times 10^{-2}$, m^2 (71.2 cm^2).

The required cross–sectional area A of each of the double I–beams in the beam must be at least 35.6 cm^2.

As a third approximation, the nomenclature assumes the next smaller standard I–beam #24, although the cross–sectional area of this I–beam is equal to: A =34,8 cm^2. That is the area is slightly less than the found required value of 35.6 cm^2, but the next after I–beam # 24 larger value of # 24a was considered in the previous second approximation.

Moment of inertia of the section of I–beam # 24 in the plane of greatest rigidity is: I_x =3 460 cm^4, the moment of inertia of the section in the plane of the minimal rigidity is: I_y =198 cm^4, the flange width is: b =11.5 mm.

The geometric characteristics of the double I–beams # 24 are: the cross–sectional area: A_2 =2×34.8=69.6, cm^2, the moment of inertia of the section of the double I–beams in the plane of greatest rigidity is: I_{x2} =2×3 460=6 920, cm^4, moment of inertia of the double I–beam section in the plane of the minimal rigidity is:

$I_{y2} = 2 \times \left(198 + \left(\frac{20}{2} + \frac{11.5}{2}\right)^2 \times 34.8\right)$=17 660, cm^4.

Moment of inertia of the section in the plane of the minimal rigidity of the double I–beams # 24 is: $I_{min} = I_{x2}$ =6 920 cm^4.

Radius of inertia of the section of the double I–beams in the plane of the minimal rigidity is: $i_{min} = \sqrt{\frac{I_{min}}{A_2}} = \sqrt{\frac{6\,920}{69.6}}$ =9.97, mm.

The maximal beam flexibility index is: $\lambda_{max} = \frac{\mu \times l}{i_{min}} = \frac{0.7 \times 12}{9.97 \times 10^{-2}}$ =84.25.

At the beam flexibility of: $\lambda = 80$, the permissible stress reduction coefficient for carbon structural steel is equal: φ_{80} =0.75, and φ_{90} =0.69 at $\lambda = 90$. For structural steel, one can find the reduction factor for the permissible stress by the linear interpolation from these data: $\varphi_3' = 0.75 + \frac{0.69-0.75}{90-80} \times (84.25 - 80)$ =0.725.

The limiting load (the load capacity of the mechanical system) is:

$[F] \leq \varphi \times \sigma_y \times A_2 = 0.725 \times 240 \times 10^6 \times 69.6 \times 10^{-4} = 1\,211 \times 10^3$, N.

The found safe permissible force closely matches the assigned load capacity of the mechanical system: F=1200 kN.

Third approximation error is:

$\Delta_{\varphi 3} = \left|\frac{\varphi_3 - \varphi_3'}{\varphi_3'}\right| \times 100\% = \left|\frac{0.702 - 0.725}{0.725}\right| \times 100\% = 3.17\%$.

The accuracy of determining the load capacity of the rod in the third approximation is sufficient and the fourth approximation is not required.

Found the geometric characteristics of the section of the double I–beams # 24: the cross–sectional area: A_2 =2×34.8=69.6, cm^2, the moment of inertia of the section in the plane of the minimal rigidity is: I_{min} =6 920 cm^4, the minimal radius of inertia of the section of the double I–beams in the plane of the minimal rigidity is: i_{min} =9.97 mm and the maximal beam flexibility is: λ_{max} =84.25.

For structural carbon steel, the range of values of the flexibility index λ of the beam at its axial compression for applying the recommendations of F.S. Yasinsky (item 5.2, Fig. 5.2.2, when the sustainability of an axially compressed rod is assessed taking into account its plastic deformation under axial compression) is: from $\lambda_0 = \frac{a-\sigma_y}{b} = 61$ to $\lambda_y = 100$.

Therefore, to find the critical force it is recommended to use the formula of F.S. Yasinsky: $F_{CR} = (a - b \times \lambda) \times A$.

If for a structural material (carbon steel) the coefficients in the formula of F.S. Yasinsky have the magnitudes: a =314 MPa and b =1.14 MPa, then the critical force corresponding to the buckling of the beam, equals: $F_{CR} = (a - b \times \lambda) \times A =$
$= (314 - 1.14 \times 84.25) \times 10^6 \times 69.6 \times 10^{-4} = 1\,517 \times 10^3$, N.

Sustainability safety factor taking into account the possibility of plastic deformation of the material of the double I–beams # 24 is equal: $n_y = \frac{F_{CR}}{F} = \frac{1\,517}{1\,200}$ =1.26.

Questions and comments:

5.4.1. The criterion of sustainability under compression of the rod, taking into account the effective axial load, the cross–sectional area, the coefficient of reduction in the permissible stress, depending on the flexibility of the rod and the conditions of its support (the pinched ends or hinged joints) and on the yield strength of the material.

5.4.2. Performing design calculations and using the method of successive approximations to determine jointly the interrelated indicators: the critical force at which the rod loses sustainability, and the cross–sectional dimensions of the rod (the double I–beam).

5.4.3. Selection of the beam profile characteristics according to the standard nomenclature, which cross–sectional size, that is the area and moment of inertia are no less than those found in the corresponding approximation according to the criterion of sustainability under compression of the beam.

5.4.4. Finding the geometric characteristics of the section and stiffness of the beam, made from a selected standard profile of connected parts for the initial or current step of approximate calculation of the reduction factor for the permissible stress in the condition of sustainability under axial compression of the beam.

5.4.5. Determining the flexibility index of a rod (the double I–beam clamped at one end and supported by the other end in a sliding hinge), finding the coefficient of permissible stress reduction using reference values of this coefficient by the linear interpolation to determine the coefficient of permissible stress reduction in the range of values of the flexibility of the rod.

5.4.6. Finding the error of the current step of approximate determination of the reduction coefficient of the permissible stress in the condition of sustainability under

compression of the rod and checking the sufficient accuracy of the approximate determination of this coefficient.

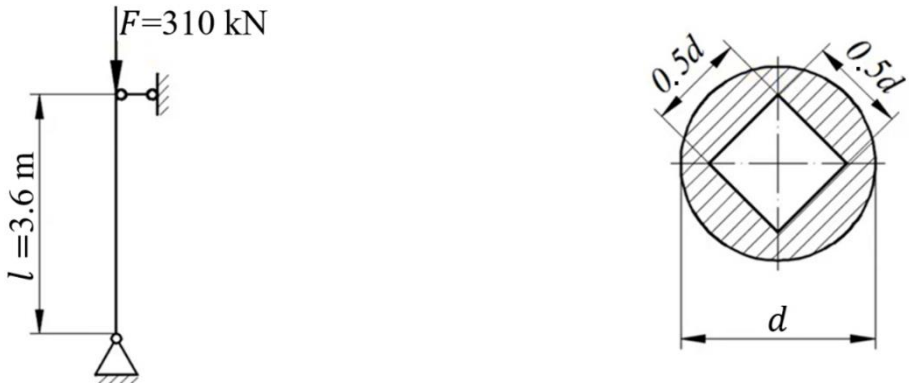
5.4.7. Determination of range of the flexibility index values of rod (the double I–beam) during its axial compression for applying the recommendations of F.S. Yasinsky, i.e. to assess the sustainability of a rod taking into account its plastic deformation under axial compression.

5.4.8. The use of the formula of F.S. Yasinsky to find the critical force corresponding to the loss of sustainability of the beam.

5.4.9. Determination of the sustainability safety factor taking into account the possibility of plastic deformation of the material of rod (the double I–beam).

5.5. Assessment of the mechanical sustainability of the shaft, the cross–sectional shape of which is specified geometrically

Exercise. The shape of the section compressed along the axis of the beam (rod element, shaft) and the characteristics of its support are shown on the diagram (Fig. 5.5.1 a). The cylindrical beam has a round cross–section on the outside and a square cutout on the inside. (Fig. 5.5.1 b).



(a) (b)

Fig. 5.5.1

The dimensions of a square cutout in the round beam are equal to half of its outer diameter d. The length of beam, the lower end of which rests on a rigid hinge, and the upper end is held by a sliding hinge, equals: l =3.6 m. The axial force that acts on upper (the sliding) end of the hinge, equals: F=310 kN. The yield stress (the proportionality limit of the elastic–strain state diagram) for structural carbon steel used is equal: σ_y= 220 MPa, the elastic modulus is equal to: E= 2.1×10^5 MPa. The exercise assigns one to find the buckling stress, the critical force corresponding to the buckling and the sustainability safety factor.

Criterion of the sustainability of rod in compression is: $\sigma = F/A \leq \varphi \times \sigma_y$.

Necessary cross–sectional area corresponding to the condition of its sustainability under equals compression: $A \geq F/(\varphi \times \sigma_y)$.

Cross–sectional area taking into account the square cutout, the size of which is equal to half the outer diameter, that is: $A = \frac{\pi \times d^2}{4} - (0.5 \times d)^2 = 0.535 \times d^2$.

The geometric characteristics of a circular cross–section in the dependence on the area of one can found as follows, the outer diameter is: $d = 1.367 \times \sqrt{A}$, the moment of inertia in the plane of minimal rigidity is: $I_x = I_y = I_{min} = \frac{\pi \times d^4}{64} - \frac{(0.5 \times d)^4}{12} =$

$= 0.04385 \times d^4 = 0.04385 \times \left(1.367 \times \sqrt{A}\right)^4 = 0.153 \times A^2$.

It is taken into account that the stiffness of a beam of a given circular cross–section with a square cutout is equal in any perpendicular directions.

The radius of inertia of the section is: $i_{min} = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{0.153 \times A^2}{A}} = 0.3912 \times \sqrt{A}$.

Maximum flexibility of the compressed rod, taking into account the assigned scheme of its fastening ($\mu = 1$) equals: $\lambda_{max} = \frac{\mu \times l}{i_{min}} = \frac{1.0 \times 3.6}{0.3912 \times \sqrt{A}} = \frac{9.20}{\sqrt{A}}$.

Finding the safe permissible force as a first approximation. The reduction factor for the permissible stress in a first approximation is taken to be equal to: φ_1=0.5, wherein the required cross-sectional area of the beam, corresponding to its sustainability in compression: $A \geq F/(\varphi \times \sigma_y)$ is determined as: $A \geq \frac{310 \times 10^3}{0.5 \times 220 \times 10^6} = 28.2 \times 10^{-4}$, m^2.

Maximum flexibility of the rod equals: $\lambda_{max} = \frac{9.20}{\sqrt{A}} = \frac{9.20}{\sqrt{28.2 \times 10^{-4}}} = 173.25$.

At beam flexibility of: $\lambda = 170$, the stress permissible reduction coefficient for carbon structural steel (item 5.4, Table. 5.4.1) is: φ_{170} =0.26 and φ_{180} =0.23 at $\lambda = 180$.

For structural steel, one can find the reduction factor for the permissible stress reduction by linear interpolation for these data:

$\varphi_1' = 0.26 + \frac{0.23 - 0.26}{180 - 170} \times (173.25 - 170)$ =0.25.

The limiting load (the load capacity of mechanical system), that is safe permissible force is: $[F] \leq \varphi \times \sigma_y \times A_2 = 0.25 \times 220 \times 10^6 \times 28.2 \times 10^{-4} = 155.1 \times 10^3$, N.

The found safe permissible force is significantly less than the assigned load capacity (F =310 kN). Initial approximation error equals:

$\Delta_{\varphi 1} = \left|\frac{\varphi_1 - \varphi_1'}{\varphi_1'}\right| \times 100\% = \left|\frac{0.5 - 0.25}{0.25}\right| \times 100\% = 100$ %.

Second approximation is required.

Coefficient of permissible stress reducing in the second approximation is taken in range of previous values: $\varphi_2 = \frac{0.5 + 0.25}{2} = 0.375$, $A \geq \frac{310 \times 10^3}{0.375 \times 220 \times 10^6} = 37.6 \times 10^{-4}$, m^2.

Maximum shaft flexibility equals: $\lambda_{max} = \frac{9.20}{\sqrt{A}} = \frac{9.20}{\sqrt{37.6\times10^{-4}}} = 150.04.$

At beam flexibility: $\lambda = 150$, the permissible stress reduction coefficient for carbon structural steel is equal: φ_{150} =0.32, and φ_{160} =0.29 at $\lambda = 160$. For structural steel, one can find the reduction factor for the permissible stress by linear interpolation of these data: $\varphi_2' = 0.32 + \frac{0.29-0.32}{160-150} \times (150.04 - 150)$ ≈0.32.

Load capacity of the mechanical system, that is safe permissible force is:
$[F] \leq 0.32 \times 220 \times 10^6 \times 37{,}6 \times 10^{-4} = 264.7 \times 10^3$, N.

The found safe permissible force is significantly less than the assigned load capacity (F =310 kN). Second approximation error equals:

$\Delta_{\varphi 2} = \left|\frac{\varphi_2 - \varphi_2'}{\varphi_2'}\right| \times 100\% = \left|\frac{0.375-0.32}{0.32}\right| \times 100\% = 17.2\ \%.$

Third approximation is required.

The coefficient for reducing the permissible stress in the third approximation is accepted in the range of previous values: $\varphi_3 = \frac{0.375+0.32}{2} = 0.348,$

$A \geq \frac{310\times10^3}{0.348\times220\times10^6} = 40.5 \times 10^{-4}$, m^2.

Maximum shaft flexibility equals: $\lambda_{max} = \frac{9.20}{\sqrt{A}} = \frac{9.20}{\sqrt{40.5\times10^{-4}}} = 144.56$.

At the flexibility: $\lambda = 140$, permissible stress reduction coefficient for carbon structural steel is equal: φ_{140} =0.36, and at $\lambda = 150$, φ_{150} =0.32. For structural steel, one can find the reduction factor for the permissible stress by linear interpolation from these data: $\varphi_3' = 0.36 + \frac{0.32-0.36}{150-140} \times (144.56 - 140)$ ≈0.342.

Load capacity of the mechanical system, that is safe permissible the force is:
$[F] \leq 0.342 \times 220 \times 10^6 \times 40.5 \times 10^{-4} = 304.7 \times 10^3$, N.

The found safe permissible force is comparable with assigned load (F =310 kN). Second approximation error equals:

$\Delta_{\varphi 3} = \left|\frac{\varphi_3 - \varphi_3'}{\varphi_3'}\right| \times 100\% = \left|\frac{0.348-0.342}{0.342}\right| \times 100\% = 1.8\ \%,$ the accuracy of determining the load capacity of the rod in the third approximation is sufficient and the fourth approximation is not required.

The found geometric characteristics of section of a given round beam with a square cutout are equal:

– the area: $A = 40.5 \times 10^{-4}$, m^2 (40.5 cm^2), the outer diameter:

$d = 1.367 \times \sqrt{A} = 1.367 \times \sqrt{40.5 \times 10^{-4}} = 0.087$, m (8.7 cm),

– the moment of inertia in the plane of minimal rigidity is:

$I_x = I_y = I_{min} = 0.153 \times A^2$=2.508×10^{-6}, m^4 (25 080 cm^4, the rigidity of a beam of a given circular cross–section with a square cutout is equal in all perpendicular directions),

– the radius of inertia of the cross–section is: $i_{min} = 0.3912 \times \sqrt{A}$ =0.0249, m and maximum beam flexibility is: λ_{max} =144.56.

For structural carbon steel, the range of values of the flexibility index λ of the beam at its axial compression for applying the recommendations of F.S. Yasinsky (item 5.2, Fig. 5.2.2, that is an the index of sustainability of a beam taking into account its plastic deformation under axial compression) has to be from: $\lambda_0 = \frac{a-\sigma_y}{b} = 61$ to $\lambda_y = 100$.

Therefore, to find the critical force it is recommended to use Euler's formula:

$F_{CR} = \frac{\pi^2 \times E}{\lambda^2} \times A = \frac{\pi^2 \times 2.1 \times 10^{11}}{144.56^2} \times 40.5 \times 10^{-4} = 401$, kN.

The safety factor is equal: $n_y = \frac{F_{CR}}{F} = \frac{401}{310}$ =1.294.

Questions and comments:

5.5.1. The criterion of mechanical sustainability of the rod under compression and the required cross–sectional area corresponding to the sustainability of the rod compressed along the axis.

5.5.2. Dependence on the circular cross–sectional area of the rod geometric characteristics: the outer diameter and moments of inertia of the cross–sectional area.

5.5.3. The coefficient of influence of length on the flexibility index of a compressed round rod, depending on the rod fastening condition, and the coefficient of permissible stress reduction, depending on the flexibility of the rod, for the structural material used (carbon steel).

5.5.4. Performing design calculations and using the method of successive approximations to determine jointly the interrelated indicators: the critical force at which the rod loses sustainability, and the required cross–sectional sizes of the rod (the area, outer diameter).

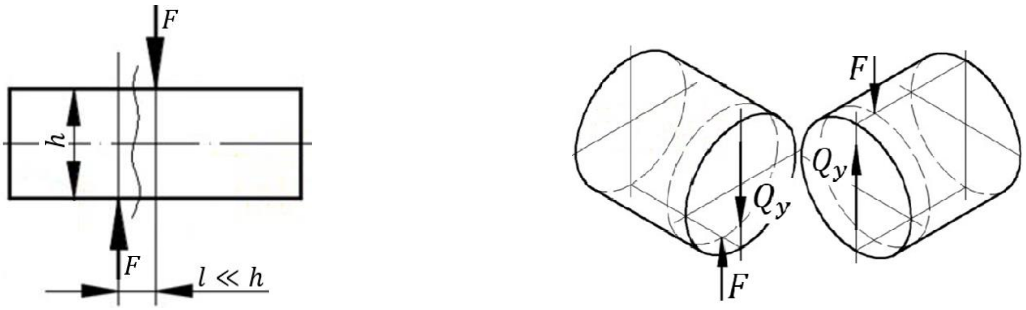
5.5.5. Finding the error of the current step of approximate determination of the coefficient of permissible stress reduction in the condition of sustainability under compression of the rod and checking the sufficient accuracy of the approximate determination of this coefficient.

5.5.6. Using the Euler's formula to find the critical force under axial compression of flexible rod, which flexibility indicator of goes beyond the range of values at which it is recommended to take into account the influence of plastic deformations on the magnitude of the critical force, for example, using the formula of F.S. Yasinsky.

Chapter 6. Shear and torsion of mechanical system elements

6.1. Mechanical shear under the influence of transverse force

Shift (the shear) is a type of mechanical load F (Fig. 6.1.1 a and b), at which only one internal force factor appears in the cross–sections of an element of the mechanical system (the beam, shaft, etc.) – the transverse force Q_y.



(a) (b)

Fig. 6.1.1

Sign of the transverse force Q_y arising at shear is not related to the direction of deformation of the beam, unlike its tension or compression, therefore the sign is chosen relative to the coordinate system used.

Typically, mechanical shift (the shear) also occurs when bending an element of the mechanical system. If the distances between the planes of action of forces F, that is the acting external force and the opposite reactive force equal to it in the magnitude, are small compared to the transverse size of the beam: $l \ll h$, one can neglect the bending considering the shear.

Tangent stress and shear deformation may be experienced by the fasteners used: bolts, rivets, dowels, splines and welds. Technologically, the cut is in the process of mechanical cutting of sheets or the rods, as well as during drilling, turning, sawing, chiselling, scraping, grinding, etc.

Transverse force Q_y, distributed in the section in the cut plane, causes shear stress at each point of the section (Fig. 6.1.2): $\tau_{xy} = \tau$.

In the sum of the cross–sectional area all arising tangential stresses give a transverse force Q_y.

An elementary shear force acts on the elementary cross–sectional area of a beam during shear (the shift): $dQ = \tau \times dA$. The total magnitude of the shearing force distributed over the section of the beam, equals: $Q = \int_A dQ = \int_A \tau \times dA$.

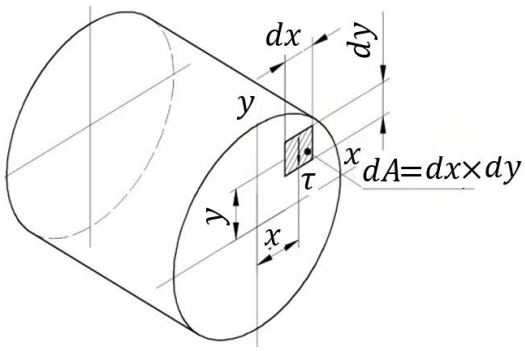


Fig. 6.1.2

If one assumes that the shear stress is distributed uniformly over the section of the beam: $\tau(A) = const$, then the cutting force will be equal to the product of the tangent stress and the cross–sectional area of the beam: $Q = \tau \times \int_A dA = \tau \times A$, that is the tangential stresses, assuming their uniform distribution over the cross–sectional area, have the value: $\tau = Q/A$.

Strength rating in shear (the shift). For the strength of a beam under shear (which is for strength at the highest tangential stresses) it is required that the effective tangential stress does not exceed a certain permissible magnitude: $\tau = Q/A \leq [\tau]$.

As a destructive tangential stress, one can consider the yield strength for tangential stresses τ_y in shear (the shift). Wherein, taking into account a certain safety factor $[n]$ the permissible shear stress equals: $[\tau] = \tau_y/[n]$.

The destructive shear stress τ_y can be found by tests of an experimental sample of mechanical system element for shear or tension/compression, since on geometric grounds for elastic deformation a proportional relationship is known between the yield limits of the material for shear and for tension/compression – by normal stresses: $\tau_y = \sigma_y/\sqrt{3}$.

With sufficient accuracy for calculations in mechanics, the dependence of the yield strengths for shear and tension/compression corresponds to the approximate formula: $\tau_y \approx \sigma_y \times 0.6$, accordingly, the ratio of the tensile strengths for tangential $[\tau]$ and normal $[\sigma]$ stresses has the form: $[\tau] \approx [\sigma] \times 0.6$.

Typical types of calculations under mechanical shear (the shift):

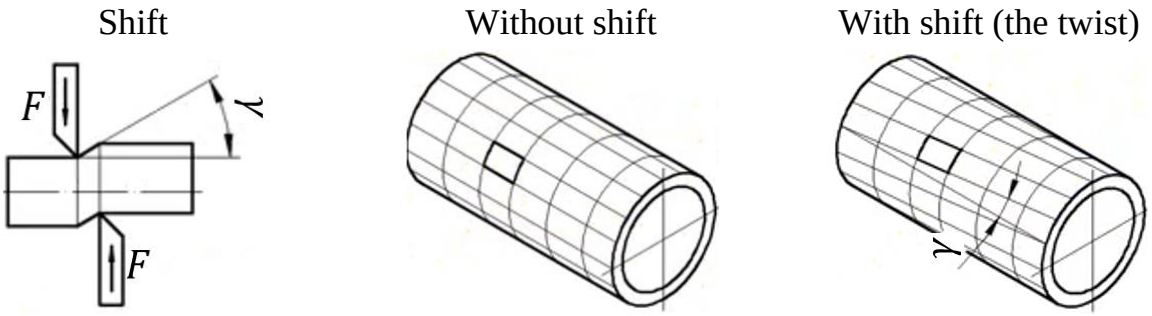
1. Verification calculation. Data: the shear force Q, the cross–sectional area of beam A and permissible shear stress $[\tau]$. The strength of the beam is checked for shear (shift).

2. Design calculation. Data: the shear force Q and permissible shear stress $[\tau]$. By the condition of the strength of beam under shear (the shift) the required area $[A]$ of its cross–section is found: $[A] \geq Q/[\tau]$.

3. Finding the maximum load (the load capacity of structure) which the beam withstands under shear (the shift). Data: the cross–sectional area of beam A and the permissible

shear stress $[\tau]$. Based on the strength condition, the permissible lateral force is determined as: $[Q] \leq [\tau] \times A$.

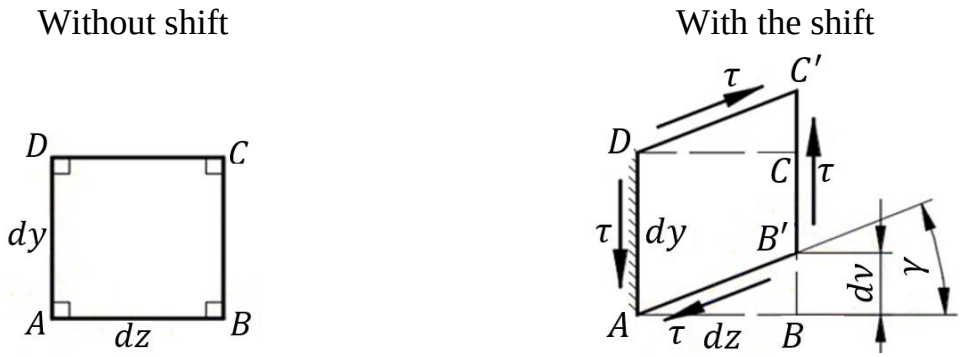
Deformations of beam under shift (the shear) are shown on scheme (Fig. 6.1.3 a).



(a) (b) (c)

Fig. 6.1.3

If one selects a small segment of unit thickness in a beam (Fig. 6.1.3 b and c, Fig, 6.1.4 b), although on at least one side of which there is a tangential stress τ, then, according to the law of pairing of tangential stresses, these stresses will also act on perpendicular faces stress τ. The initially right angles of the segment will change under influence of tangential stresses, and the angular deformation γ arises. Angular deformation occurs both in the case of shear (Fig. 6.1.3 a), and in the case of torsion of the beam (Fig. 6.1.4 c).



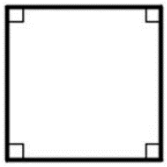
(a) (b)

Fig. 6.1.4

The corner point of segment, for example, the point B (Fig. 6.1.4) when the beam of pipe shape is torsioned, receives a displacement dv (the absolute value of shift) and occupies a displaced position B'. The shift angle γ of the segment sides, due to the smallness of this angle, is approximately equal to its tangent: $tg\ \gamma \approx \gamma$, because: $tg\ \gamma = \frac{dv}{dz}$, then $\gamma \approx \frac{dv}{dz}$.

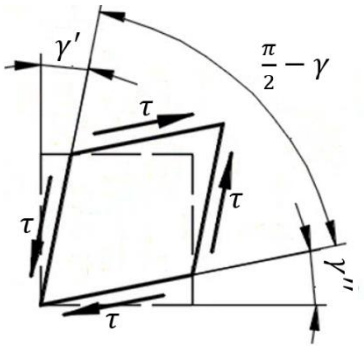
If at shear all edges of the rectangular elastic segment of beam experiencing deformation of the shear displacement (Fig. 6.1.5 a and b), then the components (γ' and γ'') of the shift angle γ one needs to add to find this angle: $\gamma = \gamma' + \gamma'' = \frac{d\omega}{dy} + \frac{dv}{dz}$.

Without shift



(a)

With shift



(b)

Fig. 6.1.5

Hooke's law in shear (the shift). It has been experimentally established that tangential stresses are proportional to angular elastic deformations, provided that they are small: $\tau = G \times \gamma$ (the Hooke's law for shear stresses), where G – the modulus of elasticity of the second type (shear modulus) – the additional elastic constant characteristic of a material, showing its rigidity under the action of tangential stresses (under torsional or shear deformations).

Mathematical dependence of the elastic modulus E in tension/compression and the elastic modulus G in shear (the shift) has the form: $G = \frac{E}{2\times(1+\mu)}$, where μ – the Poisson's ratio. For the structural materials used the Poisson's ratio is equal to: for concrete – about 0.2, for cast iron – about 0.22, for steel, glass, porcelain – from 0.23 to 0.26, for alloys of aluminum, brass, copper, organic glass and polystyrene – from 0.33 to 0.35, for lead, tin – near 0.45.

One can see that the Poisson's ratio is greater for plastic and soft materials than for brittle and hard ones, since when plastic and soft materials are deformed, the share is greater of energy spent on volume change in the total deformation energy (together with the energy of shape change), greater than for brittle and hard materials, which shape changing requires more energy than changing the shape of soft and ductile materials.

Displacement during shear (the shift) is proportional to the magnitude of the shear stress: $\frac{Q}{A} = G \times \frac{dv}{dz}$ and is determined as: $dv = \frac{Q}{G\times A} \times dz$. The displacement under shear of a mechanical element of short length l equals: $v = \frac{Q\times l}{G\times A}$.

Potential energy of deformation under shear of a beam. In the form of a mathematical increment the force acting on an elementary section segment ($dx \times dy$) under shear,

equals: $dQ = \tau \times dx \times dy$. The displacement of the face of this segment is defined as: $dv = tg\,\gamma \times dz \approx \gamma \times dz$.

Under the shear the mechanical work of the force deforming the beam turns into potential deformation energy: $dU = 1/2 \times dQ \times dv = 1/2 \times \tau \times \gamma \times dx \times dy \times dz$.

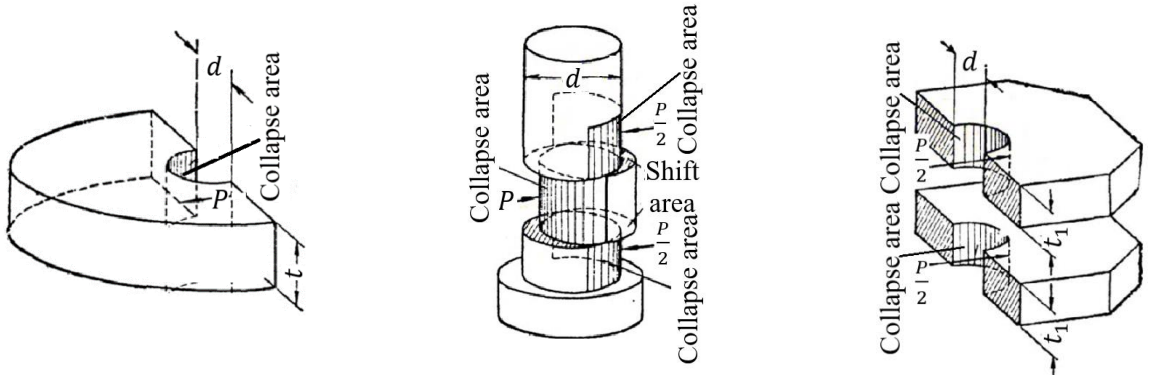
In volume of an elementary segment equal to: $dV = dx \times dy \times dz$, the specific potential energy of elastic deformation under shift is determined as:
$u = \frac{dU}{dV} = 1/2 \times \tau \times \gamma$.

The elastic deformation under shear according to the Hooke's law is determined as: $\gamma = \tau/G$, therefore, the specific potential energy of deformation in shear is: $u = \frac{\tau^2}{2 \times G}$.

Assuming that in entire volume of beam ($v = A \times l$), deformed in shear, as well as in each section of the beam, the tangent stress is constant ($\tau = Q/A$), one can find the total potential energy of deformation as: $U = u \times V = \frac{\tau^2}{2 \times G} \times V = \frac{(Q/A)^2}{2 \times G} \times A \times l = \frac{Q^2 \times l}{2 \times G \times A}$.

Crumple (the crushing) in the contact areas of the material of elements of the mechanical system. When connecting parts with rivets, bolts, dowels, etc., these fasteners exert pressure on the walls of holes and grooves, causing mechanical stress to appear in the walls. When the yield strength is exceeded, the plastic deformations of the material of the mounting holes occur. This type of damage is denominated the crumple.

The material of the fasteners is usually stronger than the material of the elements of the mechanical system being connected. In this case, the strength under shear of the fasteners themselves would be calculated, and the strength of the connected structural elements – under the crumple. As an example, the diagram shows the fastening of a cylindrical pin (Fig. 6.1.6 b) of a turnbuckle (Fig. 6.1.6 c) to an eye (Fig. 6.1.6 a), the crush area of the turnbuckle and eye and the shift area of a cylindrical fastening pin.



(a) (b) (c)

Fig. 6.1.6

The strength condition in the joints of elements of the mechanical system under crush is: $\sigma_C = F/A_C \leq [\sigma_C]$, where σ_C – the crush stress, F – the force of pressing parts,

collapsing the material of elements of the mechanical system, A_C – the area of crush over which the acting force is distributed F, $[\sigma_C]$ – the permissible stress before crush.

When cylindrical surfaces are compressed (Fig. 6.1.6), for example, when a cylindrical bushing with a diameter d is pressed against the surface of a hole of a part with a thickness δ the mechanical stress, causing crush σ_C, is distributed unevenly over the contact area. Therefore, to find the maximum stress, the projection area of the contact area onto the cut plane of the cylindrical surface is taken into account: $A_C = d \times \delta$.

Questions and comments:

6.1.1. Shear (the shift) – a type of loading in which only one internal force factor appears in the cross–sections of an element of a mechanical system (the beam, shaft, etc.) – the transverse force. The choice of the sign of force acting during the shift relative to the coordinate system used.

6.1.2. Technological application of cut (the shear) in the processing of structural materials: cutting sheets, cutting rods, drilling, turning, sawing, chiseling, scraping, grinding, etc. Typical fasteners subject to shear forces: the bolts, rivets, keys, splines, and the welds.

6.1.3. Tangential stresses arising from the action of the shear on elements of mechanical systems. Integral interdependence between shear force and shear tangential stresses. The magnitude of tangential stresses assuming their uniform distribution over the cross–sectional area.

6.1.4. The strength criterion for the shear based on the maximum tangential stress. Yield strength based on tangential stresses as part of the strength criterion. The proportional interdependence between the yield strength of structural material for shearing under tangential stresses and for tension / compression under normal stresses, as well as the ratio of strength limits under tangential and normal stresses.

6.1.5. The types of calculations for the mechanical shear: verification calculation – determination of the shear stress at assigned shear force and cross–sectional area, checking the strength according to the criterion of permissible shear stress, the design calculation – determination of the strong cross–sectional size (the area) for a given shear force and a known shear strength, finding the maximum load withstood by the beam under the influence of the shear.

6.1.6. The action of the law of pairing of tangential stresses is the action of tangential stresses of equal magnitude on the perpendicular faces of selected segment of a beam experiencing shear force, when the direction of the tangential stresses on the perpendicular faces prevents the rotation of this segment, the corners of which are deformed when the faces shift, but do not rotate.

6.1.7. Angular deformation of a beam segment, equal to the sum of the angular displacements of the segment faces. The approximate equalities of the sine and tangent of a small angle to this angle, measured in radians, used in the mathematical description of angular deformations. Translational displacement and shear deformation of a mechanical element of short length.

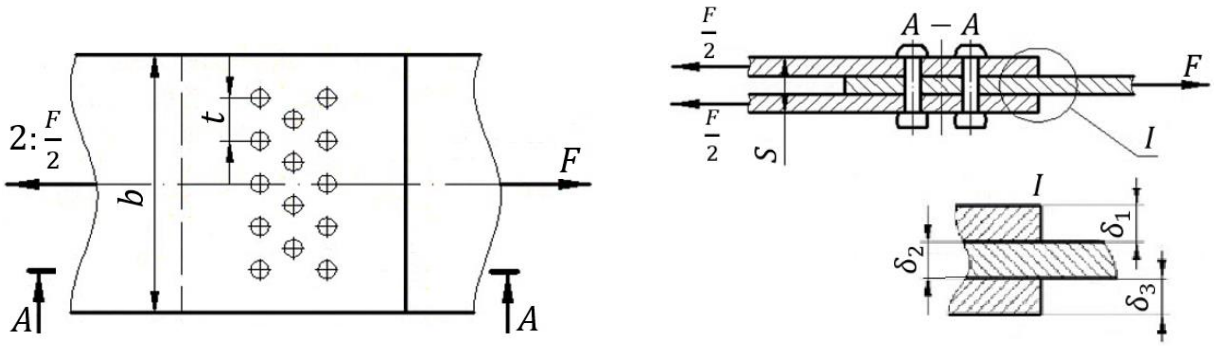
6.1.8. The Poisson's ratio and the ratio of elasticity indicators of a material with tension/compression and with shear (shift). Value of the Poisson's ratio for plastic and soft materials is greater than for brittle and hard ones, taking into account the ratio of the parts of energy: spent on changing shape during deformation and spent on changing volume.

6.1.9. The magnitude of the potential energy of deformation under shear of the beam. The work of force deforming a beam, turning into the potential energy of its elastic deformation during shear. Specific potential energy of beam deformation under shear. Total potential energy of deformation under shear provided that the tangential stresses in the cross–section of the beam are the same.

6.1.10. Crumpling (the crushing) of mechanical system elements in the contact zone. Deformation in the form of crush of the walls of holes for fasteners of elements of mechanical structures when the effective stresses exceed the yield strength of structural material. Condition of strength before crush of elements of a mechanical system at their connection points.

6.2. Calculation of the shear strength of rows of rivets when joining sheets

Exercise. A package of duralumin sheets (Fig. 6.2.1) with a total thickness S and permissible normal stress: $[\sigma]$=250 MPa connected with a multi–row staggered rivet seam.



(a) (b)

Fig. 6.2.1

The initial data: the tensile force: F=500 kN, thickness of the first sheet is: δ_1=1.0 mm, thickness of the second sheet: δ_2=1.2 mm, thickness of the third sheet: δ_3=0.8 mm, the yield strength of rivet material and sheet material under shear stress is: τ_y=360 MPa

(it is assigned the same material of the sheets and fasteners to be joined) and safety margin of rivet material: $[n]$=1.4. It is required to calculate the rivet connection for shear.

For the package of joined duralumin sheets one needs to find:

1. Required rivet diameter, which must be rounded to the nearest larger value from the following list (mm): 1.0, 1.2, 1.4, 1.6, 2.0, 2.5, 3.0, 3.5, 4.0, 5.0, 6.0, 8.0, 10.0, 12.0, 14.0, 16.0, 18.0, 20.0, 22.0, 24.0, 30.0 and 36.0.

2. Perform a test calculation of rivets for shearing and a package of connected sheets for crushing and determine the required number of rivets.

3. Set the rivet pitch in the range from 4 to 8 rivet diameter values and distribute the rivets among the rows, finding the required number of rows, take the distance between the rows equal to half the rivet pitch in the row.

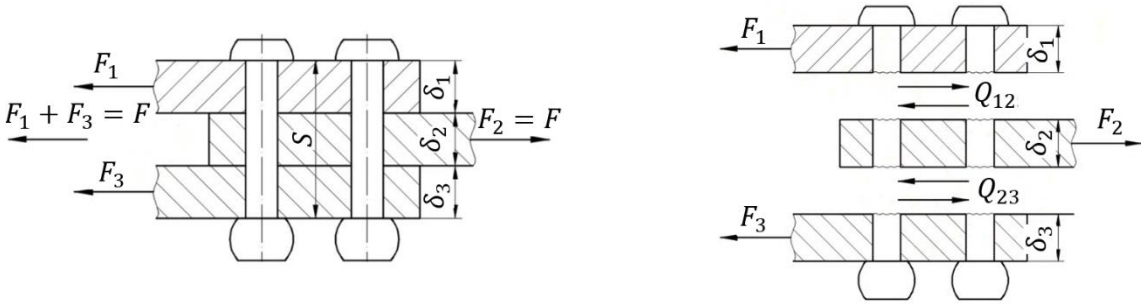
4. Find the effective tangential stress in the sections of rivets subject to the shear, the maximum crushing stress in the joined sheets and the maximum normal stress in the sheets caused by their stretching, taking into account the presence of holes for the rivets, make a conclusion about the strength of the connection of sheets, using the rivets.

Sheet package thickness is: $S = \delta_1 + \delta_2 + \delta_3$=1.0+1.2+0.8=3.0, mm.

It is recommended to determine the diameter of the rivets depending on the thickness of the sheet package as: $d = 2 \times \sqrt{S} = 2 \times \sqrt{3} = 3.46 \approx 3.5$, mm.

The shear area of the rivet is: $A = \frac{\pi \times d^2}{4} = \frac{3.1416 \times 3.5^2}{4} = 9.62$, mm^2.

The distribution of forces (F_1, F_2 and F_3) in the connected sheets (Fig. 6.2.2 a) is as follows: $F_2 = F$=500, kN, $F_1 = F \times \frac{\delta_1}{\delta_1 + \delta_3} = 500 \times \frac{1.0}{1.0+0.8} = 277.8$, kN,
$F_3 = F - F_1 = 500 - 277.8 = 222.2$, kN.



(a) (b)

Fig. 6.2.2

Transverse (the shear) forces acting on the rivets (Fig. 6.2.2 b) are equal to: $Q_{12} = F_1 = 277.8$, kN and $Q_{23} = F_3 = 222.2$, kN.

The shear strength criterion for rivets has the form: $\tau = \frac{[Q]}{n_r \times A} \leq [\tau] = \frac{\tau_y}{[n]}$.

The permissible shear stress for the rivet material is: $[\tau] = \frac{360}{1.4} = 257.1$, MPa.

The required number of rivets n_r, which ensures riveted connection (the seam) per cut, is equal to: $n_r \geq \frac{[Q]}{[\tau]\times A} = \frac{277.8\times10^3}{257.1\times9.62} = 112.3 \approx 113$.

The condition for the crushing strength of holes in connected sheets has the form: $\sigma_C = \left[\frac{F_i}{A_{ci}\times n_r}\right] \leq [\sigma_C]$, where $\left[\frac{F_i}{A_{ci}\times n_r}\right]$ – the largest value among the three (i) connected sheets, that is the largest of the values: $\frac{F_1}{A_{c1}\times n_r}$, $\frac{F_2}{A_{c2}\times n_r}$ and $\frac{F_3}{A_{c3}\times n_r}$.

The crushing areas per each rivet for the corresponding sheet in the connection are equal: $A_{c1} = d \times \delta_1 = 3.5 \times 1.0 = 3.5$, mm^2, $A_{c2} = d \times \delta_2 = 3.5 \times 1.2 = 4.2$, mm^2 and $A_{c3} = d \times \delta_3 = 3.5 \times 0.8 = 2.8$, mm^2.

The contact pressures in the joined sheets, assuming that the sheets were connected by one rivet, are: $\frac{F_1}{A_{c1}} = \frac{277.8}{3.5} = 79.4$, kN/mm^2, $\frac{F_2}{A_{c2}} = \frac{500}{4.2} = 119.0$, kN/mm^2 and $\frac{F_3}{A_{c3}} = \frac{222.2}{2.8} = 79.4$, kN/mm^2.

The permissible stress on the crumple surface of the holes for rivets in the joined sheets is equal to: $[\sigma_C] = 1.3 \times \frac{[\tau]}{0.6} = 1.3 \times \frac{257.1}{0.6} = 557.1$, MPa.

It is assumed that the permissible stress before collapse of the material $[\sigma_C]$ is 1.3 times greater than the permissible normal stress $[\sigma]$, which is related to the permissible tangential stress $[\tau]$ in magnitude: $[\sigma] \approx \frac{[\tau]}{0.6}$.

The number of rivets required for crushing strength criterion is equal to: $n_r \geq \left[\frac{F_i}{A_{ci}}\right] \times \frac{1}{[\sigma_C]} = \frac{500\times10^3}{4.2} \times \frac{1}{557.1} = 213.7 \approx 214$, pieces.

For the solution the sufficient number of rivets of 214 pieces is taken, wherein the greatest stresses arising on the cut of the rivets and on the crumple surfaces of the holes of the joined sheets are equal: $[\tau] = \frac{277.8\times10^3}{214\times9.62} = 134.9 < [\tau] = 257.1$, MPa, $\sigma_C = \frac{500\times10^3}{4.2} \times \frac{1}{214} = 556.3 \leq [\sigma_C] = 557.1$, MPa.

Questions and comments:

6.2.1. Recommended size of rivets depending on the thickness of the package of sheets being joined. Distribution of forces in sheets connected by rows of rivets.

6.2.2. The sequence of determining the characteristics of a rivet joint (the seam) and calculating strength: choosing the size of the rivets, calculating the rivet joint for shear and determining the number of rivets, determining the pitch of the rivet joint, checking the strength of the rivet seam for shear and crushing.

6.2.3. Shear strength condition for rivets. The required number of rivets to ensure the shear strength of the rivet joint (the seam). The number of rivets according to the condition of sufficient joint strength before the material of the joined sheets collapses.

6.3. Geometric and physical characteristics of torsion of a mechanical system element

The torsion is a form of mechanical loading of an element (the shaft) of mechanical system, at which only one internal force factor appears in all its sections – the torque T_z (z – the shaft axis, Fig. 6.3.1). The elements of mechanical systems designed to work under torsion usually have a solid round or annular cross–section.

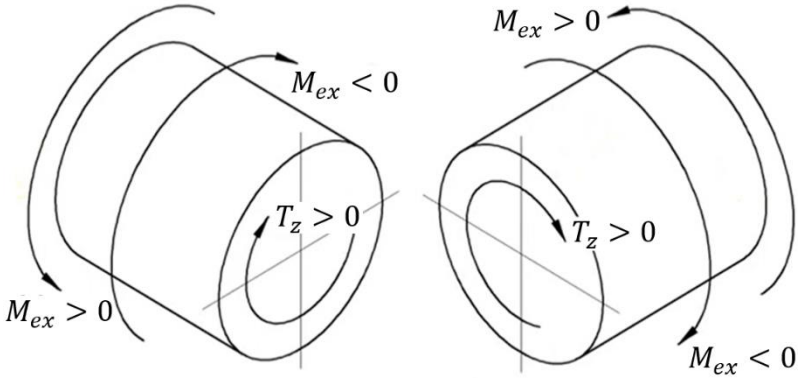


Fig. 6.3.1

Torque moment T_z in an arbitrary section of the shaft is equal in magnitude to the external moment M_{ex}, that is the sum of external moments acting only on one side of the section (in one of the directions of the shaft axis z).

The external moment M_{ex}, that is rotating (twisting) is positive ($T_z > 0$), if it tends to rotate the cut–off part counterclockwise (if viewed from the cross–section side).

Torsion geometry. To explain the geometric characteristics of torsion of an elastic shaft, one can apply a mesh of longitudinal straight lines 1 and transverse circular lines 2 to it, as shown on the diagram (Fig. 6.3.2 a) and apply a grid of radial straight lines 3 and circumferential lines 4 to the transverse section of the shaft. In the lateral projection, all drawn lines are straight, and there are right angles between the longitudinal and transverse lines. If one rotates one end section of the shaft relative to the other (Fig. 6.3.2 b), then torsional deformation will occur, which is visible from the displacements of the plotted grid lines:

1. Projections of transverse and longitudinal lines, as well as radial lines remain straight.

(a) (b)

Fig. 6.3.2

2. Circumferential lines drawn on the section are not deformed.

3. Projections of transverse lines do not tilt, and the distances between them do not change.

4. Longitudinal lines are inclined at the same angle γ, constant along the length of the shaft, rectangular segments of the mesh applied to the shaft from the outside are transformed into parallelograms.

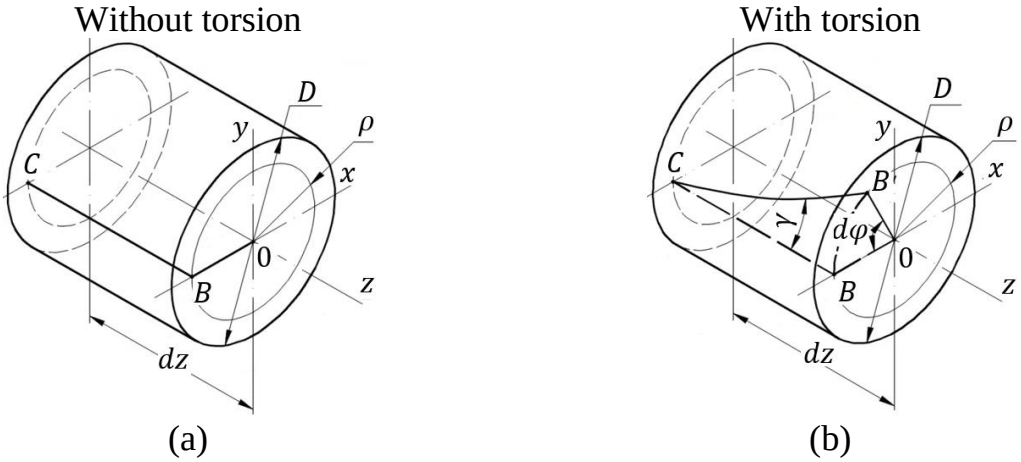
5. Mesh segments in shaft sections maintain right angles.

6. The angles of rotation of the cross–sections (Fig. 6.1.4) perpendicular to the shaft axis increase with increasing distance from the initial cross–section.

When calculating the torsional deformations of a shaft, the hypothesis of flat and rigid sections is used: the cross–sections of the shaft, which were flat before deformation, remain flat after deformation, and each section rotates when the shaft is torsed as a rigid whole (i.e. the angle of rotation at any point in the section is the same).

At each point of the shaft material during torsion the net shear occurs in one plane.

The section turning at torsion is shown on the diagram (Fig. 6.3.3). For a small rotation angle γ the tangent of the rotation angle equals to this angle can be geometrically found as: $tg\ \gamma = \frac{BB'}{BC} = \frac{BB'}{dz}$ and $BB' = tg\ \gamma \times dz = \gamma \times dz$.



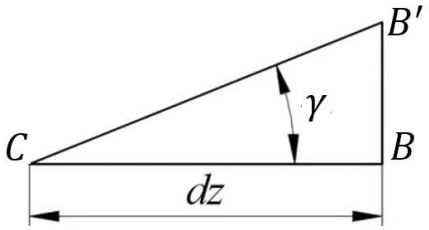
(a) (b)

Fig. 6.3.3

The section turning angle is equal: $\angle 0BB' = \rho \times d\varphi$.

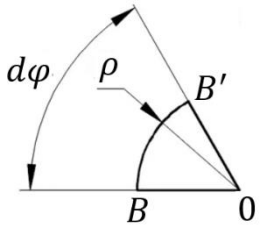
The shear angle γ (Fig. 6.3.4) depends on the angle of the section turn φ and on the section radius ρ: $\gamma = \rho \times \frac{d\varphi}{dz}$.

On the projection of the side of the shaft



(a)

In the plane of the shaft section



(b)

Fig. 6.3.4

Angular deformation γ (the shear) on the axis of the shaft subject to torsion is zero.

Physics of elastic torsion. When combining the Hooke's law: $\tau = G \times \gamma$ in relation to angular deformations and shear stresses during torsion and the geometric dependence: $\gamma = \rho \times \frac{d\varphi}{dz}$, based on the hypothesis of plane sections and elasticity, the resulting dependence of tangential stresses on angular deformations has the form: $\tau = G \times \rho \times \frac{d\varphi}{dz}$.

To find the elementary shear force (the force mathematical increment) at a certain distance ρ – radius from the shaft axis z to the considered point (Fig. 6.3.5), the vicinity dA of the point width – the increment of radius $d\rho$ and the vicinity height, proportional to the increment of the opening angle $d\alpha$, determine the area of the indicated vicinity dA. The elementary force acting on the area dA in the cross–sectional plane of the shaft, equals: $dQ = \tau \times dA$.

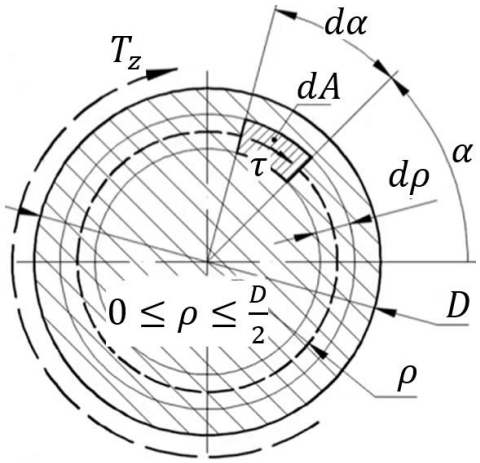


Fig. 6.3.5

The elementary moment of the force dQ, torque shaft relative to it axis z, is equal: $dT_z = \rho \times dQ = \tau \times \rho \times dA$.

The resulting moment T_z is determined by integrating over the cross–sectional area of the shaft A: $T_z = \int_A \ dT_z = \int_A \ \tau \times \rho \times dA$.

Taking into account the proportionality of tangential stresses and angular deformations according to Hooke's law: $\tau = G \times \rho \times \frac{d\varphi}{dz}$ the magnitude of the torque depending on the angle of the cross–section turn φ one can find as: $T_z = \int_A \ G \times \rho^2 \times \frac{d\varphi}{dz} \times dA$.

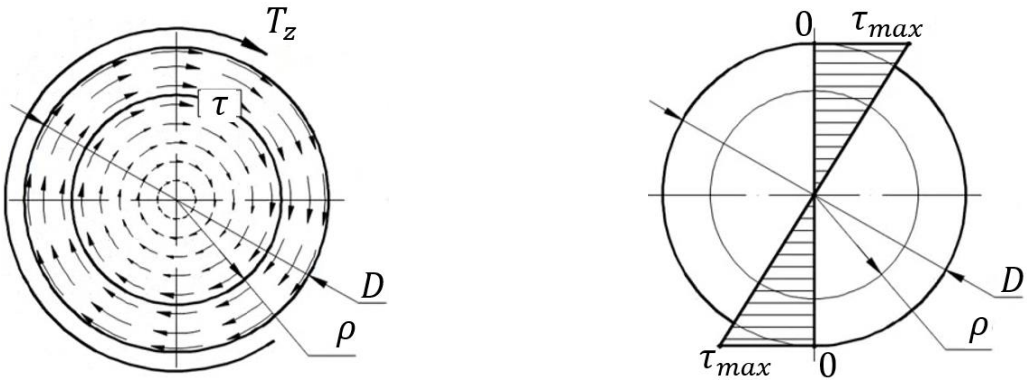
After taking factors independent of the area segment dA out of the sign of the integral, the torque T_z is determined as: $T_z = G \times \frac{d\varphi}{dz} \times \int_A \ \rho^2 \times dA = G \times \frac{d\varphi}{dz} \times I_\rho$, where $I_\rho = \int_A \ \rho^2 \times dA$ – the polar moment of inertia of the shaft cross–sectional area (the geometric characteristic of the section).

The polar moment of inertia of circular cross–section of a shaft is found as follows: $I_\rho = \int_A \ \rho^2 \times dA = \int_0^{D/2} \int_0^{2\times\pi} \rho^3 \times d\alpha \times d\rho = \int_0^{D/2} 2 \times \pi \times \rho^3 \times d\rho = \frac{\pi \times D^4}{32}$,
where dA – the element of the shaft cross–sectional area increasing with radius:
$dA = d\rho \times \rho \times d\alpha$; D – the shaft diameter ($D/2$ – radius).

The polar moment of inertia of the annular cross–sectional area of the shaft is equal to: $I_\rho = \frac{\pi \times D^4}{32} - \frac{\pi \times d^4}{32} = \frac{\pi \times D^4}{32} \times (1 - k^4)$, $k = d/D$, where d – the internal diameter of the shaft annular section.

The dependence of tangential stresses on the radius of a circular shaft one can determine using two entries of the Hooke's law on the proportionality of torque and angle of the section turn: $T_z = G \times \frac{d\varphi}{dz} \times I_\rho$ and tangent stress with the angle of the section turn: $\tau = G \times \rho \times \frac{d\varphi}{dz}$. Both formulas have common factors: $G \times \frac{d\varphi}{dz}$. If use the indicated formulas and reduce the common part, then one can find the tangent stresses τ at torsion depending on the radius ρ: $\tau = \frac{T_z}{I_\rho} \times \rho$, and the greatest tangential stress τ_{max}, corresponding to the outer radius of the shaft ($D/2$), equals: $\tau_{max} = \frac{T_z}{W_\rho}$, where W_ρ – the polar moment of resistance during shaft torsion: $W_\rho = I_\rho/(D/2)$. The polar moment of resistance for a circular shaft section is equal: $W_\rho = \frac{\pi \times D^3}{16}$.

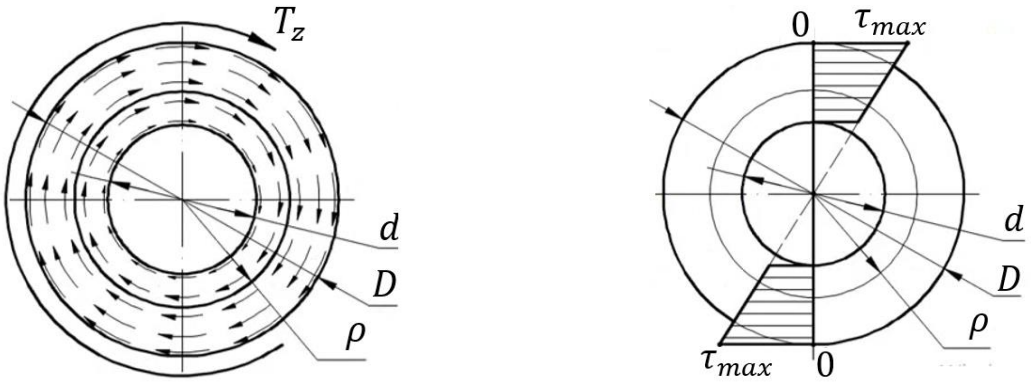
Tangent stresses τ at torsion linearly depend on the radius ρ of the location of the elementary area dA relative to the shaft axis. The distribution of tangential stresses along the radius ρ of the circular shaft section is shown on the diagram (Fig. 6.3.6). The combined effect of tangential stresses in the cross–section of the shaft during its torsion gives a torque T_z.



(a) (b)

Fig. 6.3.6

The distribution of tangential stresses τ along the radius ρ in the section of the annular shaft is shown on the diagram (Fig. 6.3.7).



(a) (b)

Fig. 6.3.7

The polar moment of resistance of an annular shaft is equal: $W_\rho = \frac{\pi \times D^3}{16} \times (1 - k^4)$.

Questions and comments:

6.3.1. Torsion is a form of mechanical loading of a shaft, in which only one internal force factor appears in all its sections – the torque.

6.3.2. The torque in an arbitrary cross–section of the shaft is equal in magnitude to the external moment, i.e. the sum of external moments acting only on one side of the cross–section (in one of the directions of the shaft axis).

6.3.3. Offset of plotted mesh lines showing the geometric shape of the torsion: the plotted lines of the mesh maintain straightness when the shaft is torsed, the longitudinal lines marked on the shaft tilt at the same angle when torsioned, the rectangular mesh elements applied to the shaft from the outside are transformed into a parallelogram shape,

circumferential lines marked on the shaft section are not deformed, the mesh segments in shaft sections maintain right angles.

6.3.4. The hypothesis of flat and rigid sections under shaft torsion: the cross–sections that were flat before deformation remain flat after deformation, and each section rotates as a rigid whole when the shaft is torsioned. At each point of the shaft material during torsion, a net shear occurs in one plane.

6.3.5. Determination of the shear angle (the deviation from the rectangular shape of the mesh segment on the surface of the shaft) depending on the radius of the shaft section and on the derivative of the angle of rotation of the shaft section along its length. Equal to zero shift on the shaft axis during its torsion (at zero radius).

6.3.6. Linear distribution of tangential stresses along the radius of the shaft section during it torsion, the Hooke's law during elastic torsion of the shaft.

6.3.7. Elementary torque (the moment increment) created by tangential stresses in the section of the shaft during its torsion, and the resulting moment determined by integrating the elementary moment over the cross–sectional area.

6.3.8. Dependence of torque on the angular deformation of the shaft (on the derivative of the angle of rotation of the section along the length of the shaft), taking into account the elasticity of the material and the linear proportionality between torque and angular deformation of the shaft according to the Hooke's law.

6.3.9. Geometric characteristics of the circular cross–section of the shaft, the polar moment of inertia of the area of the circular cross–section (solid or annular), the polar moment of resistance of the shaft.

6.3.10. Distribution along the radius of a circular cross–section of tangential stresses that act along the lines of circles in the cross–section of the shaft, both in solid one and annular.

6.4. Evaluation of shaft strength and torsional rigidity

The criterion of the shaft strength at torsion has the form: $\tau_{max} = \frac{T_z}{W_\rho} \leq [\tau]$.

Types of shaft torsional strength calculations:

1. The verification calculation. Data: the torque T_z, the polar moment of resistance of shaft section W_ρ and permissible tangential stress of the material of shaft $[\tau]$. The strength of the shaft is checked when the torque is applied to it T_z.

2. The design calculation. Data: the torque T_z and permissible tangential stress of the material of shaft $[\tau]$. By the criterion of torsional strength, the necessary polar moment of resistance of shaft section $[W_\rho]$ is sought: $[W_\rho] \geq \frac{T_z}{[\tau]}$.

3. Finding the maximum load (the torque) $[T_z]$ of a shaft subject to torsion. Data: the polar moment of resistance of shaft section W_ρ and permissible tangential stress of the shaft material $[\tau]$. According to the strength criterion: $[T_z] \le [\tau] \times W_\rho$, the permissible torque is to be find.

The permissible tangential stress $[\tau]$ for plastic materials is equal to the yield strength τ_y (the temporary stress), for brittle materials the permissible tangential stress is equal to the ultimate shear: $[\tau] = \tau_y/[n]$, taking into account the safety factor $[n]$.

Torsional rigidity of shaft. The angle of rotation of the section depends on its position along the axis of shaft (Fig. 6.3.4 b), which in the form of angle increment $d\varphi$ is: $d\varphi = \frac{T_z}{G \times I_\rho} \times dz$.

The twist angle of shaft φ, which is a relative indicator measured between shaft sections, proportional to torque T_z, the distance between sections along the z–axis of the shaft, in particular, the length of the shaft, if the angle of twist between its extreme sections is sought, and inversely proportional to the linear stiffness of shaft ($G \times I_\rho$) is determined by integration along the z–axis along the length of the shaft l and equals:

$\varphi = \int_l \frac{T_z}{G \times I_\rho} \times dz$.

When the shape and elasticity of its section are constant along the length l of shaft, for example, for a cylindrical solid or tubular shaft, the turn angle of the end section is determined as: $\varphi = \frac{T_z \times l}{G \times I_\rho}$, and the relative angle of twist is equal to: $\theta = \frac{d\varphi}{dz} = \frac{\varphi}{l} = \frac{T_z}{G \times I_\rho}$.

The rigidity of shaft one can be considered sufficient when the maximum relative twist angle θ_{max} does not exceed a certain permissible value $[\theta]$:

$\theta_{max} = \left|\frac{T_z}{G \times I_\rho}\right|_{max} \le [\theta]$, where $[\theta]$ – the permissible relative twist angle:

$[\theta] = \frac{\pi}{180^\circ} \times [\theta]^\circ$ (radian/m).

Types of calculations for shaft torsional rigidity:

1. The verification calculation of permissible relative angle of torsion: $\theta_{max} \le [\theta]$ under moment of given value of a shaft having known characteristics (the shape and cross–sectional area, sectional modulus, elasticity characteristics of the structural material).

2. Design calculation – the determination of the structural material and the necessary geometric characteristics of the shaft section (the dimensions, area, polar moment of inertia of the section) for its sufficient rigidity: $I_\rho \ge \left|\frac{T_z}{G \times [\theta]}\right|_{max}$.

3. Finding the ultimate load – the torque $[T_z]$ (load capacity of the mechanical system), at which the angle of rotation of the shaft, which has known characteristics, does not exceed the permissible value: $[T_z] \le [\theta] \times G \times I_\rho$.

Questions and comments:

6.4.1. The strength criterion of the shaft in torsion. The permissible shear stress under shaft torsion, equal to the shear stress yield strength for plastic structural materials and equal to this strength limit with a margin for brittle structural materials.

6.4.2. Types of shaft torsional strength calculations. The verification calculation – finding the highest effective tangential stresses under torsion of a shaft having known characteristics (the shape and cross–sectional area, moment of resistance), a moment of a given value, checking according to the criterion of strength of shaft in torsion.

6.4.3. Design calculation – the determination of shaft characteristics: structural material, the shape, dimensions and cross–sectional area, moment of resistance at which it is strong enough when exposed to assigned torque.

6.4.4. Finding the maximum load that the shaft with assigned geometric characteristics of cross–section and made of specified material can withstand in torsion.

6.4.5. The angle of rotation of the shaft section, proportional to the effective torque, to the distance between sections along the axis of shaft and inversely proportional to the linear stiffness of the shaft, determined by integration along longitudinal coordinate axis of shaft.

6.4.6. Criterion for sufficient of torsional rigidity of the shaft, meaning that the maximum relative torsion angle does not exceed the permissible value.

6.4.7. Types of shaft torsional stiffness calculations. The verification calculation of the permissible relative angle of twisting by the moment of a given value of a shaft having known characteristics.

6.4.8. The design calculation of the required polar moment of inertia of the shaft section to ensure its sufficient rigidity.

6.4.9. Finding the maximum load – the torque at which the angle of rotation of a shaft with known characteristics does not exceed the permissible magnitude.

6.5. Determining the torque diagram and assessing the strength and rigidity of shaft

Exercise. The shaft of variable cross-section (Fig. 6.5.1) is loaded with torques of different magnitudes and directions: $M_1 = 0.8$ kN×m, $M_2 = 1.5$ kN×m, $M_3 = 1.0$ kN×m and $M_4 = 2.5$ kN×m by sections of length: $l_1 = 1.8$ m, $l_2 = 1.8$ m, $l_3 = 1.0$ m and $l_4 = 1.2$ m.

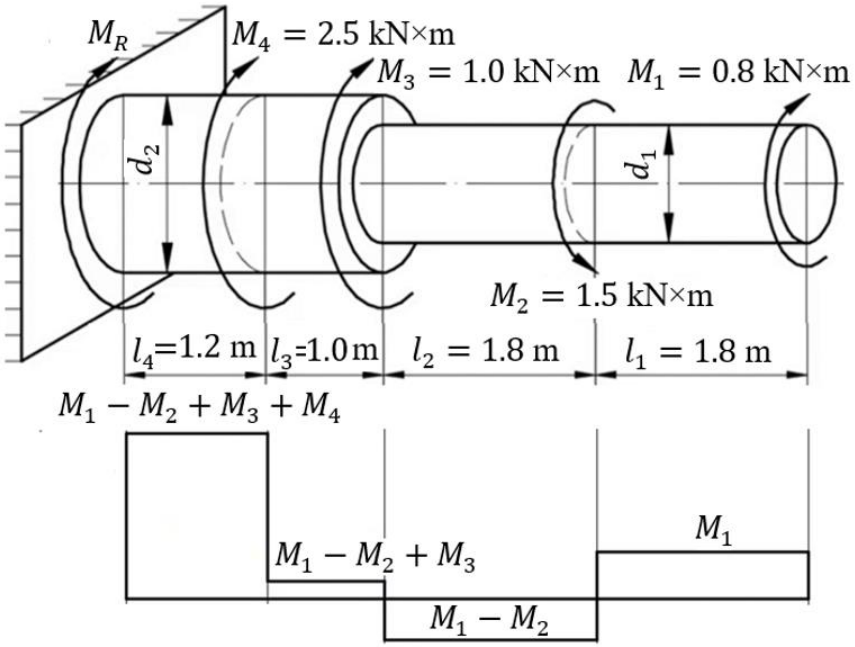


Fig. 6.5.1

The balance conditions for torques M_z in the first piece of shaft to the right of section z_1 (Fig. 6.5.2) looks like: $\sum M_{z+} = 0$, $-T_{z1} + M_1 = 0$ and $T_{z1} = M_1 = 0.8$, kN×m.

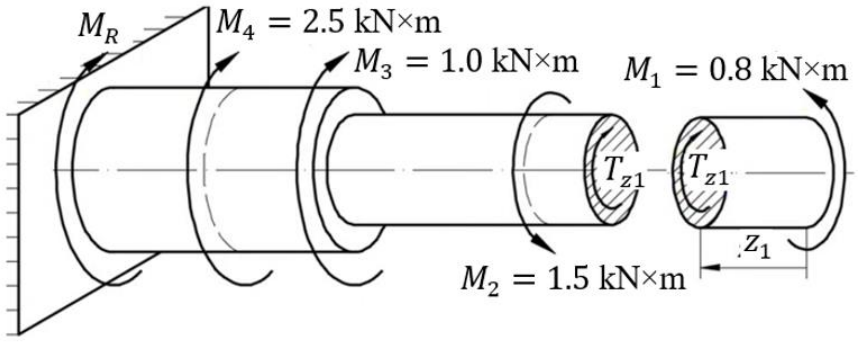


Fig. 6.5.2

On the first piece of the shaft l_1, as well as on the rest of it pieces (l_2, l_3 and l_4) in the intervals between sections in which torques are applied: M_1, M_2, M_3, M_4 and M_R, the value of total moment acting in the sections does not change: $T_{z1}(z_1) = const$.

Balance conditions for torques M_z in first piece to the left of section z_1 (Fig. 6.5.2) are: $\sum M_{z-} = 0$, $-T_{z1} + M_2 - M_3 - M_4 - M_R = 0$ and $T_{z1} = M_2 - M_3 - M_4 - M_R$.

Conditions of balance of torques M_z on the second piece to the right of the section z_2 (Fig. 6.5.3) looks like: $\sum M_{z+} = 0$, $-T_{z2} + M_1 - M_2 = 0$
and $T_{z2} = M_1 - M_2 = 0.8 - 1.5 = -0.7$, kN×m.

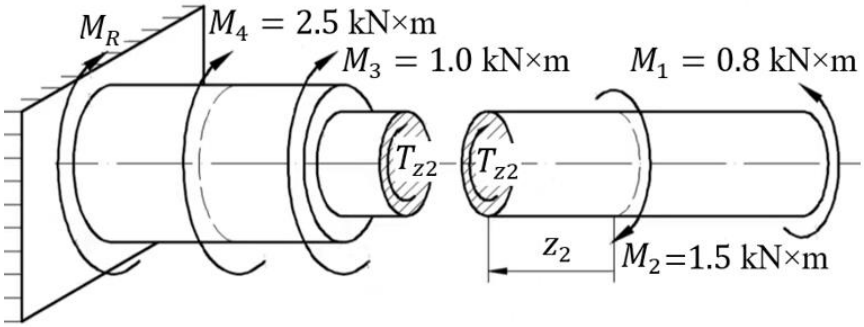


Fig. 6.5.3

Conditions of the balance of torque moments M_z in the second piece to the left of the section z_3 (Fig. 6.5.3) looks like: $\sum M_{z-} = 0$, $-T_{z2} - M_3 - M_4 - M_R = 0$ and $T_{z2} = -M_3 - M_4 - M_R$.

Conditions of the balance of torques M_z on the third piece to the right of section z_3 (Fig. 6.5.3) are: $\sum M_{z+} = 0$, $-T_{z3} + M_1 - M_2 + M_3 = 0$ and $T_{z3} = M_1 - M_2 + M_3 = 0.8 - 1.5 + 1.0 = 0.3$, kN×m.

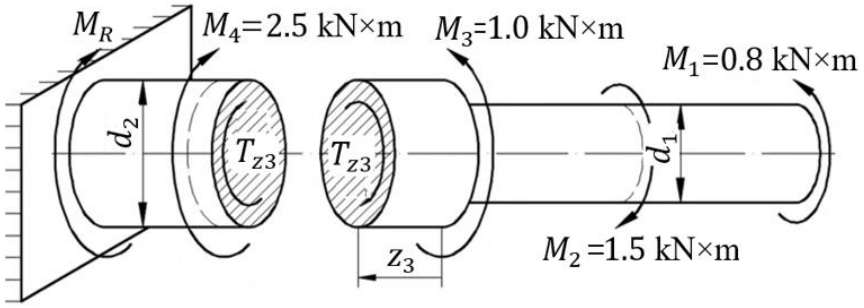


Fig. 6.5.4

Conditions of the balance of torques M_z on the third piece to the left of section z_3 (Fig. 6.5.4) are: $\sum M_{z-} = 0$, $-T_{z3} - M_4 - M_R = 0$ and $T_{z3} = -M_4 - M_R$.

Conditions of the balance of torques M_z on the fourth piece to the right of section z_4 (Fig. 6.5.5) looks like: $\sum M_{z+} = 0$, $-T_{z4} + M_1 - M_2 + M_3 + M_4 = 0$ and $T_{z4} = M_1 - M_2 + M_3 + M_4 = 0.8 - 1.5 + 1.0 + 2.5 = 2.8$, kN×m.

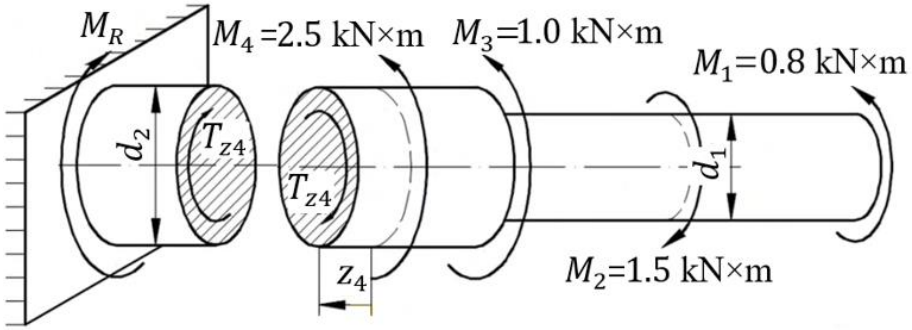


Fig. 6.5.5

Conditions of the balance of torques M_z on the third piece to the left of the section z_4 (Fig. 6.5.5) looks like: $\sum M_{z-} = 0$, $-T_{z4} - M_R = 0$ and $T_{z4} = -M_R$.

Exercise. The torques acting on the pieces of shaft are assigned (Fig. 6.5.6): $M_1 = 3$ kN×m, $M_2 = 4.5$ kN×m, $M_3 = 2$ kN×m and $M_4 = 5$ kN×m. Shaft section lengths are: $l_1 = 0.4$ m, $l_2 = 0.3$ m, $l_3 = 0.2$ m and $l_4 = 0.4$ m.

The shaft consists of first section in the shape of a round pipe, the second and third section in the shape of a round solid cylinder, and the fourth solid section of rectangular cross–section. The ratio of sizes of the rectangular section of fourth piece of the shaft is: $\frac{h}{b} = 2$. The ratio of the circular tubular section of the first piece of the shaft is: $\frac{d}{D} = 0.8$.

Shaft material – the carbon structural steel. Material yield strength at shear (shift): $G = 80\,000$ MPa. The safety factor is equal to: $[n] = 2$.

It is assigned to do the diagram of the torque, to calculate the section sizes for pieces of the shaft corresponding to the appropriate torsional strength of the shaft, to determine the diagrams of tangential stresses distribution in the sections of the shaft, to determine the diagram of rotation angles (the twisting) of the shaft.

On each piece of the shaft (Fig. 6.5.6) the diagram corresponds to a constant torque value. On the first piece of the shaft: $0 \le z_1 \le l_1$ the torque is: $T_{z1} = M_1 = 3.0$ kN×m.

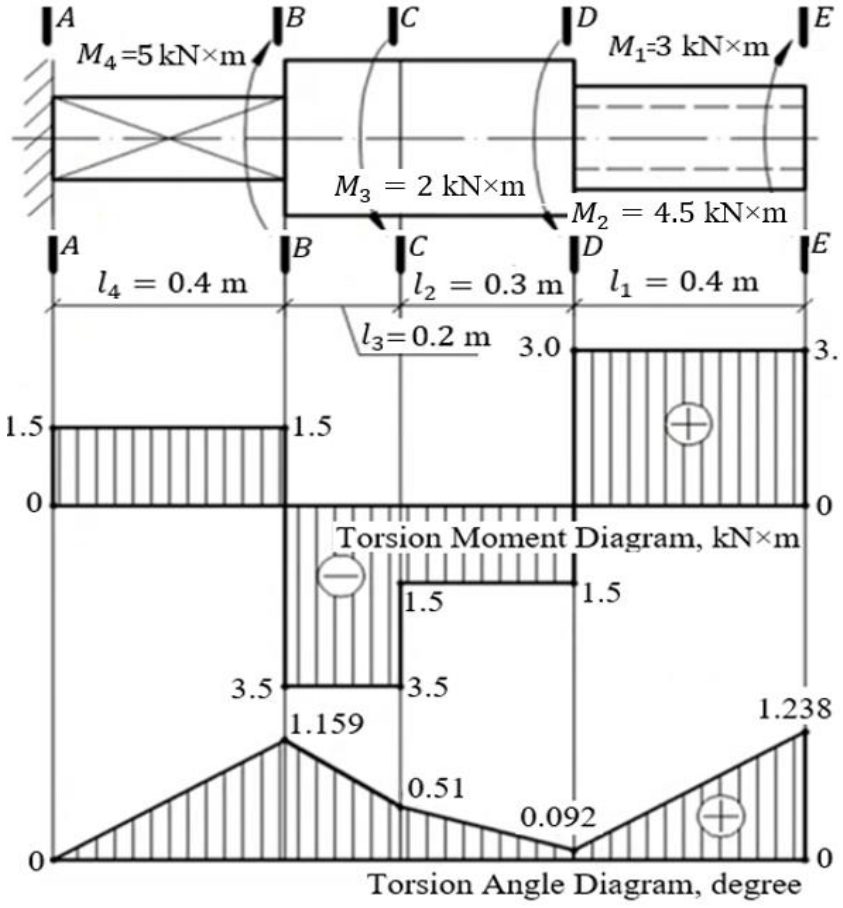


Fig. 6.5.6

The torque on second piece: $0 \le z_2 \le l_2$ is determined as:

$T_{z2} = M_1 - M_2 = 3.0 - 4.5 = -1.5$, kN×m.

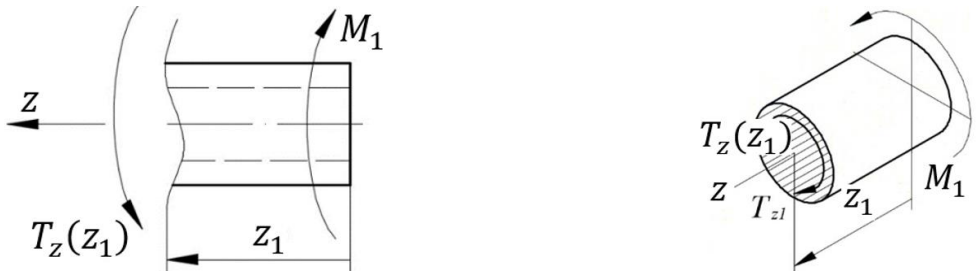
Torque on the third piece of the shaft: $0 \le z_3 \le l_3$ equals:

$T_{z3} = M_1 - M_2 - M_3 = 3.0 - 4.5 - 2.0 = -3.5$, kN×m.

Torque on the fourth piece of the shaft: $0 \le z_4 \le l_4$ is:

$T_{z4} = M_1 - M_2 - M_3 + M_4 = 3.0 - 4.5 - 2.0 + 5.0 = 1.5$, kN×m.

The signs of the torque are determined as follows: the positive direction of the torque M_1 is counterclockwise, when looking towards the torsion axis (Fig. 6.5.7).



(a) (b)

Fig. 6.5.7

Determination of strong dimensions of the shaft cross–section. The strength criterion in the shaft cross–section during torsion is: $\tau_{max} = \frac{T_z}{W_\rho} \le [\tau] = \frac{\tau_y}{[n]}$, where τ_{max} – the maximum effective shear stress, $[\tau]$ – the permissible shear stress, τ_y – the yield strength of structural material, $[n]$ – the safety factor, T_z – the maximum effective torque, W_ρ – shaft moment of resistance. For sufficient shaft strength, the sectional moment of resistance must be no less than: $W_\rho \ge \frac{T_z}{[\tau]}$.

Since for a circular shaft section the moment of resistance, which is a geometric characteristic of the cross–section, is equal to: $W_\rho = \frac{\pi \times D^3}{16}$, then the moment of resistance W_ρ of the section, found according to the strength criterion of a round solid shaft, corresponds to the diameter of the cross–section, which is equal to: $D = \sqrt[3]{\frac{16 \times W_\rho}{\pi}}$.

For a circular annular shaft cross-section, the moment of resistance is:

$W_\rho = \frac{\pi \times D^3}{16} \times (1 - k^4)$.

The moment of resistance W_ρ of the section, found according to the shaft strength criterion, corresponds to the outer diameter of the annular section: $D = \sqrt[3]{\frac{16 \times W_\rho}{\pi \times (1 - k^4)}}$, where $k = \frac{d}{D}$ – the relative internal diameter of the annular section of the shaft, d and D – the internal and external diameters of the annular section.

The polar moment of resistance of the cross–section of a rectangular shaft is related to its characteristics as: $W_\rho = \alpha \times h \times b^2$, where h and b – the major and minor sides of a rectangular shaft cross–section, α – the correction factor for shaft cross–section shape: $\alpha \approx 0.246$ at $\frac{h}{b} = 2$, the other magnitudes of coefficient α are shown in the Table 6.5.1.

Table 6.5.1

h/b	1	1.5	2	4	6	5	10	∞
α	0.208	0.231	0.246	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.229	0.281	0.299	0.307	0.131	0.333
η	1.0	0.859	0.795	0.745	0.743	0.742	0.742	0.742

The moment of resistance of rectangular shaft section is: $W_\rho = \alpha \times (h/b) \times b^3$, found by the criterion of its strength, corresponds to the section dimensions defined as: $b = \sqrt[3]{\frac{W_\rho}{\alpha \times (h/b)}}$ and $h = (h/b) \times b$.

The permissible tangential stress at the yield strength of the structural material, which is: $\tau_y = 250$ MPa, taking into account the safety factor: $[n] = 2$, is equal: $[\tau] = \frac{250}{2} = 125$, MPa.

For the first piece, which has a tubular circular cross–section and is subject to a torque: $T_{z1} = 3.0$ kN×m, the polar moment of the shaft piece resistance subject to torsion required by the strength criterion is equal to: $W_\rho \geq \frac{T_{z1}}{[\tau]} = \frac{3 \times 10^3}{125 \times 10^6} = 24 \times 10^{-6}$, m^3.

The outer diameter of the first section of the shaft with the ratio of inner and outer diameters of the annular pipe equal to: $\frac{d}{D} = 0.8$, has to be no less than: $D = \sqrt[3]{\frac{16 \times 24 \times 10^{-6}}{3.1416 \times (1 - 0.8^4)}} = 59.2 \times 10^{-3}$, m.

From the list of standard nomenclature of steel pipes for the first piece of the shaft, one can select a round pipe with an outer diameter: $D = 60$ mm and with inner diameter: $d = 0.8 \times D = 0.8 \times 60 = 48$, mm.

For the second and third pieces of the shaft, which have a general shape of a round solid section, which, according to the diagram, is affected by torque, which largest magnitude is equal to: $\left|T_{z2,3}\right| = 3.5$ kN×m, the polar moment of resistance, required by the strength criterion, is: $W_\rho \geq \frac{\left|T_{z2,3}\right|}{[\tau]} = \frac{3.5 \times 10^3}{125 \times 10^6} = 28 \times 10^{-6}$, m^3.

In this case, the total diameter of the round solid section of the second and third pieces of the shaft must be no less than: $D = \sqrt[3]{\frac{16 \times 28 \times 10^{-6}}{3.1416}} = 52.3 \times 10^{-3}$, m.

From the list of standard nomenclature for the second and third pieces of the shaft, one can select a solid cylinder with an outer diameter: $D = 53$ mm.

For the fourth section of the shaft, which has a solid rectangular cross–section subject to a torque equal to: $T_{z4} = 1.5$ kN×m, the required polar moment of resistance is: $W_\rho \geq \frac{T_{z4}}{[\tau]} = \frac{1.5\times10^3}{125\times10^6} = 12 \times 10^{-6}$, m^3.

The width of the fourth piece of shaft, which has a rectangular cross-section, has to be no less than: $b = \sqrt[3]{\frac{12\times10^{-6}}{0.246\times2}} = 29.0 \times 10^{-3}$, m. From the list of standard nomenclature for the fourth piece of shaft, one can select a value for the section width equal to: $b = 30$ mm, accordingly, the section height of the fourth piece of the shaft will be: $h = 2 \times 30 = 60$, mm.

After finding the strong dimensions of the sections of the shaft subject to torsion, and after selecting rolled metal in the standard nomenclature, to determine the effective tangential stresses, it is necessary to clarify the geometric characteristics of the sections.

The polar moment of resistance W_ρ and moment of inertia I_ρ of the cross–section of the first piece of shaft, which has the shape of a round pipe, are equal:
$W_\rho = \frac{\pi\times D^3}{16} \times (1 - k^4) = \frac{3.1416\times0.06^3}{16} \times (1 - 0.8^4) = 25.03 \times 10^{-6}$, m^3
and $I_\rho = \frac{\pi\times D^4}{32} \times (1 - k^4) = \frac{3.1416\times0.06^4}{32} \times (1 - 0.8^4) = 75.08 \times 10^{-8}$, m^4.

The polar moment of resistance W_ρ and the moment of inertia I_ρ of common cross–section of the second and third pieces of shaft, having the shape of a round solid cylinder, are equal: $W_\rho = \frac{\pi\times D^3}{16} = \frac{3.1416\times0.053^3}{16} = 29.2 \times 10^{-6}$, m^3
and $I_\rho = \frac{\pi\times D^4}{32} = \frac{3.1416\times0.053^4}{32} = 77.43 \times 10^{-8}$, m^4.

The polar moment of resistance W_ρ and the moment of inertia I_ρ of the cross–section of the fourth piece of the shaft, which has a solid rectangular cross–section, is:
$W_\rho = \alpha \times h \times b^2 = 0.246 \times 0.06 \times 0.03^2 = 13.28 \times 10^{-6}$, m^3
and $I_\rho = \beta \times h \times b^3 = 0.229 \times 0.06 \times 0.03^3 = 37.10 \times 10^{-8}$, m^4.

Coefficient β, like the coefficient α, can be determined from Table 6.5.1 depending on the aspect ratio $\frac{h}{b}$ of the rectangular section of the solid piece of the shaft when finding the polar moments: the moment of inertia of the cross–section and moment of resistance.

To do the diagrams of tangential stresses in the cross–sections of a shaft under its torsion it is necessary to find values of these stresses on the circular surfaces of the shaft and at characteristic points of non–circular surfaces. The tangential stresses in the cross–section during shaft torsion can be found using the formulas: $\tau = \frac{T_z}{I_\rho} \times \rho$ and $\tau_{max} = \frac{T_z}{W_\rho}$.

The tangential stresses in the cross–sections of a shaft having a circular tubular cross–section, at radius distances from the shaft axis ($D/2$ and $d/2$), i.e. on the outer τ_{max} and on the inner τ_{min} surfaces of the tubular cross–section are equal:

$\tau_{max} = \frac{3\times10^3}{25.03\times10^{-6}} = 119.9 \times 10^6 = 119.9$, MPa

and $\tau_{min} = \frac{T_z}{I_\rho} \times \frac{d}{2} = \frac{3\times10^3}{75.08\times10^{-8}} \times \frac{0.048}{2} = 95.9 \times 10^6 = 95.9$, MPa.

The diagram of tangential stresses in the section on the first piece of shaft is shown in the scheme (Fig. 6.5.8 a).

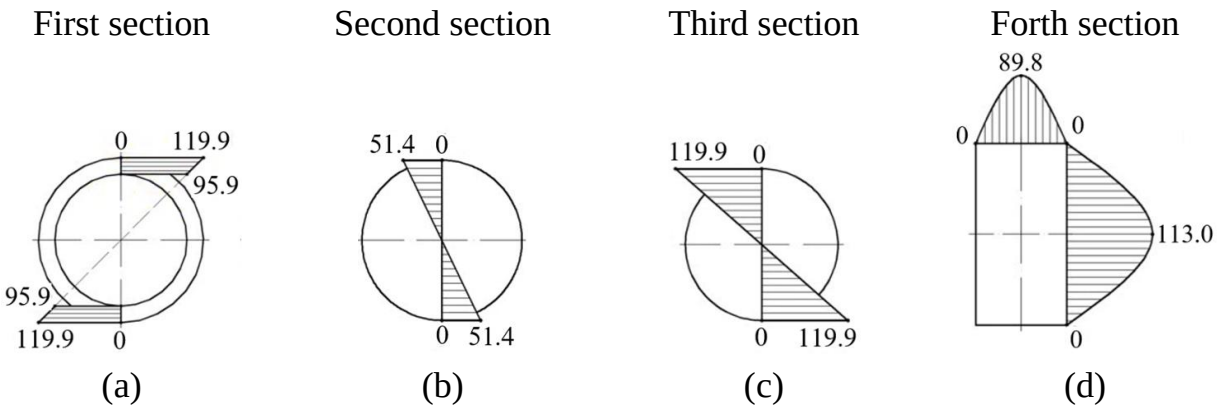


Fig. 6.5.8

Tangential stresses on the surface of the shaft on its round cylindrical pieces 2 and 3, which have a solid cross–section, at a radial distance from the shaft axis ($D/2$), i.e. the highest tangential stresses in cross–sections depend on the torques acting in these sections ($T_{z2} = -1.5$ kN×m and $T_{z3} = -3.5$ kN×m) are equal:

$\tau_{max} = \frac{-1.5\times10^3}{29.2\times10^{-6}} = -51.4 \times 10^6 = -51.4$, MPa on the second piece (Fig. 6.5.8 b)

and $\tau_{max} = \frac{-3.5\times10^3}{29.2\times10^{-6}} = -119.9 \times 10^6 = -119.9$, MPa on the third piece (Fig. 6.5.8 c).

The tangential stresses on the surfaces (in the middle of the faces) of the fourth piece of shaft, which has the rectangular shape of solid cross–section, are equal to:

$\tau'_{max} = \frac{T_z}{W_\rho} = \frac{1.5\times10^3}{13.28\times10^{-6}} = 113.0 \times 10^6 = 113.0$, MPa (Fig. 6.5.8 d)

and $\tau''_{max} = \eta \times \tau'_{max} = 0.795 \times 113.0 = 89.8$, MPa.

Since in the considered issue the permissible tangential stress of the material is equal to: $[\tau] = \frac{250}{2} = 125$, MPa, then in cross–sections in all sections of the shaft, by calculating the strength characteristics based on the highest tangential stresses during torsion of the shaft, the criterion of its strength is met.

The determining of shaft twist angles diagrams. Taking into account the linearity of the torque diagram and the uniform distribution of torque values within each piece of

the shaft, the final increment in the angle of twist of the shaft on each piece can be found as: $\Delta\varphi = \frac{T_z}{G} \times \frac{l}{I_\rho}$.

The increments in the angle of twist $\Delta\varphi$ over the shaft pieces are equal to:

$\Delta\varphi_1 = \frac{3.0\times10^3}{80\times10^9} \times \frac{0.4}{75.08\times10^{-8}} = 0.0200$, radian,

$\Delta\varphi_2 = \frac{-1.5\times10^3}{80\times10^9} \times \frac{0.3}{77.43\times10^{-8}} = -0.0073$, radian,

$\Delta\varphi_3 = \frac{-3.5\times10^3}{80\times10^9} \times \frac{0.2}{77.43\times10^{-8}} = -0.0113$, radian

and $\Delta\varphi_4 = \frac{1.5\times10^3}{80\times10^9} \times \frac{0.4}{37.10\times10^{-8}} = 0.0202$, radian.

The determining of diagram of the shaft twist angles (Fig. 6.5.6) can be done starting from the shaft embedment – the leftmost cross–section, the twist angle of which is zero: $\varphi_A = 0$. Twist angles of the remaining cross–sections of the shaft (A, B, C, D and E) are: $\varphi_B = \varphi_A + \Delta\varphi_4 = 0 + 0.0202 = 0.0202$, radian, 1.159^0,

$\varphi_C = \varphi_B + \Delta\varphi_3 = 0.0202 - 0.0113 = 0.0089$, radian, 0.510^0,

$\varphi_D = \varphi_C + \Delta\varphi_2 = 0.0089 - 0.0073 = 0.0016$, radian, 0.092^0

and $\varphi_E = \varphi_D + \Delta\varphi_1 = 0.0016 + 0.0200 = 0.0216$, radian, 1.238^0.

Based on the found values of the twist angles of the sections, the diagram of the shaft twist angles is done.

Questions and comments:

6.5.1. The balance of moments twisting a shaft of variable cross–sectional pieces. Determination of the shaft embedding reaction.

6.5.2. The torsional strength criterion of a variable–section shaft by moments, acting on the shaft distributed over it pieces. Polar moment of inertia and polar moment of resistance of the shaft section – for round solid section, the tubular and the rectangular.

6.5.3. Determination of the diagram of torques acting on the shaft and diagrams of the distribution of tangential stresses in the sections of the shaft.

6.5.4. Shaft torsional rigidity. Diagram of rotation angles (the twisting) of the shaft, determined by shaft length pieces, and finding the angles of rotation relative to the shaft support fastening. Condition for the sufficient shaft rigidity.

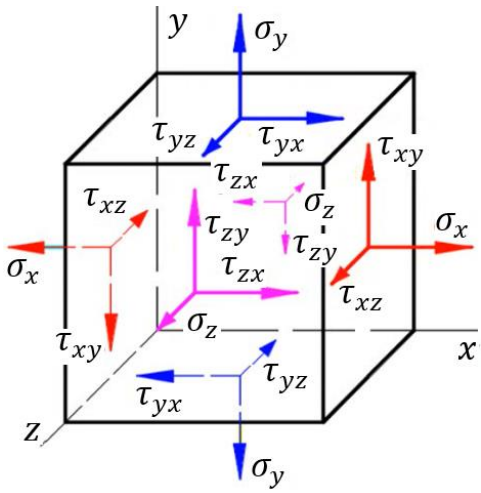
6.5.5. The sequence of the design calculation of a shaft subject to torsion, the formation of the equilibrium equations, the determination of support reactions at embedding of a statically definable shaft, the determination of the torque diagram and the diagrams of distribution of tangential stresses in the shaft section and the determining of the section dimensions according to the shaft strength criterion.

Chapter 7. Stress–strain state of elastic elements of mechanical systems, hypotheses of plasticity and destruction of elements, main areas and main stresses

7.1. Stresses on inclined areas of sections of mechanical system elements

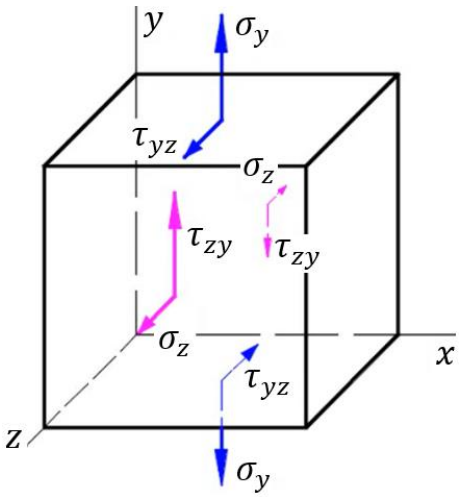
The action of stresses on the volume element of an elastic material, taking into account the balance of normal stresses: σ_x, σ_y and σ_z and pair equilibrium in terms of the magnitude of tangential stresses: $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$ shown on the diagram of the action of stresses on the faces of an elementary loaded element at its three–dimensional stress state (at stress–strain state) and at plane stress state (Fig. 7.1.1 a).

Three–dimensional stress state



(a)

Plane stress state



(b)

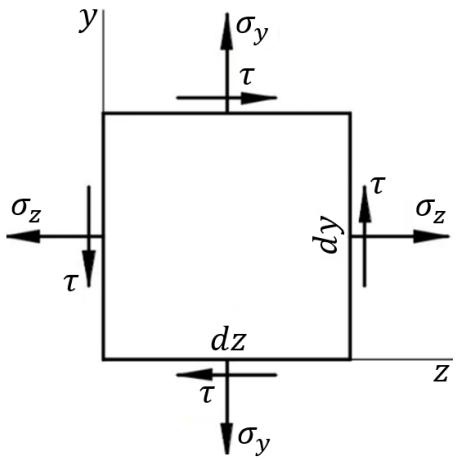
Fig. 7.1.1

In particular, the volume element of an elastic material is in a plane stressed state (Fig. 7.1.1 b and Fig. 7.1.2 a) when the stresses acting in one of the orthogonal directions are equal to zero (Fig. 7.1.1 b, for example, in the direction x: $\sigma_x = 0$ and $\tau_{xy} = \tau_{yx} = \tau_{xz} = \tau_{zx} = 0$).

In practical issues, the version of a plane stress state can serve as an actual simplification of the diagram of a three–dimensional stress state: σ_y, σ_z and τ, in cases where such simplification of the loading scheme is possible.

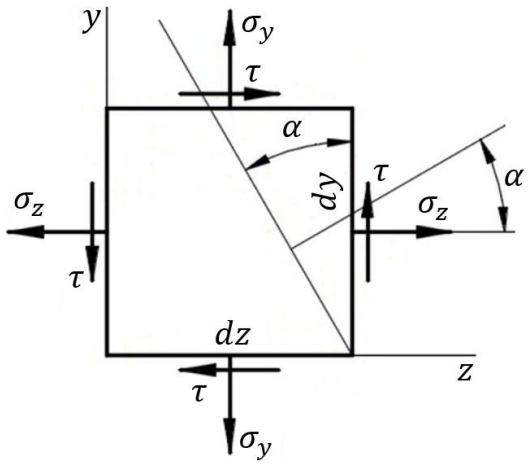
Depending on the inclination α of the elementary area dy (Fig. 7.1.2 b) one can find its changed value (the length dl and area dA, Fig. 7.1.3 a) and stress acting on the inclined area (σ_α and τ_α). To do this, consider the condition of the balance of the part of the element cut off by the inclined site.

Projection of the stress state onto a plane



(a)

Mechanical stresses on different sides of an inclined site

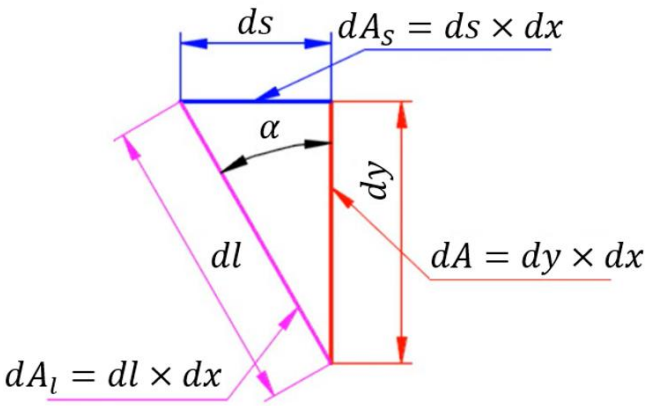


(b)

Fig. 7.1.2

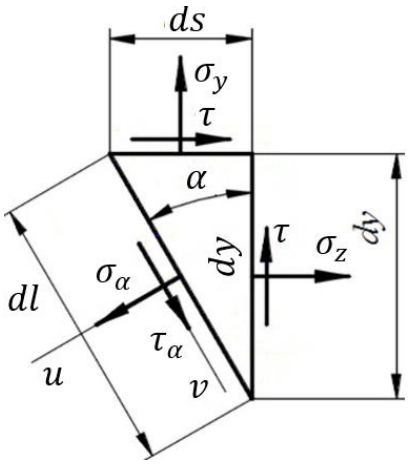
The ratio of the length dl and the initial size dy with the slope α of elementary site and the ratio of the horizontal size ds to the size dy are (Fig. 7.1.3 a):
$dl/dy = 1/\cos\alpha$ and $ds/dy = 1/\operatorname{tg}\alpha$.

Side lengths and areas of elementary surfaces



(a)

Stresses in an inclined area

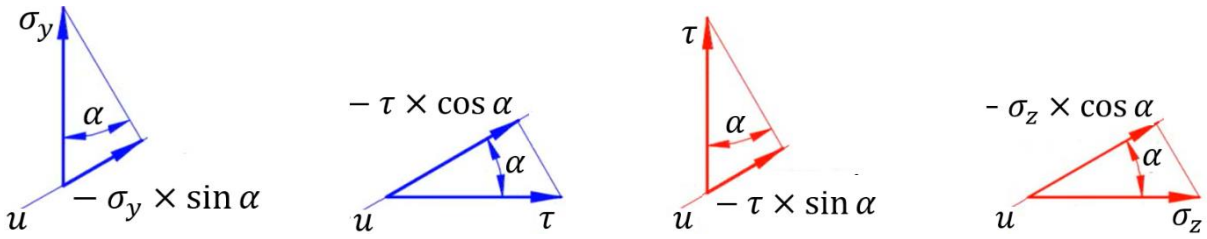


(b)

Fig. 7.1.3

In a plane stress state, to find normal stress σ_α in an inclined elementary site, the equilibrium condition is required of the element cut off by the site in projection on the normal axis u (Fig. 7.1.4 a and b) to the inclined site: $\sigma_\alpha \times dA_l - \tau \times dA_S \times \cos\alpha - -\sigma_y \times dA_S \times \sin\alpha - \tau \times dA \times \sin\alpha - \sigma_z \times dA \times \cos\alpha = 0$.

When finding projections of stresses on directions inclined to the direction axes the vector schemes are used (Fig. 7.1.4 a–d).



(a) (b) (c) (d)

Fig. 7.1.4

Condition of the balance of the element cut off by the elementary site (Fig. 7.1.3 b), if the increment of the x coordinate in the plane of parallel sections dx is taken out as a common factor, has the form: $(\sigma_\alpha \times dl - \tau \times ds \times \cos\alpha - \sigma_y \times ds \times \sin\alpha - -\tau \times dy \times \sin\alpha - \sigma_z \times dy \times \cos\alpha) \times dx = 0.$

Determining the lengths of the sides of the element cut off by the site with its original size dy: $\sigma_\alpha \times \frac{dy}{\cos\alpha} - \tau \times dy \times \frac{\sin\alpha}{\cos\alpha} \times \cos\alpha - \sigma_y \times dy \times \frac{\sin\alpha}{\cos\alpha} \times \sin\alpha - -\tau \times dy \times \sin\alpha - \sigma_z \times dy \times \cos\alpha = 0$, to convert one can perform a reduction by a common factor dy, as well as a multiplication by a common factor $\cos\alpha$:
$\sigma_\alpha - \tau \times \sin\alpha \times \cos\alpha - \sigma_y \times \sin^2\alpha - \tau \times \sin\alpha \times \cos\alpha - \sigma_z \times \cos^2\alpha = 0.$

The resulting equation for finding the normal stress in an inclined elementary site can be transformed using the formula for the sine of a double angle:
$2 \times \sin\alpha \times \cos\alpha = \sin(2 \times \alpha)$, then in a plane stress state the normal stress σ_α in an inclined elementary site taking into account the balance of the part cut off from the loaded element, equals: $\sigma_\alpha = \sigma_y \times \sin^2\alpha + \sigma_z \times \cos^2\alpha + \tau \times \sin(2 \times \alpha).$

In a plane stress state, to find the tangent stress τ_α in an inclined elementary site, the condition of the balance of the part of the element cut off by the elementary site in the projection of forces on the tangent axis v has the form: $(\tau_\alpha \times dl + \tau \times ds \times \sin\alpha - -\sigma_y \times ds \times \cos\alpha - \tau \times dy \times \cos\alpha + \sigma_z \times dy \times \sin\alpha) \times dx = 0,$ to transform one can determine the lengths of the sides of the element cut off by the site with its original size dy, reduce by the common factor dy, and also multiply by the common factor $\cos\alpha$:
$\tau_\alpha + \tau \times \sin^2\alpha - \sigma_y \times \sin\alpha \times \cos\alpha - \tau \times \cos^2\alpha + \sigma_z \times \sin\alpha \times \cos\alpha = 0.$

Using the formulas for the sine of a double angle: $2 \times \sin\alpha \times \cos\alpha = \sin(2 \times \alpha)$ and for the cosine of a double angle: $\cos^2\alpha - \sin^2\alpha = \cos(2 \times \alpha)$, in a plane stress state, one can determine the shear stress τ_α in an area inclined at an angle α:
$\tau_\alpha = \frac{\sigma_y - \sigma_z}{2} \times \sin(2 \times \alpha) + \tau \times \cos(2 \times \alpha).$

Sum of normal stresses in mutually perpendicular sections: σ_α and $\sigma_{\alpha+\pi/2}$. When stretching an element of a mechanical system (the rod, Fig. 7.1.5 a and b), the normal stress in the cross–section is equal: $\sigma_z = \sigma = F/A$, and the shear stress is zero: $\tau = 0.$

(a) (b)

Fig. 7.1.5

At the same time, in the longitudinal section of the rod both normal and shear stress are equal to zero, that is the longitudinal section of the rod is not loaded.

Wherein, in the section of the rod inclined at angle α the normal and tangential stresses are equal: $\sigma_\alpha = 0.5 \times \sigma \times \cos^2 \alpha$ and $\tau_\alpha = -0.5 \times \sigma \times \sin(2 \times \alpha)$.

If the angle of inclination of the section under axial load of an element of the mechanical system is $\pm 45^0$ (that is $\pm\pi/4$), then the normal and the shear stresses in the inclined section are equal to: $\sigma_{\pi/4} = 0.5 \times \sigma$ and $\tau_{\pi/4} = \mp 0.5 \times \sigma$.

The sum of normal stresses in mutually perpendicular sites: σ_α and $\sigma_{\alpha+\pi/2}$, inclined at an angle α and at an angle: $\alpha + \pi/2$ are:

$$\sigma_\alpha + \sigma_{\alpha+\pi/2} = \sigma_y \times \sin^2 \alpha + \sigma_z \times \cos^2 \alpha + \tau \times \sin(2 \times \alpha) +$$
$$+\sigma_y \times \sin^2(\alpha + \pi/2) + \sigma_z \times \cos^2(\alpha + \pi/2) + \tau \times \sin(2 \times \alpha + \pi) =$$
$$= \sigma_y \times \sin^2 \alpha + \sigma_z \times \cos^2 \alpha + \tau \times \sin(2 \times \alpha) +$$
$$+\sigma_y \times \cos^2 \alpha + \sigma_z \times \sin^2 \alpha - \tau \times \sin(2 \times \alpha) = \sigma_y + \sigma_z.$$

Conclusion, the sum of normal stresses in mutually perpendicular sections: σ_α and $\sigma_{\alpha+\pi/2}$ does not change when these sections are rotated together: $\sigma_\alpha + \sigma_{\alpha+\pi/2} = \sigma_y + \sigma_z$.

Questions and comments:

7.1.1. The action of mechanical stresses on the volume element of an elastic material in its three–dimensional stress state and in the plane stress state. Scheme of a plane stress state as a possible simplification of a three–dimensional stress state, if such a simplification is possible.

7.1.2. Normal and shear stresses acting on an inclined site. The balance condition for a part of a volume element when it is cut off by an inclined site. Finding stress projections on directions inclined to the axes using the vector schemes.

7.1.3. Normal and tangential stress in a platform inclined at a certain angle, taking into account the equilibrium of the part cut off from the loaded element.

7.1.4. Equal in magnitude normal and tangential stresses, provided that the angle of inclination of the section under axial load of an element of the mechanical system is equal to $\pm 45^0$ (that is $\pm\pi/4$).

7.1.5. The sum of normal stresses in mutually perpendicular sections of a volume element of an elastic material, which remains unchanged when these sections are rotated together at any angle, in which the tangential stresses are equal in magnitude according to the law of the tangent stress pairing.

7.2. Main areas and main stresses in the three–dimensional stress state of a mechanical system material

Finding the angle α of rotation to the coordinate axes of sections of loaded element of the volume of an elastic material in which there are no tangential stresses. If normal stresses in perpendicular directions are known, and tangent stresses depending on the angle of inclination of the section are determined using the formula:
$\tau_\alpha = \frac{\sigma_y - \sigma_z}{2} \times \sin(2 \times \alpha) + \tau \times \cos(2 \times \alpha)$, then the angle of inclination of the section α_0, in which the tangent stresses are equal to zero one can determine how:
$\mathrm{tg}(2 \times \alpha_{0\tau}) = \frac{2 \times \tau}{\sigma_y - \sigma_z}$.

Depending on the stresses in the given sections, in other sections, which in relation to the given ones are inclined at the found angle $\alpha_{0\tau}$, the tangent stresses are equal to zero: $\tau_\alpha = 0$. The sites in sections in which there are no shear stresses are called main sites. The main sites, in which there are no tangential stresses ($\tau_\alpha = 0$), are present in both the plane and volumetric stress states.

Both normal stresses (σ_y and σ_z), acting in mutually perpendicular sections, in the general case do not vanish in total, except for the special case of pure shear.

Depending on the angle of rotation of the section α_0, in which the site is located, the highest value of normal stress necessarily corresponds to zero values of the derivative: $\frac{d\sigma_{\alpha 0}}{d\alpha} = 0$, to find which the dependence of normal stresses on the angle of inclination of the site is used: $\sigma_\alpha = \sigma_y \times \sin^2 \alpha + \sigma_z \times \cos^2 \alpha + \tau \times \sin(2 \times \alpha)$,
$\frac{d\sigma_\alpha}{d\alpha} = 2 \times \sigma_y \times \sin\alpha \times \cos\alpha - 2 \times \sigma_z \times \sin\alpha \times \cos\alpha + 2 \times \tau \times \cos(2 \times \alpha)$.

Equating the derivative to zero: $\frac{d\sigma_\alpha}{d\alpha} = 0$, one can determine the angle of rotation of the section $\alpha_{0\sigma}$, in which the elementary area is located, with the highest value of normal stress: $(\sigma_y - \sigma_z) \times \sin(2 \times \alpha_{0\sigma}) + 2 \times \tau \times \cos(2 \times \alpha_{0\sigma}) = 0$ and $\mathrm{tg}(2 \times \alpha_{0\sigma}) = \frac{2 \times \tau}{\sigma_z - \sigma_y}$.

The found the section inclination angle $\alpha_{0\sigma}$, in which the normal stress has the greatest magnitude, coincides with the section inclination angle $\alpha_{0\tau}$, in which the tangent

stresses are equal to zero, which is on the main areas in which normal stresses have the greatest values.

Normal stresses on the main areas are denoted main stresses. For a three-dimensional stress state, the main stresses are indicated: 1, 2 and 3, that is: $\sigma_1 > \sigma_2 > \sigma_3$.

Angles of rotation of sections with the highest values of tangential stresses. Using the dependence of tangent stress on the angle of inclination of the section:
$\tau_\alpha = \frac{\sigma_y - \sigma_z}{2} \times \sin(2 \times \alpha) + \tau \times \cos(2 \times \alpha)$, finding the derivative with respect to the angle of inclination of the section and equating the derivative to zero:
$\frac{d\tau_\alpha}{d\alpha} = (\sigma_y - \sigma_z) \times \cos(2 \times \alpha_{0\tau}) + 2 \times \tau \times \sin(2 \times \alpha_{0\tau})$, the angle of rotation of the section $\alpha_{0\tau}$, corresponding to the highest shear stress, one can determine how:
$\mathrm{tg}(2 \times \alpha_{0\tau}) = \frac{\sin(2\times\alpha_{0\tau})}{\cos(2\times\alpha_{0\tau})} = \frac{\sigma_y - \sigma_z}{2\times\tau}$, that is: $\mathrm{tg}(2 \times \alpha_{0\tau}) = \frac{1}{\mathrm{tg}(2\times\alpha_{0\sigma})} = \frac{1}{\left(\frac{2\times\tau}{\sigma_z - \sigma_y}\right)}$.

Sections with maximum tangential stresses are inclined at an angle of 45^0 (that is $\pi/4$) relative to the plane of the main sites, on which the tangential stresses are equal to zero.

In the sites, which have rotation angles: $\alpha_{0\tau}$ and $(\alpha_{0\tau} + \pi/2)$ in relation to specified sections, the tangent stresses take extreme values – the largest or smallest.

Tangent stresses in perpendicular sections are equal to each other according to the law of pairing of tangential stresses: $\tau_{\alpha_{0\tau}} = \tau_{\alpha_{0\tau}+\pi/2}$.

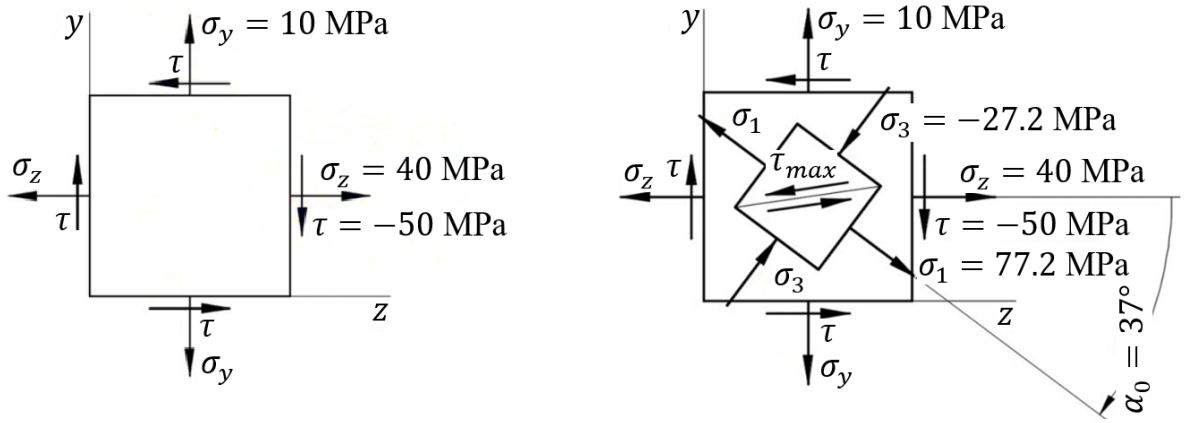
The values of main stresses. To transform the formulas, one reduces all trigonometric functions used to the tangent of a double angle, using known mathematical equations:
$\sin(2 \times \alpha_0) = \pm\frac{\mathrm{tg}(2\times\alpha_0)}{\sqrt{1+\mathrm{tg}^2(2\times\alpha_0)}}$, $\cos(2 \times \alpha_0) = \pm\frac{1}{\sqrt{1+\mathrm{tg}^2(2\times\alpha_0)}}$,
$\cos^2\alpha_0 = 0.5 \times (1 + \cos(2 \times \alpha_0))$ and $\sin^2\alpha_0 = 1 - \cos^2\alpha_0$,
and, replacing the trigonometric functions in the original formula:
$\sigma_\alpha = \sigma_y \times \sin^2\alpha + \sigma_z \times \cos^2\alpha + \tau \times \sin(2 \times \alpha)$,
one can find the main stresses as: $\sigma_{13} = \frac{\sigma_y + \sigma_z}{2} \pm \frac{1}{2} \times \sqrt{(\sigma_z - \sigma_y)^2 + 4 \times \tau^2}$.

The resulting formula gives two values of the main stresses (the largest and the smallest), one can assign indices to them: 1, 2 or 3. In a plane stress state, the third magnitude of the normal stress in the main areas, that is second main stress σ_2 is zero and:
$\sigma_1 > \sigma_3$.

The magnitude of maximum tangent stress equals:
$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \times \sin(2 \times \pi/2) = \frac{\sigma_1 - \sigma_3}{2}$.

Exercise 1. Finding the main normal stresses and the greatest tangential stresses in a plane stress state (Fig. 7.2.1 a). Given are the normal stresses and the tangent stress acting on the elastic element of the mechanical system: $\sigma_y = 10$ MPa, $\sigma_z = 40$ MPa and

$\tau = -50$ MPa. It is required to determine the angle of inclination of the sections with the main sites α_0, the values of the main stresses σ_{13} and the maximal shear stress τ_{max}.



(a) (b)

Fig. 7.2.1

The rotation angle of the sites with main sites is determined as:

$\text{tg}(2 \times \alpha_{0\sigma}) = \frac{2\times\tau}{\sigma_z - \sigma_y} = \frac{2\times(-50)}{40-10} = -3.333$, $2 \times \alpha_{0\sigma} = -73.30°$

and $\alpha_{0\sigma} = -36.65° \approx -37°$ (Fig. 7.2.1 b).

The main stresses are as follows: $\sigma_{13} = \frac{\sigma_y+\sigma_z}{2} \pm \frac{1}{2} \times \sqrt{(\sigma_z - \sigma_y)^2 + 4 \times \tau^2} =$

$= \frac{10+40}{2} \pm \frac{1}{2} \times \sqrt{(40-10)^2 + 4 \times (-50)^2} = 25 \pm 52.2$, MPa, because $\sigma_1 > \sigma_2 > \sigma_3$, then for a plane stressed state the values of the main stresses are equal: $\sigma_1 = 77.2$ MPa and $\sigma_3 = -27.2$ MPa, and the maximal tangent stress is:

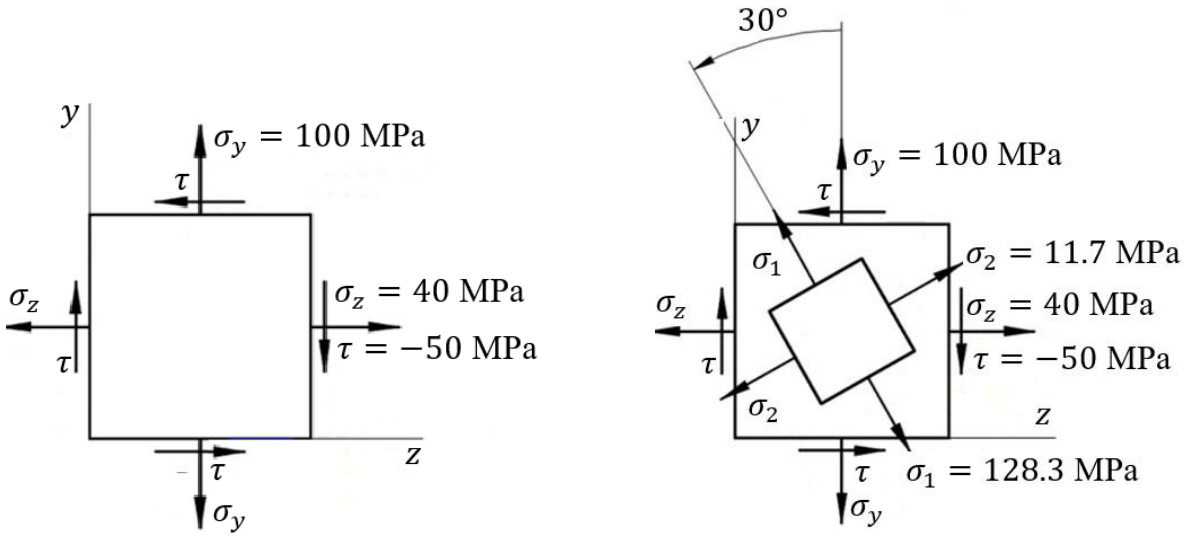
$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{77.2-(-27.2)}{2} = 52.2$, MPa.

Exercise 2. Finding the main normal stresses and the biggest tangential stresses in a plane stress state (Fig. 7.2.2 a).

The normal stresses and the tangent acting on the elastic volume element are assigned: $\sigma_y = 100$ MPa, $\sigma_z = 40$ MPa and $\tau = -50$ MPa.

It is required to determine the angle of inclination of the sections with the main sites α_0, the values of the main stresses σ_{12} and the maximal tangent stress τ_{max}.

The angle of the sections inclination with the main sites (Fig. 7.2.2 b) is determined as: $\text{tg}(2 \times \alpha_{0\sigma}) = \frac{2\times(-50)}{40-100} = 1.667$, $2 \times \alpha_{0\sigma} = -59.04°$ and $\alpha_{0\sigma} = -29.52° \approx -30°$.



(a) (b)

Fig. 7.2.2

The turn angle of the main stress σ_1 from the direction of the specified maximal stress $\sigma_y = 100$ MPa to the left on 30° (Fig. 7.2.3).

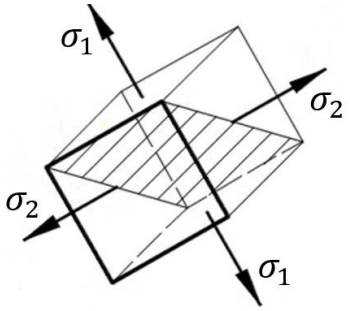


Fig. 7.2.3

The main stresses are equal: $\sigma_{13} = \frac{\sigma_y+\sigma_z}{2} \pm \frac{1}{2} \times \sqrt{(\sigma_z - \sigma_y)^2 + 4 \times \tau^2} =$

$= \frac{100+40}{2} \pm \frac{1}{2} \times \sqrt{(40-100)^2 + 4 \times (-50)^2} = 70 \pm 58.3$ MPa, for a plane stress state the values of the main stresses are: $\sigma_1 = 128.3$ MPa and $\sigma_2 = 11.7$ Mpa ($\sigma_3 = 0$), that's why the maximal tangent stress is: $\tau_{max} = \frac{\sigma_1-\sigma_3}{2} = \frac{128.3-0}{2} = 64.2$ MPa.

Questions and comments:

7.2.1. Determination of the angle of inclination to coordinate axes of the sections of the loaded element of the volume of the elastic material in which site the main areas are located, in which are no tangential stresses in plane and in three–dimensional stress state.

7.2.2. Determination of the angle of the site inclination in which the largest normal stress acts, using the necessary condition for the maximum of the normal stress function,

depending on angle of inclination of the site in the form of a criterion for the derivative to be equal to zero.

7.2.3. The main stresses (the largest normal stresses) acting in the main sites, at which shear (the tangent) stresses are zero. Ordering the main stresses by their magnitude.

7.2.4. The angle of inclination of the section in which the maximum tangent stress acts, determined using the necessary condition of maximum of tangential stress function depending on the angle of rotation of the section in the form of criterion for the derivative to be equal to zero.

7.2.5. The angle of rotation of the sections is 45^0 (that is $\pi/4$) with maximum tangential stresses relative to the plane of the main sites of the loaded element of the elastic volume, at which the tangential stresses are zero.

7.2.6. Trigonometric functions (the sine, cosine and tangent) of double angles used to transform formulas when finding main stresses – the largest normal stresses acting in the main sites.

7.2.7. Determination of two main stresses (the first and third in order of stress magnitude) in a plane stress state in which the third value (second principal stress) is zero.

7.2.8. Determination of maximal shear stress in a plane stress state when the main normal stresses are known (the highest and lowest ones, denoted as first and third main stresses).

7.2.9. The sequence of finding the main stresses (the maximal normal stresses) and the maximal tangential stress in the plane stressed state of the elastic element: the determining of the angle of inclination of sections with the main sites, the determining of the main normal stresses and the maximal tangential stress.

7.3. Generalized Hooke's law for the three–dimensional stress state of an elastic material

The dependences of deformations (ε_x, ε_y and ε_z) in three directions under a three–dimensional stress state on normal stresses (σ_x, σ_y and σ_z) considering the Poisson's ratio μ one can indicate by the deformation ε_x example along the x axis, arising from these stresses: $\varepsilon_x' = \frac{\sigma_x}{E}$, where E – the modulus of elasticity of the material,

$\varepsilon_x'' = -\mu \times \varepsilon_y'' = -\mu \times \frac{\sigma_y}{E}$ and $\varepsilon_x''' = -\mu \times \varepsilon_z''' = -\mu \times \frac{\sigma_z}{E}$,

$\varepsilon_x = \varepsilon_x' + \varepsilon_x'' + \varepsilon_x''' = \frac{\sigma_x}{E} - \mu \times \frac{\sigma_y}{E} - \mu \times \frac{\sigma_z}{E} = \frac{1}{E} \times (\sigma_x - \mu \times (\sigma_y + \sigma_z))$.

The Hooke's law (the dependence of strains: ε_x, ε_y and ε_z on the stresses: σ_x, σ_y and σ_z) under a three–dimensional stress state has the form:

$\varepsilon_x = \frac{1}{E} \times (\sigma_x - \mu \times (\sigma_y + \sigma_z))$, $\varepsilon_y = \frac{1}{E} \times (\sigma_y - \mu \times (\sigma_x + \sigma_z))$

and $\varepsilon_z = \frac{1}{E} \times (\sigma_z - \mu \times (\sigma_x + \sigma_y))$.

The angular shear strains (γ_{xy}, γ_{yz} and γ_{zx}) depending on tangential stresses (τ_{xy}, τ_{yz} and τ_{zx}) under a three–dimensional stress state are: $\gamma_{xy} = \frac{\tau_{xy}}{G}$, $\gamma_{yz} = \frac{\tau_{yz}}{G}$ and $\gamma_{zx} = \frac{\tau_{zx}}{G}$.

In the main axes perpendicular to the sections in which the tangent stresses are equal to zero, according to the Hooke's law, linear deformations dependent on normal stresses are: $\varepsilon_1 = \frac{1}{E} \times (\sigma_1 - \mu \times (\sigma_2 + \sigma_3))$, $\varepsilon_2 = \frac{1}{E} \times (\sigma_2 - \mu \times (\sigma_1 + \sigma_3))$
and $\varepsilon_3 = \frac{1}{E} \times (\sigma_3 - \mu \times (\sigma_1 + \sigma_2))$.

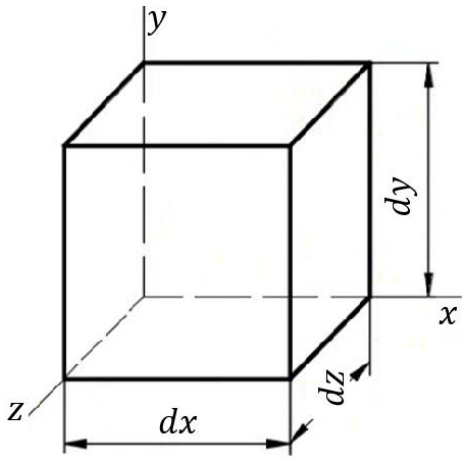
The Hooke's law for the option of plane stress state in yz coordinates has the form: $\varepsilon_y = \frac{1}{E} \times (\sigma_y - \mu \times \sigma_z)$, $\varepsilon_z = \frac{1}{E} \times (\sigma_z - \mu \times \sigma_y)$, $\varepsilon_x = -\frac{\mu \times (\sigma_y + \sigma_z)}{E}$ and $\gamma_{xy} = \frac{\tau_{xy}}{G}$.

For a plane stress state at solving equations one can multiply all terms in the second equation by the Poisson's ratio μ and add the second equation to the first:
$E \times (\varepsilon_y + \mu \times \varepsilon_z) = \sigma_y - \mu \times \sigma_z + \mu \times \sigma_z - \mu^2 \times \sigma_y$,
$E \times (\varepsilon_y + \mu \times \varepsilon_z) = \sigma_y \times (1 - \mu^2)$ and $\sigma_y = \frac{E}{1-\mu^2} \times (\varepsilon_y + \mu \times \varepsilon_z)$.

Thus, one can find the normal stress σ_y in a plane stress state depending on the deformations (ε_y and ε_z). Similarly, one can find the normal stress σ_z, and the Hooke's law in the form of stress (σ_y, σ_z and τ_{yz}) dependences on strains (ε_y, ε_z and γ_{yz}) for a plane stress state this takes the forms: $\sigma_y = \frac{E}{1-\mu^2} \times (\varepsilon_y + \mu \times \varepsilon_z)$,
$\sigma_z = \frac{E}{1-\mu^2} \times (\varepsilon_z + \mu \times \varepsilon_y)$ and $\tau = G \times \gamma$.

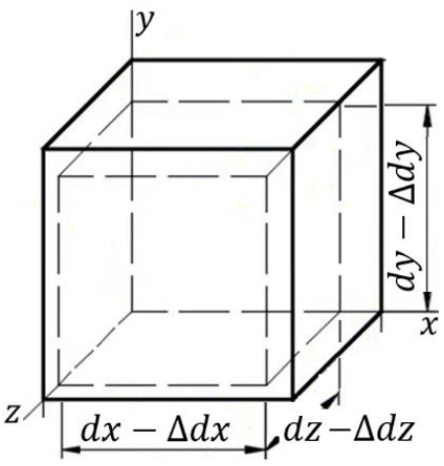
The change under a three–dimensional stress state in the volume of an elastic element shaped like a parallelepiped is shown on the scheme (Fig. 7.3.1).

The elastic element before deformation



(a)

The elastic element after deformation



(b)

Fig. 7.3.1

The volume of an elastic element shaped like a parallelepiped before deformation is equal: $dV = dx \times dy \times dz$, and after deformation the volume of an elastic element is: $dV - \Delta dV = (dx - \Delta dx) \times (dy - \Delta dy) \times (dz - \Delta dz)$, or, taking into account the relative smallness of the deformations and their products, neglecting the magnitudes of the terms in which the deformations (Δdx, Δdy and Δdz) are factors more than once, the volume of the elastic element after deformation is equal:

$dV - \Delta dV = dx \times dy \times dz - \Delta dx \times dy \times dz - \Delta dy \times dx \times dz -$

$-\Delta dz \times dx \times dy.$

Relative change in volume θ of an elastic element during its deformation equals:

$\theta = \frac{\Delta dV}{dV} = \frac{\Delta dx \times dy \times dz + \Delta dy \times dx \times dz + \Delta dz \times dx \times dy}{dx \times dy \times dz} =$

$= \frac{\Delta dx \times dy \times dz}{dx \times dy \times dz} + \frac{\Delta dy \times dx \times dz}{dx \times dy \times dz} + \frac{\Delta dz \times dx \times dy}{dx \times dy \times dz} = \frac{\Delta dx}{dx} + \frac{\Delta dy}{dy} + \frac{\Delta dz}{dz} = \varepsilon_x + \varepsilon_y + \varepsilon_z,$

that is: $\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

For an elastic material, taking into account the dependence of stresses on deformations according to the Hooke's law, the relative change in volume θ depending on the main stresses one can find as:

$\theta = \frac{1}{E} \times (\sigma_1 - \mu \times (\sigma_2 + \sigma_3) + \sigma_2 - \mu \times (\sigma_1 + \sigma_3) + \sigma_3 - \mu \times (\sigma_1 + \sigma_2)),$

or $\theta = \frac{1-2\times\mu}{E} \times (\sigma_1 + \sigma_2 + \sigma_3)$.

Thus, materials characterized by the Poisson's ratio equal to: $\mu = 1/2$ (that is 0.5) are incompressible in volume. The Poisson's ratio cannot be greater than 0.5, since in this case the volumetric deformation would become negative with a positive sum of the main stresses: $(\sigma_1 + \sigma_2 + \sigma_3)$.

In order for under a three–dimensional stress state to change only the volume of the loaded element, and the ratio of dimensions does not change, it is required that all the main stresses are equal to each other: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0$, under this condition the tangential stresses and the angular shear strains are equal to zero:

$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$ and $\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$.

Questions and comments:

7.3.1. Deformation in each of the three directions (the Hooke's law under a three–dimensional stress state), depending on the normal stress in this direction and on stresses in perpendicular directions (which are also perpendicular to each other).

7.3.2. The angular shear strains under a three–dimensional stress state.

7.3.3. The Hooke's law in the main axes perpendicular to sections in which the shear stresses are zero.

7.3.4. The Hooke's law for a plane stress state (the dependence of strains on stresses for a plane stress state and the dependence of stresses on strains).

7.3.5. Changing the volume of an elastic element under three–dimensional stress state. Relative change in volume under a three–dimensional stress state, corresponding to the sum of relative linear deformations of the volume element in the axes of the main stresses, relative change in volume as a function of main stresses.

7.3.6. Volumetrically incompressible linearly elastic materials, which correspond to a Poisson's ratio equal to ½ (the limit value of the Poisson's ratio).

7.3.7. Change in the volume of an elastic element while maintaining the ratio of the element's dimensions in its three–dimensional stress state, when the main stresses are equal to each other, and the tangential stresses and angular shear deformations are zero.

7.4. Specific potential energy of deformation of an elastic material in its stress–strain state

Since the specific potential energy of deformation (which is the amount of energy per unit volume of an elastic material in a stressed–strain state) characterizes the state of the material at the point, having no volume and deformable dimensions, then the potential energy does not depend on any direction of the axes of the coordinate system or on their turning.

That's why to find the specific potential energy of deformation (which dimension, in the International system of physical units is: J/m^3) it is more convenient to make mathematical transformations in the coordinate system in which acting main stresses are: σ_1, σ_2 and σ_3, where: $\sigma_1 > \sigma_2 > \sigma_3$, and the result will correspond to any coordinate system. In the main coordinate system, the specific potential energy of elastic deformation is equal: $u = \frac{1}{2} \times (\sigma_1 \times \varepsilon_1 + \sigma_2 \times \varepsilon_2 + \sigma_3 \times \varepsilon_3)$.

Using the relationships between stresses and strains according to the Hooke's law: $\varepsilon_1 = \frac{1}{E} \times (\sigma_1 - \mu \times (\sigma_2 + \sigma_3))$, $\varepsilon_2 = \frac{1}{E} \times (\sigma_2 - \mu \times (\sigma_1 + \sigma_3))$ and $\varepsilon_3 = \frac{1}{E} \times (\sigma_3 - \mu \times (\sigma_1 + \sigma_2))$, the specific potential strain energy u one can find as: $u = \frac{1}{2 \times E} \times (\sigma_1 \times (\sigma_1 - \mu \times (\sigma_2 + \sigma_3)) + \sigma_2 \times (\sigma_2 - \mu \times (\sigma_1 + \sigma_3)) + +\sigma_3 \times (\sigma_3 - \mu \times (\sigma_1 + \sigma_2)))$, the final value of specific potential energy of deformation u equals: $u = \frac{1}{2 \times E} \times (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2 \times \mu \times (\sigma_1 \times \sigma_2 + \sigma_2 \times \sigma_3 + \sigma_3 \times \sigma_1))$.

The part of the specific potential energy of deformation u_V, spent on volume change, equals: $u_V = \frac{1}{2 \times E} \times \left(\sigma_0^2 + \sigma_0^2 + \sigma_0^2 - 2 \times \mu \times (\sigma_0 \times \sigma_0 + \sigma_0 \times \sigma_0 + \sigma_0 \times \sigma_0)\right) = \frac{1}{2 \times E} \times 3 \times \sigma_0^2$, where $\sigma_0 = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ and $u_V = \frac{1 - 2 \times \mu}{6 \times E} \times (\sigma_1 + \sigma_2 + \sigma_3)^2$.

By subtracting from the total value of the deformation energy that part of it that is spent on changing the volume u_V, one can find the remaining part of the specific potential deformation energy u_F, spent on changing the shape:

$u_F = \frac{1+\mu}{3\times E} \times (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \times \sigma_2 - \sigma_2 \times \sigma_3 - \sigma_3 \times \sigma_1)$.

The total potential energy of deformation is equal to the sum of energy u_V, leading to change the material volume, and the energy u_F to a change the shape: $u = u_V + u_F$.

Relationships between elastic constants of materials. If one consider only the state of pure shear, in which the shear stresses act in sections, and normal stresses are equal to zero, then the specific potential energy calculated from shear stresses is equal: $u = \frac{1}{2} \times \tau \times \gamma$, or, with replacement of the angular deformation under shear stress according to the Hooke's law, as: $\gamma = \frac{\tau}{G}$, the specific potential energy u, determined from tangential stresses, is: $u = \frac{\tau^2}{2\times G}$.

The angle of inclination of the main sites, in which the greatest shear stresses act, in relation to the original main sites corresponding to the highest normal stresses, is equal: $\mathrm{tg}(2 \times \alpha_0) = \frac{2\times\tau}{0} = \infty$, $2 \times \alpha_0 = \pi/2$ and $\alpha_0 = \pi/4$ (that is 45°, Fig. 7.4.1).

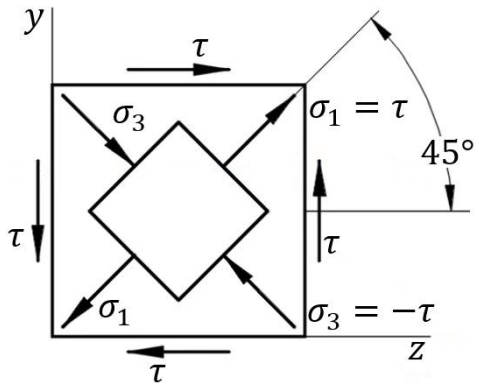


Fig. 7.4.1

In a plane stress state, the main normal stresses and the greatest shear stresses are mutually dependent: $\sigma_{13} = \pm\frac{1}{2} \times \sqrt{4 \times \tau^2} = \pm\tau$.

The specific potential energy of deformation in a plane stress state, found from the main stresses and the greatest tangential stress, is equal to:

$u = \frac{1}{2\times E} \times (\sigma_1^2 + \sigma_3^2 - 2 \times \mu \times \sigma_1 \times \sigma_3) =$

$= \frac{1}{2\times E} \times (\tau^2 + (-\tau)^2 - 2 \times \mu \times \tau \times (-\tau)) = \frac{\tau^2\times(1+\mu)}{E}$.

Since the magnitude of the specific potential energy of deformation u does not depend on the choice of coordinate system, then the values of the specific potential energy found from the main normal stresses (σ_1 and σ_3), acting on the main sites in the sections

in which the tangential stresses are equal to zero, and at the highest tangential stresses τ, acting in sections inclined to the main areas at the angle $\pi/4$ (that is $45°$), in which normal stresses are equal to zero.

That's why: $\frac{\tau^2}{2\times G} = \frac{\tau^2\times(1+\mu)}{E}$, and the ratio of elastic modules: at compression \ tension E (I kind) and at shift G (II kind) and Poisson's ratio μ is: $G = \frac{E}{2\times(1+\mu)}$.

Questions and comments:

7.4.1. The potential energy of elastic deformation in a three–dimensional stress state, independent of the directions in the coordinate system, characterizes the stresses at a point that does not have deformable dimensions, which, when finding the potential energy, justifies the possibility of any choice of coordinate system, in particular such a coordinate system, the axes of which coincide in directions with main stresses.

7.4.2. Specific potential energy of elastic deformation per unit volume equal to half the sum of the products of stresses and deformations along each of the coordinate axes associated with the directions of the main stresses.

7.4.3. Specific potential energy of elastic deformation per unit volume as a function of main stresses (when deformations in the direction of main stresses are redefined through these stresses).

7.4.4. Total specific potential energy of elastic deformation and part of the energy, associated with a change the volume of a deformable elastic element experiencing a three–dimensional stress state, and part of the energy spent on changing its shape.

7.4.5. Specific potential energy of elastic deformation under pure shear, in which only the tangential stresses act in sections, and the normal ones are equal to zero.

7.4.6. The specific potential energy of elastic deformation in a plane stress state as a function of the main stresses and the largest tangential stresses.

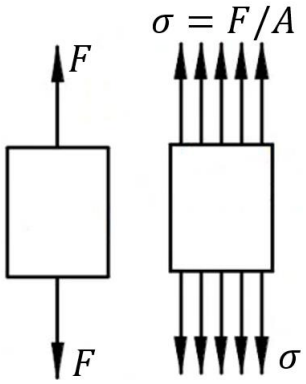
7.4.7. The ratio of the elastic modulus of a material under compression/tension and under shear (the shift), determined by an equal value of the specific potential energy of deformation for different data: by the specific potential energy, estimated from the highest shear stresses acting in sections inclined to the main sites at an angle of $45°$, and by the amount of energy found from the stresses on the main sites, when the tangential stresses are zero and the normal stresses are equal to the main ones.

7.5. Hypotheses of plasticity and destruction of elastic material of a mechanical system under stress–strain state

Under a three–dimensional stress state, as well as in a plane stress state, the material at each point is loaded more intensely, and possibly less intensely, than under uniaxial tension/compression. To assess the strength of an elastic material, certain theoretical hypotheses of fracture or plasticity can be used, showing under what loading conditions the material goes into a plastic state or collapses.

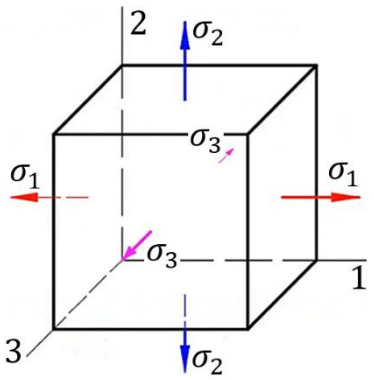
At uniaxial tension, normal stress σ acts in the material of the sample under study (Fig. 7.5.1 a).

Normal stresses in uniaxial tension



(a)

Normal stresses in the main directions 1, 2 and 3



(b)

Fig. 7.5.1

Within the framework of the fracture and plasticity hypotheses, a complex stress state (three–dimensional or flat) is compared with uniaxial tension. The stress that occurs in uniaxial tension/compression of a mechanical element, in which the proximity of the material to a dangerous state is the same as in a complex stress state, is denoted equivalent stress σ_E. The strength criterion at a complex stress state has the form: $\sigma_E \leq [\sigma]$, where $[\sigma]$ – the permissible stress allowed for considered material in uniaxial tension.

In accordance with the hypothesis of the maximal normal stresses, the dangerous state at the point of an elastic body occurs when the largest main stress in it absolute magnitude reaches the limit value determined at tensile tests. In a complex stress state, the equivalent stress σ_E is the largest normal stress (modulo σ_1 or σ_3) among the main stresses: σ_1, σ_2 and σ_3, where: $\sigma_1 > \sigma_2 > \sigma_3$ (Fig. 7.5.1 b). The maximum normal stresses hypothesis does not take into account other stresses acting in the material, except the greatest normal stress. The maximum normal stresses hypothesis gives a completely correct assessment of the tensile strength of brittle materials.

The strain under simple (the uniaxial) tension equals: $\varepsilon = \frac{\Delta l}{l} = \frac{\sigma}{E}$. According to the hypothesis of the largest axial deformations, the dangerous stress–strain state occurs when the maximum modulus of the main deformation ε_1 reaches the limit value found by tensile tests.

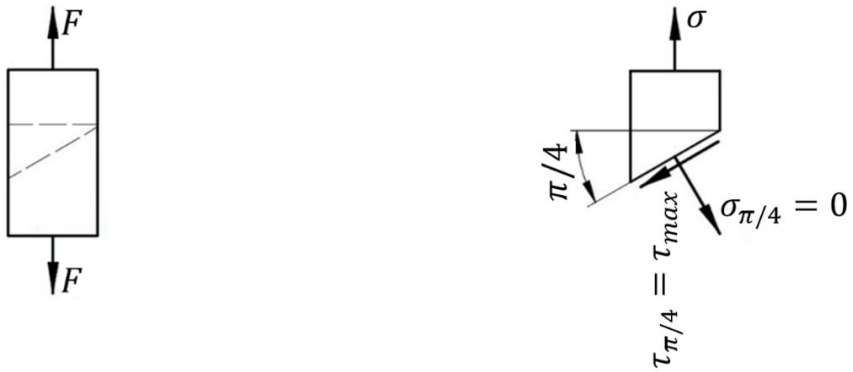
At the maximum modulus of tensile strain ε_1, determined by stresses as: $\varepsilon_1 = \frac{1}{E} \times (\sigma_1 - \mu \times (\sigma_2 + \sigma_3))$, the equivalent stress magnitude (the modulus) one can find, as: $\sigma_E = \sigma_1 - \mu \times (\sigma_2 + \sigma_3)$.

At the maximum compressive strain ε_3 the equivalent stress is calculated in the same way.

The hypothesis of the largest axial deformations is considered to be to a certain extent one–sided and associated with the assumption of the destruction of an elastic material under loading by normal stresses, which in a number of cases does not correspond to the results of experimental studies.

The hypothesis of the largest axial deformations can serve as a correct assessment of the strength of brittle materials, and at small values of the Poisson's ratio, this hypothesis corresponds quite closely to the estimate of the largest normal stresses.

According to the hypothesis of the highest tangential stresses, a dangerous state of the material occurs when the maximum tangent stress τ_α (at: $\sigma_\alpha = 0$) reaches the limit value determined at the tensile test (Fig. 7.5.2). In the axial tension or compression, the maximum tangent stress occurs in a section inclined by $\pi/4$ (that is 45°) relative to the planes of the main sites, wherein the maximum tangent stress equals: $\tau_{\pi/4} = \sigma/2$.



(a) (b)

Fig. 7.5.2

In a complex stress state, in particular, in plane stress state, the maximum tangent stress depending on the main stresses (σ_1 and σ_3) one can determine how: $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$. Assessing the complex stress state according to the hypothesis of maximum tangential

stresses as equally dangerous with axial tension, that is equating: $\frac{\sigma_1-\sigma_3}{2}=\frac{\sigma}{2}$, one can find the equivalent stress as: $\sigma_E=\sigma_1-\sigma_3$.

The hypothesis of the highest tangential stresses is quite well confirmed by experiment and is based on observations of the transformation of the material into a plastic state and the destruction of many types of metal structural materials. At the verifications, the maximum tangential stress hypothesis usually has an error in the safety margin, which is the actual failure occurs at a slightly greater load than that estimated by the maximum shear stress hypothesis. The hypothesis of the highest tangential stresses is recommended to be used to assess the strength of structures and parts made of plastic materials. The disadvantage of the hypothesis of the highest tangential stresses is the absence of one of the main stresses σ_2 in the equivalent stress criterion used.

A dangerous state at a point of a loaded material, according to the hypothesis of specific potential energy of shape change, occurs when this specific energy reaches the limiting value determined during a tensile test. At simple (the uniaxial) stretching, the specific potential energy of change in shape is equal: $u_F=\frac{1+\mu}{3\times E}\times\sigma^2$. In a complex stress state, the specific potential energy of the shape change equals:
$u_F=\frac{1+\mu}{3\times E}\times(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\times\sigma_2-\sigma_2\times\sigma_3-\sigma_3\times\sigma_1)$.

By equating the values of the specific potential energy of deformation for states assessed as equally dangerous: simple (the uniaxial) tension and complex stress state, one can determine the equivalent stress according to the hypothesis of specific potential energy of deformation: $\sigma_E=\sqrt{\frac{1}{2}\times((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2)}$.

The hypothesis of the highest specific potential energy of shape change u_F is well confirmed by experiment for plastic materials. Within the framework of this hypothesis, only part of the deformation energy is taken into account, since there is no doubt that with a change in volume without violating the proportions of dimensions (which is under equilateral compression), the plastic state of the material is not achieved at any stress value.

Questions and comments:

7.5.1. The reason for the difference between the fracture and plasticity hypotheses is that in a volumetric stress state, the material at each point of the loaded volume element experiences stresses that are not equal to the stresses that arise during uniaxial tension \ compression.

7.5.2. The criterion for the strength of an elastic material under a complex stress state (the three–dimensional). The stress arising in uniaxial tension \ compression of a

mechanical element, which is the equivalent stress when the proximity of the material to a dangerous state is the same as in a complex stress state.

7.5.3. Corresponding to the assessment of the strength of brittle materials is the hypothesis of destruction and plasticity, when a dangerous state at the point of a loaded elastic volume occurs when the largest principal stress in absolute value reaches the limiting value corresponding to uniaxial tension \ compression.

7.5.4. The suitability for assessing the strength of brittle materials of the hypothesis of the greatest axial deformations when the main deformation reaches the limiting value corresponding to uniaxial tension/compression.

7.5.5. The correspondence of hypotheses for assessing strength based on the maximum main stresses and the greatest deformations in the direction of the main stresses for brittle materials with a small Poisson's ratio.

7.5.6. Limiting volumetric stress state in accordance with the hypothesis of the highest tangential stresses, when the highest tangential stress reaches the limiting value found in a section inclined at an angle of $45°$, with respect to the plane of the main sites under uniaxial tension \ compression. Replacement of the tangential stress by the main stresses in the criterion of greatest tangential stresses.

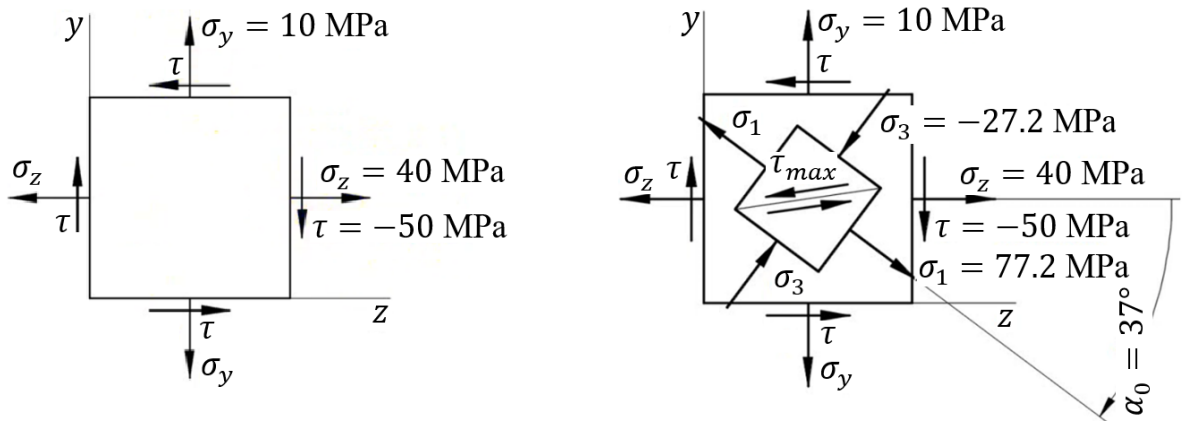
7.5.7. Correspondence of the hypothesis of the highest tangential stresses to experimental observations of destruction of plastic materials in their complex stress state and a certain excess of the withstand–able loads over those found from the highest tangential stresses, which goes into the safety margin.

7.5.8. The limiting three–dimensional stress state corresponding to observations for plastic materials in accordance with the hypothesis of specific potential energy of shape change, which for this state in the form of equivalent stress is determined depending on the main stresses, on the elasticity of the material and on its Poisson's ratio, when the limiting values of this specific energy are found in uniaxial tension \ compression.

7.6. Finding the main stresses and deformations of the elastic element and checking the strength using fracture hypotheses for plane stress state

Exercise. The normal stresses and the tangent one acting in the elastic element (Fig. 7.6.1 a) are assigned: $\sigma_y = 10$ MPa, $\sigma_z = 40$ MPa and $\tau = -50$ MPa, as well as: the elastic modulus: $E = 2.1 \times 10^5$ MPa, the Poisson's ratio: $\mu = 0.28$, the tensile strength: $[\sigma] = 100$ MPa. It was previously determined that the angle of the sections inclination with the main sites is equal: $\alpha_{0\sigma} = -37°$, the main stresses are equal: $\sigma_1 = 77.2$ MPa and $\sigma_3 = -27.2$ MPa (because the stress state is plane, then the second main stress is zero: $\sigma_2 = 0$), the maximal tangent stress is: $\tau_{max} = 52.2$ MPa. The exercise assigns one to

find the main deformations, the specific potential energy of deformations and to check the strength of the elastic element.



(a) (b)

Fig. 7.6.1

The main deformations of the elastic element (ε_1, ε_2 and ε_3) are:

$$\varepsilon_1 = \frac{1}{E} \times \left(\sigma_1 - \mu \times (\sigma_2 + \sigma_3)\right) = \frac{1}{2.1 \times 10^5} \times \left(77.2 - 0.28 \times (0 - 27.2)\right) =$$
$$= 4.039 \times 10^{-4},$$
$$\varepsilon_2 = \frac{1}{E} \times \left(\sigma_2 - \mu \times (\sigma_1 + \sigma_3)\right) = \frac{1}{2.1 \times 10^5} \times \left(0 - 0.28 \times (77.2 - 27.2)\right) =$$
$$= -0.667 \times 10^{-4},$$
$$\varepsilon_3 = \frac{1}{E} \times \left(\sigma_3 - \mu \times (\sigma_1 + \sigma_2)\right) = \frac{1}{2.1 \times 10^5} \times \left(-27.2 - 0.28 \times (77.2 + 0)\right) =$$
$$= -2.325 \times 10^{-4}.$$

The volumetric deformation of an elastic element θ (the relative volume change) is: $\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = (4.039 - 0.667 - 2.325) \times 10^{-4} = 1.048 \times 10^{-4}$.

The total specific potential energy of deformation of the elastic element is:

$$u = \frac{1}{2 \times E} \times (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2 \times \mu \times (\sigma_1 \times \sigma_2 + \sigma_2 \times \sigma_3 + \sigma_3 \times \sigma_1)) =$$
$$= \frac{1}{2 \times 2.1 \times 10^5} \times (77.2^2 + 0^2 + (-27.2)^2 -$$
$$-2 \times 0.28 \times (77.2 \times 0 + 0 \times (-27.2) + 77.2 \times (-27.2))) = 187.5 \times 10^{-4}, \text{ mJ/mm}^3.$$

Specific potential energy of change in volume of an elastic element equals:

$$u_V = \frac{1 - 2 \times \mu}{6 \times E} \times (\sigma_1 + \sigma_2 + \sigma_3)^2 = \frac{1 - 2 \times 0.28}{6 \times 2.1 \times 10^5} \times (77.2 + 0 - 27.2)^2 =$$
$$= 8.7 \times 10^{-4}, \text{ mJ/mm}^3.$$

The specific potential energy of the shape changing of an elastic element equals:

$$u_F = \frac{1 + \mu}{3 \times E} \times (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \times \sigma_2 - \sigma_2 \times \sigma_3 - \sigma_3 \times \sigma_1) = \frac{1 + 0.28}{3 \times 2.1 \times 10^5} \times$$
$$\times \left(77.2^2 + 0^2 + (-27.2)^2 - 77.2 \times 0 - 0 \times (-27.2) - 77.2 \times (-27.2)\right) =$$
$$= 178.8 \times 10^{-4}, \text{ mJ/mm}^3.$$

The verification of the strength of the elastic element according by the hypothesis of the maximum normal stresses: $\sigma_E = \sigma_1 = 77.2 < [\sigma] = 100$, MPa, therefore, the strength of the elastic element according to the considered criterion is ensured.

The verification of strength according to hypothesis of largest axial deformations: $\sigma_E = \sigma_1 - \mu \times (\sigma_2 + \sigma_3) = 77.2 - 0.28 \times (0 - 27.2) = 84.16 < [\sigma] = 100$, MPa.

The verification of strength according to the hypothesis of maximum tangential stresses: $\sigma_E = \sigma_1 - \sigma_3 = 77.2 - (-27.2) = 104.4 > [\sigma] = 100$, MPa, that is, the strength of the elastic element according to the hypothesis of maximum tangential stresses is not ensured.

The verification of strength according to the hypothesis of the highest specific potential energy of shape change: $\sigma_E = \sqrt{\frac{1}{2} \times ((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2)} =$

$= \sqrt{\frac{1}{2} \times ((77.2 - 0)^2 + (0 - (-27.2))^2 + (-27.2 - 77.2)^2)} =$

$= 93.8 < [\sigma] = 100$, MPa.

In addition to the considered hypotheses of strength (the fracture and plasticity) of the elastic element: the maximum normal stresses, the greatest axial deformations, the greatest tangential stresses and the greatest specific potential energy of shape change, other strength hypotheses are also known.

Questions and comments:

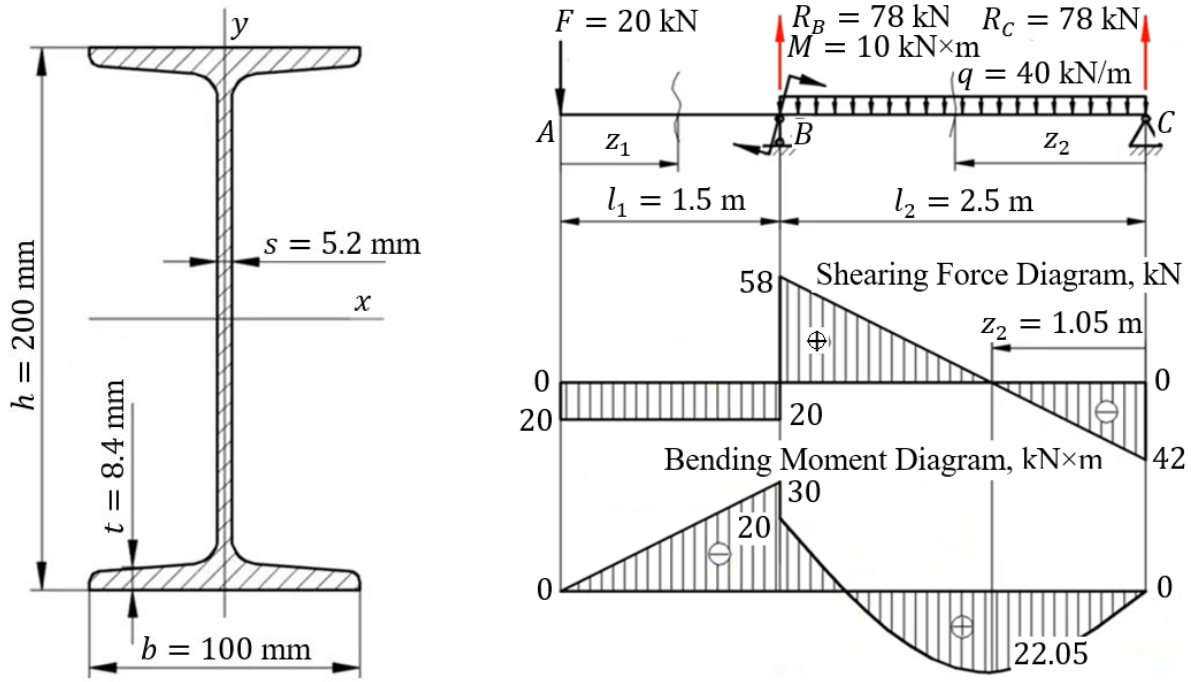
7.6.1. The sequence of determining the main deformations, specific potential energy of deformations and checking the strength of the elastic element at a given volumetric stress state.

7.6.2. Determination of the main indicators of the volumetric stress state of an elastic element: the main deformations depending on the main stresses and volumetric deformation, as well as the specific potential energy of deformation of the elastic element and the specific potential energy of change in the volume of the elastic element.

7.6.3. Checking the strength of an elastic element under a given three–dimensional stress state according to the following criteria: the highest normal stress, the maximum axial deformation (as a function of the main stresses and the Poisson's ratio), the highest tangential stresses (as a function of the main stresses) and the highest specific potential energy of deformation – as a function of the main stresses.

7.7. Testing the strength of a material under a complex stress state – with the combined action of bending and shear on a mechanical system

Exercise. Check the strength of the I–beam (Fig. 7.7.1 a), taking into account the normal and tangential stresses acting at critical points and the hypotheses of strength (the plasticity and fracture) of the elastic material of the beam. Data: the cross–sectional area of I–beam profile from the standard range with a wall height of 200 mm is: $A = 26.8$ cm^2, the moment of inertia of beam cross–section is: $I_x = 1840$ cm^4, the moment of resistance of beam section under bending is: $W_x = 184$ cm^3, static moment of the part of the area cut off by a line passing through the center of the sectional area is: $S_x = 104$ cm^3, the permissible normal stress is: $[\sigma]$ = 160 MPa.



(a) (b)

Fig. 7.7.1

Studied loaded beam is statically definable, and its strength would be analyzed after finding the reactions in the supports and determining the diagrams (Fig. 7.7.1 b) of bending moments and shearing forces.

Strength check by normal stresses. The normal stress σ in a layer of the beam section located at a distance y from the neutral axis of the beam section layer and the maximum normal stress are determined as: $\sigma = \frac{M}{I_x} \times y$ and $\sigma_{max} = \frac{M_{max}}{I_x} \times y_{max} = \frac{M_{max}}{W_x}$.

The dangerous section in the beam – the section of the greatest bending moment M_{max} is located on support B, to the left of this support, that is from the point B of the moment application: $M = 10$ kN×m, the dangerous points at which the greatest normal

stresses σ_{max} act are located on the outer surface on the upper flange of the I–beam – farthest from the neutral axis x, passing through the geometric center of the cross–sectional area of the beam. The bending moment in the indicated section B is equal: $M_{max} = 30$ kN×m, and the moment of resistance is: $W_x = 184$ cm^3.

The highest normal stress acting at the critical points of the dangerous section B of an I–beam, equals: $\sigma_{max} = \frac{30\times10^3}{184\times10^{-6}} = 163.0 \times 10^6$, Pa – exceeds the maximum permissible normal stress: $\sigma_{max} = 163.0$ MPa $> [\sigma] = 160$, MPa.

That is, the strength of beam at maximum normal stresses is insufficient. But in terms of relative value, the found maximum normal stress exceeds the permissible value within the limits of the technical error in determining the mechanical stress, that is, the excess of the maximum stress is equal to 1.88 %:

$\left|\frac{\sigma_{max}-[\sigma]}{[\sigma]}\right| \times 100\ \% = \left|\frac{163-160}{160}\right| \times 100\ \% = 1.88\ \% < 5\ \%$.

Thus, within the limits of error, the strength of the I-beam one can consider sufficient according to the assessment by maximum normal stresses.

The diagram (Fig. 7.7.2) shows the distribution of normal stresses in the critical section B.

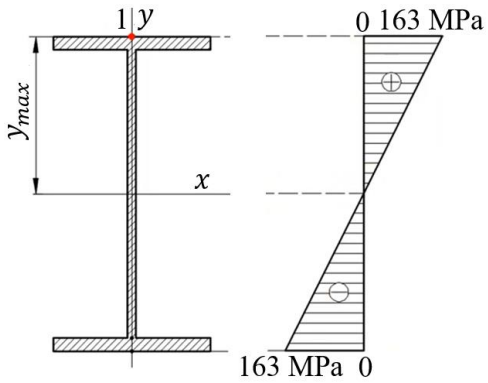


Fig. 7.7.2

The verification of strength based on the highest tangential stresses.

When finding tangent stress, the formula is used of D.I. Zhuravsky: $\tau = \frac{Q\times S_x}{I_x\times b}$, where Q – the shear force, S_x – the static moment of the cut–off area, that is part of the sectional area of the beam, located in the direction to the periphery of the section from its geometric center of the area, outside the line equidistant from the geometric center of the cross–section, at the level of which the shear stress is determined, I_x – the moment of inertia of cross–section, b – the width of the section on a line equidistant from the geometric center of the cross–section, at the level of which the shear stress is determined.

Dangerous cross–section of a beam (that is the section of greatest shear force Q_{max}) located on support B, to the right of this support. The dangerous points in this section, in which the greatest tangent stresses τ_{max} act, are located on a line passing through the

geometric center of the I–beam section (in the horizontal plane of symmetry of the I–section). In accordance with the shear force diagram the greatest value of the force acts to the right of the section of the beam on the support B, equals: $Q_{max} = 58$ kN.

The initial data for calculating the shear stress at the point # 2 (see Fig. 7.7.3, the connection of the I–beam wall to the shelf) are: $A_3 = b \times t = 10 \times 0.84 = 8.4$ cm^2, $y_{c3} = \frac{h}{2} - \frac{t}{2} = \frac{200}{2} - \frac{8.4}{2} = 95.8$ mm, $b'(y_3) = 100$ mm, $b''(y_3) = s = 5.2$ mm, $S_{x3} = 9.58 \times 8.4 = 80.5$ cm^3, $I_x = 1840$ cm^4.

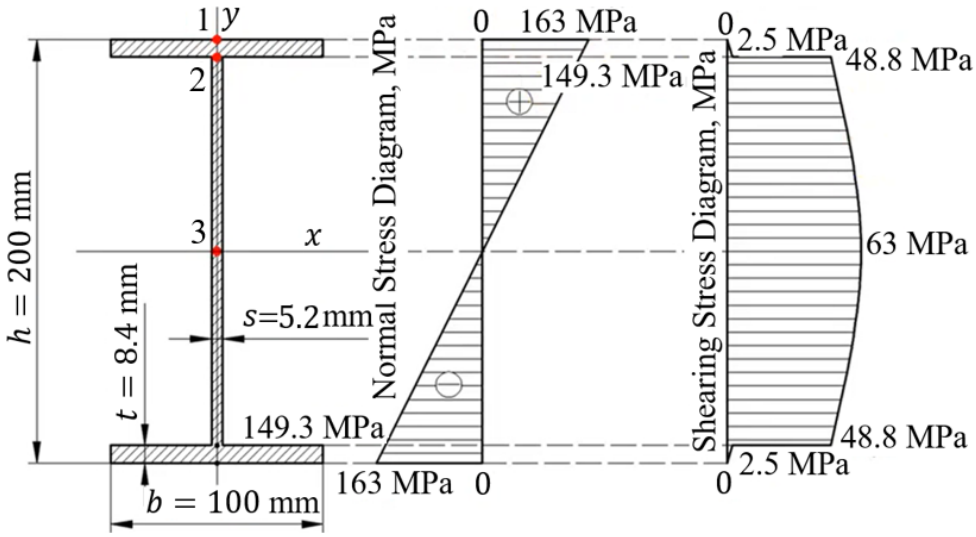


Fig. 7.7.3

The tangent stresses acting at the point # 2 of dangerous section of the I–beam are: $\tau_3' = \frac{Q_{max} \times S_{x3}}{I_x \times b'} = \frac{58 \times 10^3 \times 80.5 \times 10^{-6}}{1840 \times 10^{-8} \times 0.1} = 2.5 \times 10^6$, Pa and

$\tau_3'' = \frac{Q_{max} \times S_{x3}}{I_x \times b''} = \frac{58 \times 10^3 \times 80.5 \times 10^{-6}}{1840 \times 10^{-8} \times 5.2 \times 10^{-3}} = 48.8 \times 10^6$, Pa.

The initial data for calculating the maximum shear stress are: $S_{x\,max} = 104$ cm^3, $I_x = 1840$ cm^4 and $b = s = 5.2$ mm.

The limiting shear stress one can find in proportion to the value of the limiting normal stress: $[\tau] \approx 0.6 \times [\sigma] = 0.6 \times 160 = 96.0$, MPa.

The maximum tangent stress of the I–beam, acting at point # 3 of the dangerous section equals: $\tau_{max} = \frac{Q_{max} \times S_{x\,max}}{I_x \times b} = \frac{58 \times 10^3 \times 104 \times 10^{-6}}{1840 \times 10^{-8} \times 5.2 \times 10^{-3}} = 63.0 \times 10^6$, Pa – less than the maximum shear stress: $\tau_{max} = 63.0 < [\tau] = 96.0$, MPa

The stresses, acting at critical points of the dangerous section of the I–beam are indicated in the Table 7.7.1 and on the graph (Fig. 7.7.3).

Table 7.7.1

The point	σ, MPa	τ, MPa
1	163.0	0.0
2	149.3	2.5 / 48.8
3	0.0	63.0

The verification of strength by main stresses. The main stresses at the point # 2 section in areas in which there are no shear stresses are equal:

$\sigma_{13} = \frac{\sigma}{2} \pm \frac{1}{2} \times \sqrt{\sigma^2 + 4 \times \tau^2} = \frac{149.3}{2} \pm \frac{1}{2} \times \sqrt{149.3^2 + 4 \times 48.8^2} = 74.7 \pm 89.2$, MPa,

the values of the main stresses are equal: $\sigma_1 = 163.9$ MPa and $\sigma_3 = -14.5$ MPa for the plane stress state because $\sigma_1 > \sigma_2 > \sigma_3$.

Equivalent stress according to the hypothesis of the greatest tangential stresses equals: $\sigma_E = \sigma_1 - \sigma_3 = 163.9 - (-14.5) = 178.4$, MPa, which exceeds the permissible stress: $[\sigma]$ = 160 MPa. Thus, the strength of the I–beam at the assigned load is insufficient according to the hypothesis of the highest tangential stresses.

Exceeding the permissible stress is more than 5 %:

$\left|\frac{\sigma_E - [\sigma]}{[\sigma]}\right| \times 100\ \% = \left|\frac{178.4 - 160}{160}\right| \times 100\ \% = 11.5\ \% > 5\ \%.$

Questions and comments:

7.7.1. The sequence of testing a beam according to strength criteria (the hypotheses of plasticity and fracture), taking into account the normal and tangential stresses acting at its critical points, loaded with concentrated and distributed forces, applied moment and reactions of supports, and experiencing bending moments and transverse (the shear) forces.

7.7.2. Determination of normal stresses at critical points of the beam cross–section in which the greatest bending moment acts.

7.7.3. Checking by the criterion of the highest tangential stresses of strength in the section of the beam in which the greatest shear (the transverse) force acts. Distribution of tangential stresses in the cross–section of a beam and their determination using the formula of D.I. Zhuravsky, the diagram of tangential stresses in the beam cross–section. Ultimate shear stress dependence on the normal stress strength.

7.7.4. Finding the main stresses at critical points of the cross–section of a beam, taking into account the normal and tangential stresses acting at these points. The equivalent stress and strength check using the criterion of the highest tangential stress.

7.8. Determination of load when assessing the overall longitudinal strength of a ship

The general loads acting on the ship's hull influence the choice of strong cross–sectional dimensions of structures – the thickness of sheet structural elements and used parts made from rolled profiles. A significant influence of general loads is typical for the hulls of ships of large size (the displacement and length), experiencing extreme bending moments, both in calm water – depending on the of filling state with cargo, and in rough seas, as well as with a small relative height of the side or having limited draft, which is typical for inland navigation ships, of mixed navigation or coastal, to which number usually the harbor ships belong, including the bunkering ones.

Strength calculations taking into account the general loads acting on the hull should cover the main structural connections of ship: the deck plating streaks, the bottom plating, the second bottom, side plating, the longitudinal structural elements (the keel, stringers, carlines, longitudinal stiffeners, etc.), the sheets of the longitudinal bulkheads and double sides.

For the practical purposes, the overall longitudinal strength of the hull of a sea ship is usually assessed by the combination of several conditions, which can be specified either as a certain value of the withstanding load, or in the form of criteria for the strength of the loaded structural material, which cover, for example, the strength under the fatigue failure.

The following criteria and conditions that are common when checking the overall longitudinal strength of ship can be mentioned (a and b):

a) the assessment of fatigue failure under repeated cyclic bending of a ship experiencing an alternating wave bending moment simultaneously with a constant bending moment in calm water; avoiding fatigue failure of the material affects the value of the safety factor, i.e. the ratio of the yield strength of a structural material to the equivalent stress found for the ultimate static load according to one or another hypothesis of the material strength;

b) joint verification of the overall strength of the hull for bending and shearing under the action of the maximum possible sum of absolute values (without taking into account the signs) of the wave bending moment and the moment when the bottom hits the water surface with possible slamming, as well as the bending moment in calm water.

In some cases, when checking the overall strength, it may also be necessary to take into account the torque acting on the hull (the oblique–symmetric bending), which is typical for assessing the strength of connecting structures of multihull ships.

Exercise. The exercise assigns one to find the distributed load, to determine a diagram of shear forces and bending moments for a ship intended for delivery of the liquefied natural gas (www.icedesign.info, the project by International Contract Engineering Ltd,

ICE Marine Design, LNG– bunkering tanker), having the following characteristics: the length over all (LOA) 122 m, the waterline length: L =116 m, the breadth: B =20 m, the depth: H =11 m, cargo load line draft: T =5.2 m, the displacement at summer load line draft: Δ=9 150 ton, the cargo capacity 8 000 m^3 of liquefied natural gas.

Other elements of the ship used in the exercise: the shapes of the theoretical hull, shown on the drawing (Fig. 7.8.1), and on the fragment of the Microsoft EXCEL table (Fig. 7.8.2) with the ordinates y_{ij} of the theoretical hull – that is, the distances in the transverse direction from the centerline plane of the ship to the intersection point of each i–th waterline and each j–th frame section.

The ship hull

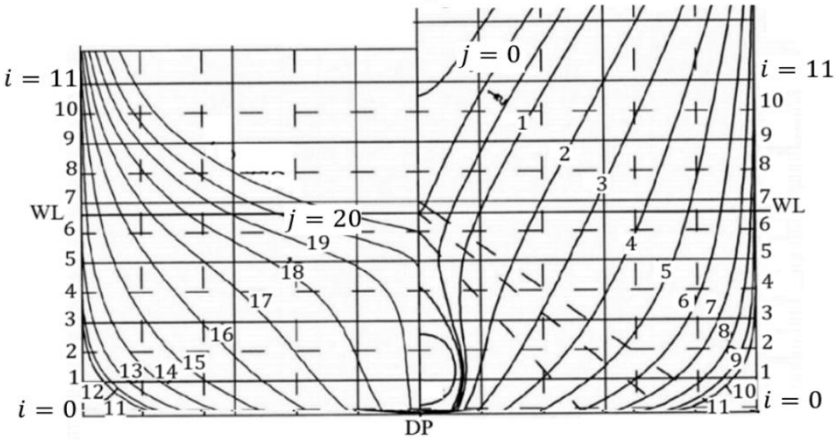


Fig. 7.8.1

	U	V	W	X	Y	Z	AA	AB	AC	AD
29	Y-table		WL	*j*						
30			0	1	2	3	4	5	6	7
31	Section	0	0.57	0.74	0.65	0.40	0.00	0.00	0.00	0.11
32	*i*	1	0.65	0.83	0.83	0.74	0.74	0.83	1.01	1.40
33		2	0.83	1.11	1.30	1.49	1.80	2.11	2.43	2.87
34		3	1.20	1.80	2.22	2.76	3.31	3.77	4.12	4.60
35		4	1.90	2.87	3.66	4.36	4.96	5.45	5.82	6.32
36		5	2.87	4.36	5.33	6.07	6.70	7.09	7.34	7.73
37		6	4.24	5.94	6.70	7.60	8.13	8.39	8.52	8.79
38		7	5.33	7.09	7.99	8.65	9.05	9.19	9.32	9.46
39		8	6.45	7.99	8.79	9.32	9.59	9.73	9.86	10.00
40		9	6.96	8.79	9.32	9.73	9.86	9.86	10.00	10.00
41		10	7.60	9.05	9.73	10.00	10.00	10.00	10.00	10.00
42		11	7.60	9.05	9.73	10.00	10.00	10.00	10.00	10.00
43		12	7.21	8.92	9.59	9.86	10.00	10.00	10.00	10.00
44		13	6.57	8.39	9.19	9.46	9.73	9.86	10.00	10.00
45		14	5.33	7.34	8.26	8.79	9.32	9.59	9.73	9.73
46		15	4.01	5.82	6.57	7.60	8.39	8.79	9.19	9.32
47		16	2.76	4.01	5.08	5.94	6.83	7.47	8.13	8.65
48		17	1.59	2.43	2.98	3.66	4.72	5.82	6.96	7.73
49		18	0.74	1.01	1.40	1.80	2.43	3.77	5.45	6.70
50		19	0.11	0.17	0.21	0.25	0.57	1.59	3.66	5.45
51		20	0.00	0.00	0.00	0.00	0.00	0.05	2.00	4.24

Fig. 7.8.2

In the given fragment of the Microsoft EXCEL program the ordinates y_{ij} of the surface points of the theoretical body are specified in meters. Given the usual symmetry of ship, the theoretical drawings are developed in three projections: «Body Plan», which is shown in the diagram above – one half of the projection has numbered cross–sections of the ship as viewed from the bow, and the second half – from the stern, and the projections not shown: the «Side» and the «Half Breadth Plan».

The composition of liquefied natural gas usually transported by ship includes 90 % of methane (from 85 to 95 %) which has a mass density of 0.668 ton/m^3, as well as 10 % mixture of gases: the ethane with a mass density of 1.263 ton/m^3, propane with a density of 1.872 ton/m^3, butane and nitrogen. The total mass density of liquefied natural gas can be approximately estimated as: $0.9 \times 0.668 + 0.1 \times \frac{1.263+1.872}{2} = 0.758$, ton/m^3. The total mass of the liquefied natural gas cargo on the ship with a volume of 8 000 m^3 is equal to: $8\,000 \times 0.758 = 6\,064$, ton. Taking into account the technology for transporting liquefied natural gas, cargo tanks are usually loaded completely, and if it is required to transport and transfer cargo in parts, it is necessary to use several cargo tanks.

On the ship, the characteristics of which are specified to check the overall strength, there are two cargo tanks of the same size – in the middle part of the ship, i.e. towards the stern of the bow superstructure (Fig. 7.8.3, the scheme of the ship general arrangement), and in the stern of the ship, with a capacity of 4 000 m^3 each, i.e. 3 032 ton – when each tank is completely filled.

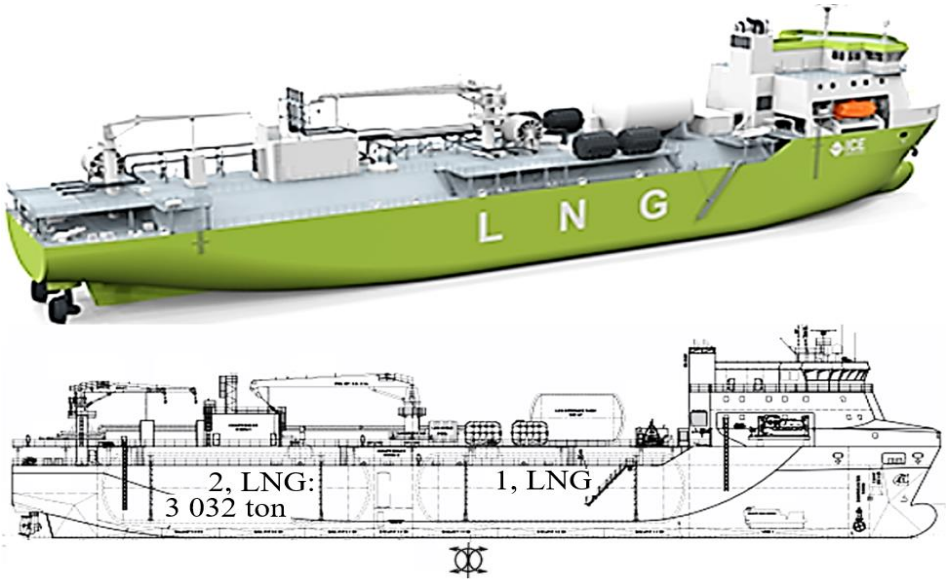
To check the overall longitudinal strength of the ship it is necessary to consider a partial load, assuming that this loading option corresponds to the maximum shear forces and bending moments in critical cross–sections when the stern cargo tank is fully loaded and there is no cargo in the middle cargo tank (it is assumed that the cargo from the medium tank was unloaded).

If the ship is in the ballast, which needs to be shifted in the direction to the bow to exclude the unacceptable trim, and the displacement with cargo is approximately equal to: $9\,150 - 3\,032 = 6\,118$, ton, which corresponds to the volumetric displacement of the: $W = \frac{6\,118}{1.025} = 5\,969$, m^3.

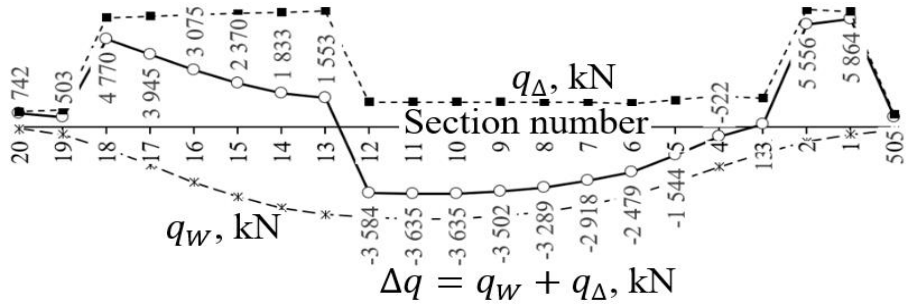
The distance between each two theoretical waterlines of the seven ones at draft of: T =5.2 m, i.e. when the ship drafted by the load line, one can the distance between each two theoretical waterlines determine as: $\Delta T = \frac{T}{7} = \frac{5.2}{7}$ =0.743, m. According to ordinates of the surface point of the theoretical hull y_{ij} (Fig. 7.8.1) at the intersection of each i–th waterline and each j–th frame (the cross-section line) with a draft along the sixth waterline, considering the partial loading of the ship in the option when checking the overall

strength, one can determine the immersed area S_j of each theoretical cross-section using the method of trapezoid rule as: $S_j = 2 \times \Delta T \times \left(\sum_{i=0}^{6} y_{ij} - \frac{y_{0j}+y_{6j}}{2}\right)$.

Diagram of the distribution along the length of the ship of Archimedes forces q_W acting on it proportional the curve of cross-sectional areas S_j (the displacement curve of the areas) is shown in the graph (Fig. 7.8.3 b).



(a)



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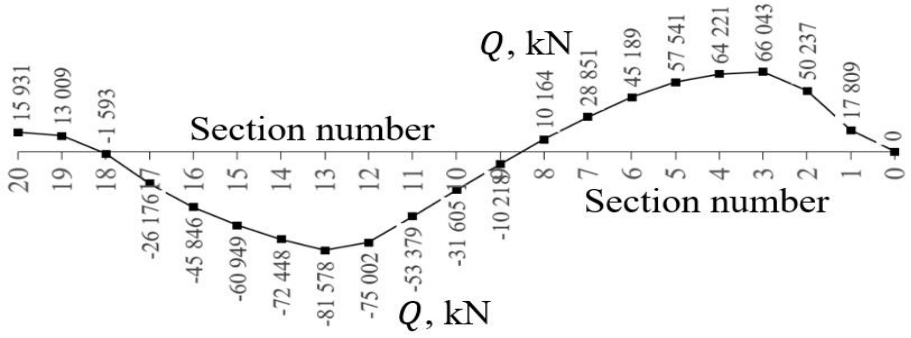

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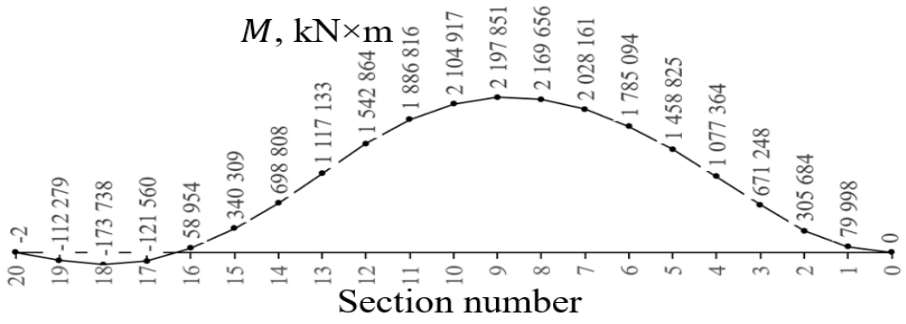


(d)

Fig. 7.8.3

With the use of the Microsoft EXCEL program and the distribution of the immersed areas of the cross–sections of hull along the ship length the submerged volume by method of trapezoid one determines as: $W = \Delta L \times \left(\sum_{j=0}^{20} S_j - \frac{S_0+S_{20}}{2}\right)$, where:

$\Delta L = \frac{L}{20} = \frac{116}{20}$ =5.8, m – the theoretical spacing, i.e. the equal distance between adjacent cross–sections of the theoretical hull, including zero section.

The volumetric displacement of the ship is found equal to 5 969 m3 during par-tial loading under the assumption that with such loading option the greatest shear forces and bending moments occur in critical cross–sections, when the aft cargo tank is fully loaded and the cargo is not available in the middle cargo tank.

The static moment of the immersed volume can be found numerically using the method of trapezoid and the distribution along the length of the immersed cross–sectional areas of the hull, as: $M_W = \Delta L \times \left(\sum_{j=0}^{20} S_j \times \left(\frac{L}{2} - \Delta L \times j\right) - \frac{L}{2} \times \frac{(S_0-S_{20})}{2}\right)$.

When the stern cargo tank is loaded the static moment of the immersed volume found by the Microsoft EXCEL program is equal to (minus) $-11\,851$ m^3, and the abscissa of the center of immersed volume, located aft from the midship plane is equal to:

$x_w = \frac{M_W}{W} = \frac{-11\,851}{5\,969} = -1.99$, m.

The general forces act on the ship in calm water due to the unequal distribution of weight and the Archimedes forces in accordance with the distribution of the immersed volume S_j, and additional cyclic forces arise due to the uneven distribution of the im-mersed volume of ship in rough seas.

The distribution of the weight of ship for estimating the overall strength in the ab-sence of detailed information one can approximately found as the sum of the distributions of the light weight of the ship: ($9\,150 - 6\,064 - 1000 = 2\,086$, ton, that is the weight 20 464 kN) and the variable weight of the ship – its deadweight, which includes the weight of the cargo (3 032 ton – when partially filled with cargo – 50 % of liquefied natural gas in the stern tank), as well as the weight of the supplies and the ballast (about 1,000 ton), including, the fuel.

A cargo of the mass 3 032 ton (weighing 29 744 kN) of liquefied natural gas in the stern cylindrical tank (i.e. located closer to the stern of the ship) is distributed evenly along the length of the hull section in which this tank is located, in an area from 14.5 to 49.3 m in stern from the midship frame. The length of the membrane cargo tank is 34.8 m.

For one theoretical spacing (that is the distance between the cross–sections of the theoretical hull: ΔL =5.8 m) there is a load mass of 505.3 ton (i.e. the weight: 505.3×9.81=4 957.3 kN per 5.8 m).

To balance the trim with partial (the half) load that is asymmetrical relative to the midship frame, in order to prevent the bottom in the bow of the ship from coming out of the water and to eliminate the possibility of slamming (the impacts by the bottom in the bow on the water surface in rough sea conditions), the use of ballast is required and the displacement of the supplies with a total mass of about 1 000 ton (the weight 9 810 kN) in bow ballast tanks located along the length of ship in the range from 2.9 to 24.5 m aft from zero theoretical frame (i.e. from the bow perpendicular).

In the plane of first theoretical frame (5.8 m aft from the bow perpendicular) is the center of mass of half (500 ton) of this ballast, in the plane of second theoretical frame (11.6 m from the bow perpendicular) is the center of mass of the other half of the ballast.

The mass of the superstructure with a bow (about 20 % of the ship's light weight, i.e. of 2 086 ton) equals 417.2 ton (the weight 4 093 kN) and is distributed over a quarter section of the ship's length – 29 m in the aft direction from the zero theoretical frame (the bow perpendicular).

Rest of the ship mass ($\Delta_E = 2\,086 - 417.2 = 1\,668.8$, ton, the weight 16 371 kN) is distributed along its length in accordance with the recommended scheme of the Byles (Fig. 7.8.4) for marine displacement ships, having the full shapes of the hull contours.

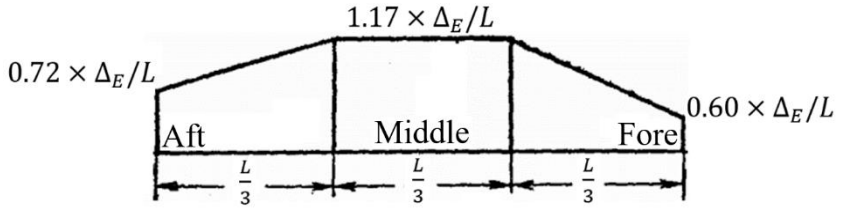


Fig. 7.8.4

The total values of the distributed weight of the ship q_Δ, including the weight of cargo, the ballast and supplies (with fuel), the weight of the hull and part of the hull – the superstructure on the bow are shown in the diagram (Fig. 7.8.3 b), which is combined with the diagram of the Archimedes forces acting on the ship q_W, proportional to the distribution of displacement volume along the length of the ship, i.e. the immersed areas S_j of theoretical frames, taking into account the theoretical spacing – the distance between the adjacent frames (5.8 m), he mass density of sea water 1.025 ton/m^3 and the acceleration

of gravity (9.81 m/s^2). As a result, the indicated diagram also shows a diagram of the total distributed load acting on the ship: $\Delta q = q_W + q_\Delta$, taking into account the minus sign of the distributed force of the ship weight q_Δ.

The function of the distributed total load Δq, acting on the ship, is a derivative of the transverse (the shear) force Q (Fig. 7.8.3 c), and the shear force is a derivative of overall bending moment M (Fig. 7.8.3 d). The values of the shear force and the bending moment at zero theoretical frame are equal to zero. To find the values of shear forces and bending moments in the cross–section of each j–th theoretical frame when determining the diagrams, one can use the numerical integration and the method of trapezoid:
$Q_j = Q_{j-1} + \frac{\Delta q_j + \Delta q_{j-1}}{2} \times \Delta L$, $Q_0 = 0$, $j = 1, \ldots, 20$ and $M_j = M_{j-1} + \frac{Q_j + Q_{j-1}}{2} \times \Delta L$, $M_0 = 0$.

In the partial loading option of the ship considered to determine the external forces (the bending moment and the shear force) to assess the overall longitudinal strength, under the assumption that this loading option corresponds to the maximum shear forces and bending moments in critical sections when the stern cargo tank is fully loaded, there is no cargo in the middle cargo tank, and the ship without significant trim is in ballast, which, like the supply (the fuel) is shifted to the bow tanks, the bending moment has the greatest value (about 2 200 000 kN×m) corresponding the 9–th theoretical frame (Fig. 7.8.3 d), in this cross–section the shear force is zero. The greatest shear force (minus 81 580 kN) acts in the cross–section of 13–rd theoretical frame (Fig. 7.8.3 c), in which the bending moment is relatively small (about 1 117 000 kN×m).

In the cross–section at first third of the ship length both the bending moment (1 785 000 kN×m) – at the 6–th theoretical frame section and the shear force (approximately 45 200 kN) are. Combined action of this bending moment and the shear force can be considered as a sign of a critical cross–section during overall bending. The check of the ship overall strength under the action of both normal stresses under general bending and the tangential stresses under shearing is of interest, considering the methodological motivation of the exercise too.

The cyclic action of loads acting on ship in sea, which can lead to fatigue failure, usually affects the value of the safety factor, i.e. the ratio of the yield strength of a structural material to the equivalent stress found for the ultimate static load according to one or another hypothesis of the strength of the material.

Since, in order to avoid slamming (the impacts of waves on bottom when ship bow end comes out of the water in rough water). When loading the aft cargo tank the ballast is taken into the bow tanks, where supply reserves, including fuel, are also shifted, the possibility of slamming is not considered to have an impact on the critical value of the bending moment and the shear force, which are influenced by the location of the load and ballast,

taken into account when determining the diagrams and when finding the largest values of the bending moment and the shear force.

In addition to the cyclic action of loads, which is taken into account when determining the safety factor, and in addition to the increase in the static bending moment and the shear force with ballast reception depending on the position of the load, which, on the other hand, eliminates slamming, the overall strength of the ship and the magnitude of the critical load are affected by influence also the position of ship relative to the wave profile, and the greatest values of the bending moment and the shear force, which arise, for example, when the ship is positioned at the top of a wave, the length of which is commensurate with the length of the ship, differ in the proportion of the dynamic coefficient (without slamming) from the bending moment and the shear force acting at a certain load of the ship on calm water, which was discussed above when determining the load diagrams.

Therefore, to check the overall strength of the ship below it is proposed to consider bending and shearing in the section of the 6–th theoretical frame (approximately one third of the length from the bow perpendicular), when the static bending moment is equal to: 1 785 000 kN×m (Fig. 7.8.3 d), and the shear force is: 45 200 kN (Fig. 7.8.3 c) and in the cross–section located closer to the midship frame of the ship (on the 9–th theoretical frame), in which the found value of the total bending moment on calm water is the largest (2 197 851 kN×m), although the shear force in this section is practically zero.

Questions and comments:

7.8.1. The greatest influence of the general loads acting on a sea displacement ship due to the uneven distribution along the length of ship of its weight and the Archimedes forces in calm water and on waves, typical for ships of large size (length), as well as for ships with a limited draft and a relatively small depth, for example, inland, mixed or coastal ships.

7.8.2. The structural elements of a ship experiencing normal and shear stresses during bending and shearing under the influence of weight and the Archimedes forces unevenly distributed along the length of the ship: the deck plating streaks, bottom plating, second bottom plating, side plating, longitudinal braces (keel, stringers, carlines, longitudinal stiffeners, etc.), sheets of longitudinal bulkheads and double sides.

7.8.3. Common criteria and conditions when checking the overall longitudinal strength of a ship: the determination of safety factor of the material taking into account fatigue failure under repeated cyclic bending of the ship experiencing the alternating wave bending moment, joint verification of the overall strength of the hull in bending and shear for the limit state, taking into account the impact possibility of the bottom on the surface of water during slamming, as well as the bending moment in calm water.

7.8.4. The obliquely symmetrical bending of connecting structures, considered when assessing the overall strength of multi–hull ships (the catamarans, trimarans, floating semi–submersible platforms, etc.).

7.8.5. Finding the immersed cross–sectional areas by numerical integration of the ordinates of the points of intersection of the frames with the theoretical waterlines using the method of trapezoid in the EXCEL, calculating the area of diagram of the distribution of immersed volume along the length of ship hull.

7.8.6. Numerical integration by the method of trapezoid the curve of distribution of immersed volume along the length of the ship hull, that is finding the immersed volume of the ship and the coordinates of the center of the immersed volume.

7.8.7. The approximate determination of the distribution along ship length of the sum of main components of it weight: the empty ship, cargo and ballast in the absence of detailed data on the design of ship.

7.8.8. The distribution of ballast when the ship is partially loaded in order to balance its trim and eliminate slamming – the impacts of the bottom in the bow on the water surface in waves. When calculating the load in the option to exclude slamming the combining the prerequisites, which is associated with the loads from ballast and supplies (fuel, other) received to the bow of the ship, considering when finding the overall bending moment and the shear force.

7.8.9. Approximate determination of the light weight of a ship distributed along the length, using the recommended scheme of Byles for finding the bending moment and the shear force when checking the overall longitudinal strength of the ship.

7.8.10. Diagram of the total distributed load acting on the ship, as the sum of diagrams: the distributed Archimedes force and the weight force, taking into account the minus sign of the distributed force of the ship weight.

7.8.11. Using the method of trapezoid for numerical integration of distributed load when determining the general diagram of the shear forces and for integrating the diagram of shear forces when determining the diagram of bending moments.

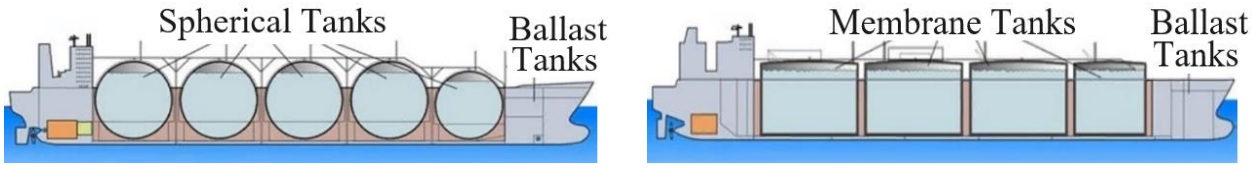
7.8.12. Location in middle part of the ship hull of critical cross–sections considered when checking the overall strength, differing little in structural elements, for example, the sections of cylindrical insert of the ship hull or close to the insert, in which the greatest general bending moment or maximum shear force acts together with sufficiently high bending moment.

7.8.13. Systematic methodological approach that eliminates duplication of overall strength while taking into account the following conditions: the cyclic action of loads and the corresponding safety factor of the material – the ratio of the yield strength and equivalent stress at critical points, selected in critical cross–sections; additional load arising when receiving ballast to exclude unacceptable trim and slamming; the static position of

the ship on top of a wave, the length of which is commensurate with length of the ship, taken into account by use of dynamic coefficient of the total load found in calm water.

7.9. Checking the strength and determining the strong cross–sectional sizes of the structural elements of the ship, taking into account the general loads acting on the hull

When checking the overall strength of the ship hull structures, it is necessary to take into account the purpose of the ship and the conditions for transporting cargoes. For example, the maritime transport of liquefied natural gas is usually carried out in spherical tanks (Fig. 7.9.1 a) or in the membrane ones (Fig. 7.9.1 b), which are the capacities for cargoes built into ship.



(a) (b)

Fig. 7.9.1

The ships transporting liquefied natural gas are double–hulled, that is, in addition to the outer hull and its extended structural elements, which are subject to the load from general bending, they also have internal hulls – the cargo tanks built into the outer hull. Cargo tanks fully or partially loaded with liquefied natural gas are to be able to withstand local loads from the liquid cargo they contain, which may move if the cargo tanks are partially filled. The requirements for the design of cargo tanks for liquefied natural gas by maritime classification societies include, in addition to their sufficient local strength, tightness and thermal insulation performances. When transporting liquefied natural gas, an important operational condition is the full loading of tanks, therefore, for logistical efficiency, the ship is usually equipped with several cargo tanks so that this condition can be met when transporting different amounts of cargo without partially filling them.

The construction of these ships is characterized by a tendency to increase the tonnage and number of ships for transporting the liquefied natural gas in membrane–type cargo tanks. The volume of the ship premises is better used when placed the membrane–type cargo tanks than when using spherical tanks. In addition, the membrane tanks are more technologically advanced during construction than the spherical ones.

At the same time, the spherical tanks due to their geometric shape are characterized by slightly better thermal insulation and mechanical performances than the membrane ones under the influence of weight of the cargo and the inertial forces, including under the

hydrodynamic shock loads, if the liquid cargo moves when cargo tank is partially filled and the sloshing occurs – the impacts of liquid cargo on flat membrane bulkheads.

To protect the bulkheads of membrane cargo tanks from sloshing and to improve thermal insulation performances the various design solutions can be used, for example, the two–layer design of flat pneumatic panels, including the heat–insulating layers and the sealed membrane, which have not only low thermal conductivity, small thickness and low mass, but also the ability to redistribute local loads over large areas of membrane tank floors that occur when hitting flat surfaces by liquid cargo in cases of partial filling of cargo tanks, secured by the patent # 2817739 on August 14, 2023 (Russian Federal Service for Intellectual Property, the authors: Alexey S.Ogai, Kirill S.Tolmachev, Sergei A.Ogai, Anatoly I.Azovtsev, Michael V.Voyloshnikov and Sergei A.Ogai – junior, Fig. 7.9.2).

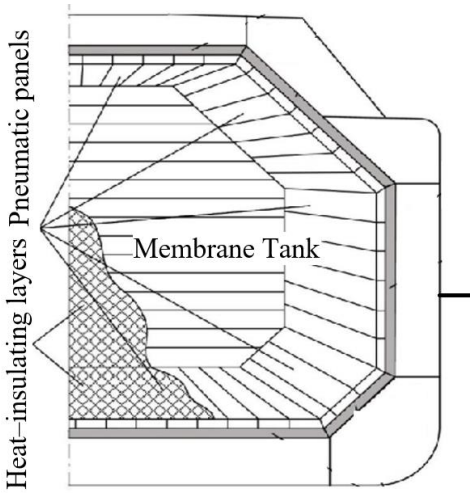


Fig. 7.9.2

Exercise. It is assigned to check the overall longitudinal strength of the ship's hull, the characteristics of which are given in the previous section (item 7.8), under the influence of a load in the section of the 6–th theoretical frame (approximately one third of the length from the bow perpendicular), when the ship is partially loaded (50 % cargo, with a fully filled aft cargo tank and with ballast in the bow of the ship), the static bending moment in calm water is equal to: 1 785 000 kN×m, and the shear force is: 45 200 kN, as well as under the influence of overall bending moment of the ship hull in the cross–section, located closer to the midship frame (on the 9–th theoretical frame), in which the value of overall bending moment in calm water is the greatest (2 197 851 kN×m), and the shear force is practically zero.

The dynamic load coefficient (without slamming) can be recommended to be taken equal to: 1.25, and the effect of cyclic loads affecting the possibility of fatigue failure of the structural material must be taken into account by choosing a safety factor in relation to the yield strength of the structural material.

Under the action of static bending moment (1 785 000 kN×m) at 6 theoretical frames, taking into account the dynamic coefficient, the working moment is:
1 785 000 × 1.25 = 2 231 250, kN×m, and the working shearing force:
45 200 × 1.25 = 56 500, kN.

The liquefied natural gas delivery ship has two membrane–type cargo tanks with a capacity of 3 032 ton (4 000 m^3) each. The first cargo tank (34.8 m long) is located in the middle part of the ship, slightly offset in the bow direction and occupies an area from 29.5 to 64 m along the length of the ship from it bow perpendicular, and the second is located in an area from 14.5 to 49.3 m aft from the midship section.

The structures of cargo tanks, which have a relatively small length along the length of the ship, are not included in the elements to be taken into account when finding stresses from shear forces and bending moments during the overall bending of the ship hull. The structural elements experiencing load during the overall bending of the ship are highlighted with a solid line shown in the half cross–section (Fig. 7.9.3).

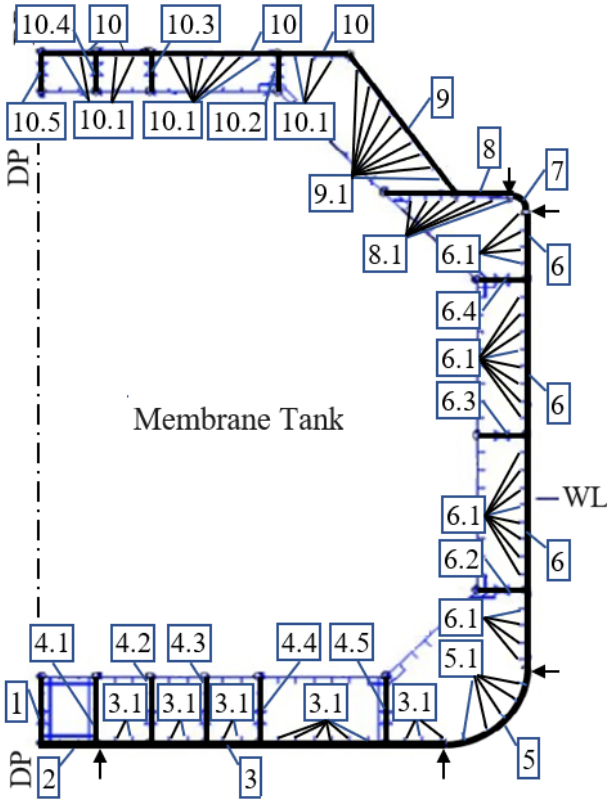


Fig. 7.9.3

The half–width structural section diagram shows the following structural elements of the hull experiencing load during the overall bending: the vertical keel (1), the horizontal keel (2), the bottom streaks of plating (3), the bottom longitudinal stiffeners (3.1, 15 items – within the half–width of the ship), the bottom stringers (4.1–4.5), the chine streaks of plating (5), the stiffeners of the chine streaks (5.1, 5 items), the side streaks (6), the side

longitudinal stiffeners (6.1, 24 items – within the half–width of the ship), the side stringers (6.2–6.4), the deck stringer (7), main deck flooring (8), the longitudinal stiffeners of main deck (8.1, 7 items – within the half–width of the ship), the deck inclined sheet (9), the stiffening ribs of the deck inclined sheet (9.1, 9 items – within the half–width of the ship), the trunk deck plating (10), the stiffeners of the trunk deck (10.1, 13 items – within the half–width of the ship), the carlines (10.2–10.5).

It is advisable to determine the geometric characteristics of the transverse structural section located in the middle part of the ship hull (within the cargo spaces and the cylindrical insert) in the Microsoft EXCEL program, and the table is shown as a fragment of a worksheet (Fig. 7.9.4).

	B	C	D	E	F	G	H	I	J	K
46	Thickness s, mm		Quantity	F, mm^2	z_C, m	S_Z, mm^2×m	z_{C0}, m	I_C, mm^2×m^2	I_0, mm^2×m^2	σ, MPa
47	Vertical keel (1)	50	1.6 m	79 200	0.8	63 360	-4.81	1 833 992	16 896	-104.7
48	Horizontal keel (2)	41	2.4 m	99 000	0	0	-5.61	3 118 089	0	-122.1
49	Bottom streaks of plating (3)	33	16.8 m	475 200	0	0	-5.61	14 966 825	0	-122.1
50	Bottom longitudinal stiffeners (3.1)	13	30 items	118 606	0	0	-5.61	3 735 599	0	-122.1
51	Bottom stringers (4.1–4.5)	33	10 items	528 000	0.8	422 400	-4.81	12 226 610	113	-104.7
52	Chine streaks of plating (5)	33	2 items	165 876	0.581	96 442	-5.03	4 197 995	384 423	-109.4
53	Longitudinal stiffeners of chine streak (5.1)	13	10 items	39 535	0.581	22 986	-5.03	1 000 562	91 624	-109.4
54	Side streaks of plating (6)	33	2 items	594 000	6.1	3 623 400	0.49	141 390	4 009 500	10.6
55	Side longitudinal stiffeners (6.1)	13	48 items	189 770	6.1	1 157 595	0.49	45 171	1 280 946	10.6
56	Side stringers (6.2–6.4)	33	6 items	178 200	6.24	1 111 968	0.63	70 253	1 112 396	13.7
57	Deck stringer (7)	33	2 items	41 469	10.85	450 132	5.24	1 139 741	6 007	114.0
58	Main deck flooring (8)	30	2 items	130 680	11.0	1 437 480	5.39	3 793 548	0	117.2
59	Longitudinal stiffeners of the main deck (8.1)	12	14 items	46 531	11.0	511 837	5.39	1 350 751	0	117.2
60	Deck inclined sheet (9)	30	2 items	204 391	12.4	2 534 451	6.79	9 417 403	133 536	147.6
61	Stiffening ribs of deck inclined sheet (9.1)	12	18 items	59 825	12.4	741 831	6.79	2 756 463	39 086	147.6
62	Trunk deck plating (10)	30	12.8 m	380 160	13.8	5 246 208	8.19	25 486 476	0	178.1
63	Stiffeners of trunk deck plating (10.1)	12	26 items	86 414	13.8	1 192 513	8.19	5 793 321	0	178.1
64	Carlines (10.2–10.5)	30	7 items	71 400	13.5	963 900	7.89	4 442 416	3 742	171.5
65			TOTAL:	3 488 257		19 576 504		95 516 606	7 078 267	
66				F_Σ, mm^2		$S_{Z\Sigma}$, mm^2×m		$I_{C\Sigma}$, mm^2×m^2	$I_{0\Sigma}$ mm^2×m^2	

Fig. 7.9.4

In the EXCEL calculation table, the sizes (the thickness s of sheets and racks of profile elements) and quantity are specified and the following geometric characteristics are determined (based on the full width of the ship, taking into account that the symmetrical cross–section diagram Fig. 7.9.3 shows half the width): the sectional area F of a sheet element or group of elements (in particular, the groups of stiffeners of the bottom, side or deck flooring); the applicate z_C from the main plane (that is, from the comparison axis located in the bottom plane) to the center of the cross–sectional area of the structural element (or group of elements); static moment of area of a structural element of ship hull section: $S_Z = F \times z_C$ relative to the main plane; the applicate z_{C0} from the neutral axis of the cross–section of the ship hull – the horizontal $y - y$, axis passing through the center of the cross–sectional area of the ship's hull, to the center of the cross–sectional area of the structural element: $z_{C0} = z_C - z_{C\Sigma}$, where: $z_{C\Sigma} = \frac{S_{Z\Sigma}}{F_\Sigma} = \frac{19\,576\,504}{3\,488\,257} = 5.61$, m, F_Σ and $S_{Z\Sigma}$ – the total area of the elements in the cross–section of the ship hull and their total static

moment relative to the main plane; the own moment of inertia of the area I_C of each structural element in the cross–section of the ship hull and the total own moment of inertia of the structural elements $I_{C\Sigma}$; the transfer moment of inertia of the area of the structural element in the cross–section of the ship's hull relative to the neutral axis of the cross–section of the ship's hull $y-y$: $I_0 = F \times z_{C0}^2$ and the total transfer moment of inertia $I_{0\Sigma}$.

The total moment of inertia of the ship's hull cross section as part of the sum of the proper $I_{0\Sigma}$ and transfer $I_{C\Sigma}$ moments of the inertia od ship transverse section is:
$I_{C\Sigma} + I_{0\Sigma} = 95\,516\,606 + 7\,078\,267 = 102\,594\,873$, mm^2×m^2.

Normal stresses in the longitudinal connections of ship's hull under the action of general bending moment M, equal 2 231 250 kN×m, depending on the applicate z of the cross-section point are determined in the EXCEL program (Fig. 7.9.4) in accordance with the Navier formula, as: : $\sigma = \frac{M \times z_{C0}}{I_{C\Sigma}+I_{0\Sigma}} = \frac{2\,231\,250 \times z_{C0}}{102\,594\,873} \times 10^3$, MPa.

The width of the plating sheets of the bottom, sides, the chine, vertical and horizontal keel, decks, longitudinal inclined sheets and stringers is shown by the dimensional lines on the diagram of half the cross–section of the ship hull (Fig. 7.9.5).

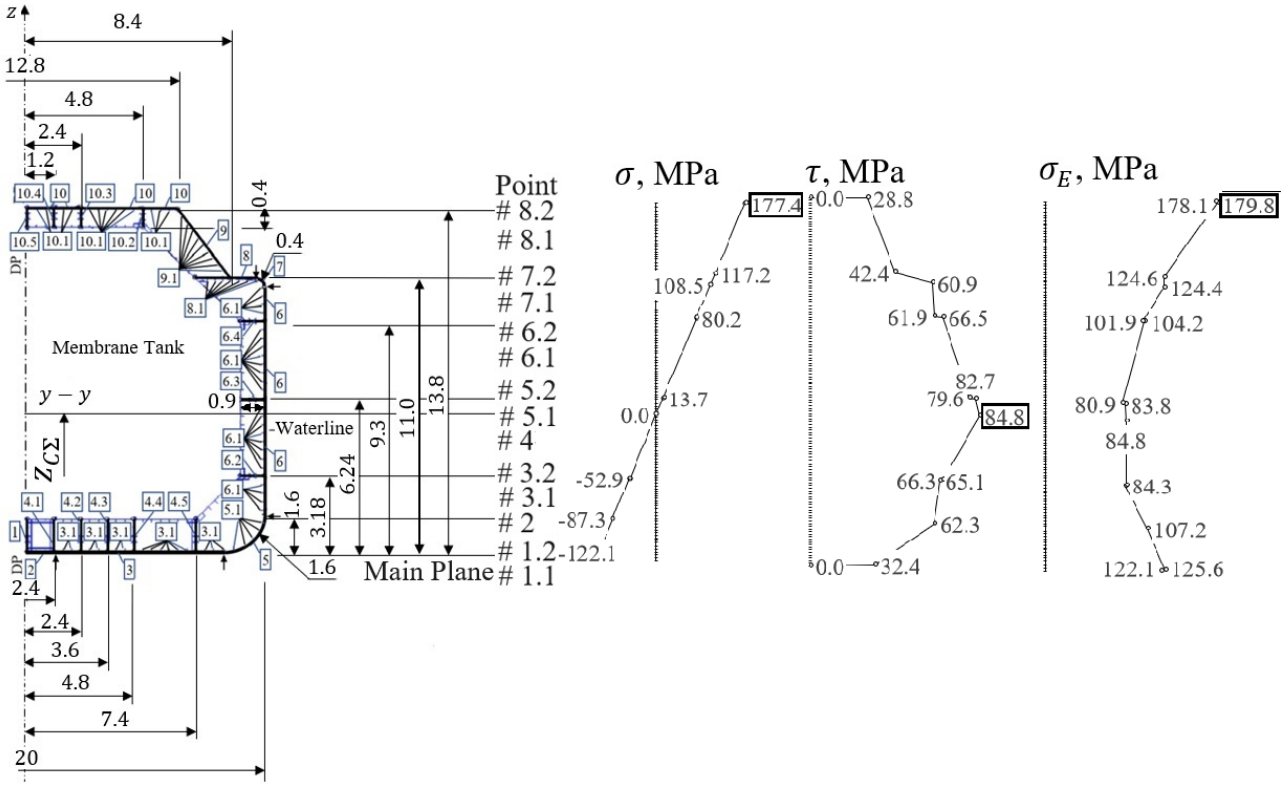


Fig. 7.9.5

When determining the geometric characteristics of the cross–section of a ship's hull for each structural element, taking into account its shape and cross–sectional dimensions (the thickness of the element and the width of its cross–section) within the approximate limits acceptable for general calculations, it is convenient to adhere to certain assumptions:

1. The thicknesses of sheets of structural elements in the cross–section of the hull and the walls of profile parts are determined in millimeters (mm), their cross-sectional areas – in the square millimeters (mm^2), the static moments of area: $mm^2 \times m$, the moments of inertia of sections of structural elements: $mm^2 \times m^2$. When finding the geometric characteristics of sections, the scale relationships of dimensions are taken into account in accordance with those shown in the table on a fragment of the Microsoft EXCEL worksheet (Fig. 7.9.4).

2. When determining the areas and other geometric characteristics of structural elements of the cross–section of ship that do not intersect the plane of symmetry (the diametrical), such as stringers, the carlines, side plating, the chine streak, deck inclined sheet, groups of stiffeners that strengthen the sustainability of the listed sheet structural elements, their area when performing calculations is doubled taking into account the symmetry of the ship.

3. When determining the areas and geometric characteristics of other structural elements that intersect the plane of symmetry, as well as those that are in the plane of symmetry (the vertical keel and diametrical carlines on the deck), the symmetry of ship is taken into account by the dimensions of the sections of these structural elements.

4. When determining the applicate of the center of the area of each element, the thickness of the sheets is equal to zero, and the center of the cross–sectional area of a flat sheet structural element is located in the middle of its width.

5. The proper moment of inertia of the area of each structural element relative to an axis lying in the plane of this element, when determining the moment of inertia of the cross–section of the ship hull, is taken equal to zero, and only proper moment of inertia of the cross–section of those elements, which do not lie on the axis relative to which the moment of inertia of the section is sought.

6. When determining coordinates (the applicates) of center of the cross–sectional area of ship the applicates of centers of the cross–section areas of a group of stiffeners installed on the bottom, sides and on the deck to increase the sustainability of these floors and plating sheets (the bottom and sides) and deck flooring are combined with the coordinates of the centers of the cross–section areas of the sheet elements on which each group of stiffeners is installed.

7. The own moments of inertia of the half bulb plate profile stiffener groups installed on the bottom, sides and on the deck when determining the moment of inertia of the cross–section of ship are taken equal to zero and only the transfer moments of inertia of these stiffener groups are taken into account. This introduces a slight underestimation of the overall strength and rigidity of the hull, which, when checking the overall longitudinal strength of the ship, serves to increase the safety margin.

8. The thickness of the external elements of the hull is included in the calculation of the geometric characteristics of cross–section of the ship without taking into account the thickness reserve for corrosion to perform a strength check, which is thereby carried out for the hull of the ship minus the expected corrosion wear or other wears (the residual deformations of structural elements etc.).

Finding the proper moment of inertia of the section of a sheet structural element relative to an axis perpendicular to the plane of the sheet is carried out using the formula: $I_0 = s \times y^3/12$, where s – the sheet thickness, y – the width of sheet cross–section in the direction perpendicular to the axis, relative to which the proper moment of inertia is sought.

The sections of a sheet curved along a circle (radius R) – the chine of the plating streak and deck stringer, when their geometric characteristics are found, have the shape of a thin–walled semi–ring. Taking into account the symmetry of the ship, two quarters of the annular section on each chine form a quarter–ring, the same applies to quarter–ring shapes of each of the deck stringers sections.

The area of semi–ring is: $F = \pi \times R \times s$, where s – the thickness of semi–ring; the applicate of center of the square of semi–ring relative to the cutting line of the second half of the ring is: $z_C = 2 \times R/\pi$; the own moment of inertia of the area of semi–ring relative to the line passing through the center of the area of semi–ring section parallel to the cutting line of the second half of ring is: $I_0 = R^3 \times s \times \frac{\pi^2+8}{2\times\pi}$.

In the construction of metal ships, in addition to flat rolled sheets and curved stamped sheet parts, the profiled rolled products are widely used, the cross–section of which has the shape of a standard half bulb plate, the geometric characteristics of different sizes, which are shown in the Table 7.8.1 (the Interstate standard 21937–76 dated 01.01.1978).

The cross–sectional area of the half bulb plate, depending on the thickness of its wall s one can find approximately using the empirical formula, which is created by approximating standard values, as: $F \approx 8.9 \times s^2 + 254 \times s - 950$, mm^2 (the thickness of wall of half bulb plate s is indicated in millimeters, Fig. 7.9.6).

In the Microsoft EXCEL program in the table, which is shown as a fragment of a worksheet (Fig. 7.9.4), the following are indicated as the characteristics of a group of stiffeners having the shape of a standard half bulb plate: the width of the half bulb plate stand, the number of stiffeners (twice in relation to the quantity shown in the diagram, Fig. 7.9.3, taking into account the symmetry of the ship's hull) and the area of the stiffeners in each group installed on the bottom, the chine streaks, on the sides, on the main deck, on the inclined deck plate and on trunk deck of cargo tanks.

Table 7.8.1

#	s, mm	h, mm	y_0, mm	F, cm^2	i_x, cm
5	4	50	3.13	2.86	1.55
5.5	4.5	55	3.41	3.47	1.70
6	5	60	3.74	4.28	1.85
7	5	70	4.40	5.07	2.17
8	5	80	5.07	5.84	2.49
9	5.5	90	5.65	7.03	2.81
10	6	100	6.29	8.63	3.11
12	6.5	120	7.55	11.13	3.76
14a	7	140	8.82	14.05	4.39
14b	9	140	8.53	16.85	4.38

#	s, mm	h, mm	y_0, mm	F, cm^2	i_x, cm
16a	8	160	9.99	17.94	5.02
16b	10	160	9.75	21.11	5.00
18a	9	180	11.13	22.18	5.67
18b	11	180	10.83	25.78	5.65
20a	10	200	12.35	27.36	6.29
20b	12	200	12.06	31.36	6.28
22a	11	220	13.53	32.82	6.93
22b	13	220	13.20	37.22	6.91
24a	12	240	14.71	38.75	7.56
24b	14	240	14.41	43.55	7.54

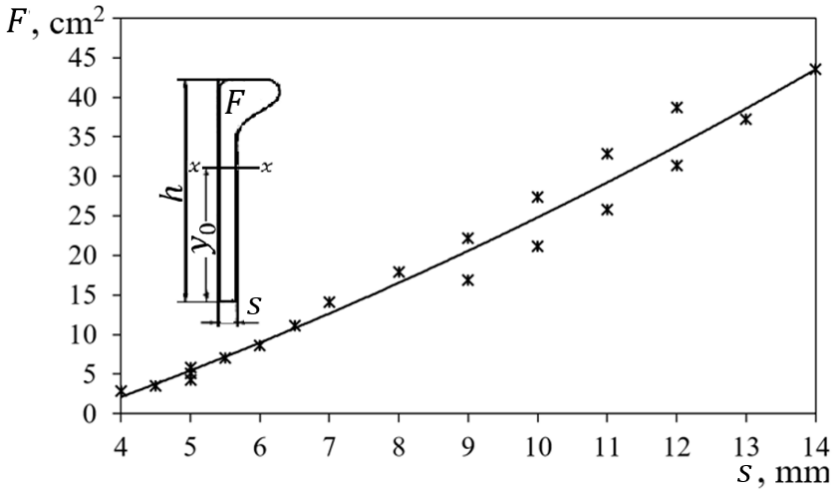


Fig. 7.9.6

The moment of inertia of the cross–section of the ship hull is equal to the sum of its own and transfer moments of inertia relative to the horizontal axis $y - y$, passing through the center of the cross–sectional area of the ship's hull:

$I_y = I_{C\Sigma} + I_{0\Sigma} = 95\,616\,606 + 7\,078\,267 = 102\,594\,873$, mm^2×m^2.

Applicate of the neutral axis of the cross–section of the ship hull $y - y$ is:

$z_{C\Sigma} = \frac{S_{Z\Sigma}}{F_\Sigma} = \frac{19\,576\,504}{3\,488\,257} = 5.61$, m.

A fragment of the Microsoft EXCEL worksheet (Fig. 7.9.4) also contains a determination of normal stresses σ, that arise under general bending and tension of longitudinal structural connections falling within the cross–section. The values of normal stresses σ, indicated in the table Microsoft EXCEL, acting at the centers of the cross-sectional area of each longitudinal structural connection, with the overall bending of the vessel by a

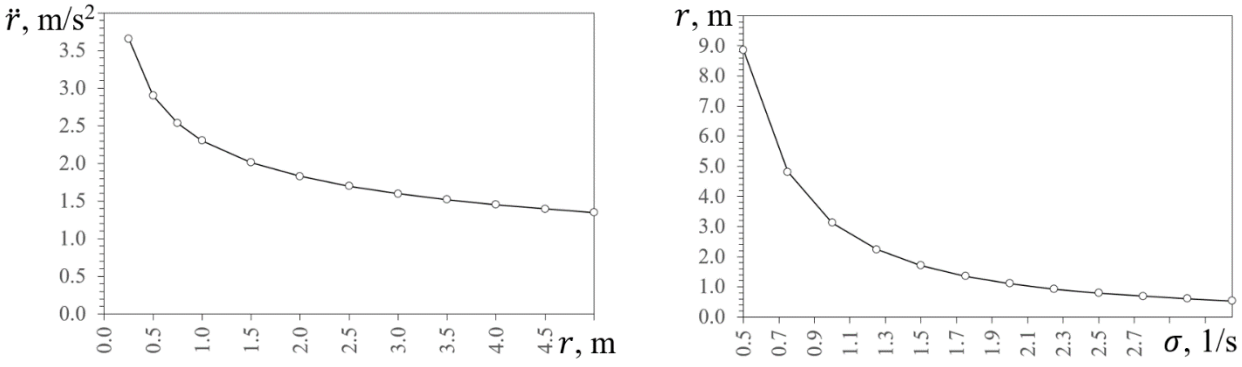
moment M taking into account the moment of inertia I_y of the cross–sectional area are: $\sigma = \frac{M \times z_{C0}}{I_y}$.

Uneven distribution along the length of the ship of its weight forces and the Archimedes forces of still water is the cause for the total bending moment found in the 6–th theoretical frame section (approximately one third of the length from the bow perpendicular, Fig. 7.8.3) is equal to: 1 785 000 kN×m and the shear force: 45 200 kN.

When a ship is exposed to moving waves, it is also necessary to take into account the dynamic load factor. It would be mentioned that for such objects as an entire ship, the calculation of elements of mechanical systems and forces is usually associated with methodological difficulties and it is usually required, along with the calculation as it is, to use the empirical values too. At the same time, when using these expert estimates, it is advisable to have a general vision of the proportions between physical magnitudes, for example, the ratio of the accelerations experienced by ship when moving on waves and the acceleration of gravity (9.81 m/s^2), acting on the ship and on the water in the absence of waves.

The amplitude of vertical acceleration can be found by double differentiation with respect to time of the wave ordinate function: $z = r \times \sin(\sigma \times t - k \times x)$, where r is the amplitude of the wave profile of a certain frequency σ and the wave number: $k = \frac{2 \times \pi}{\lambda_W}$.

The amplitude of the acceleration is equal to: $r \times \sigma^2$. Determining the wave amplitude depending on its length using the Zimmerman's empirical formula: $r = 0.085 \times \lambda_W^{3/4}$, m, taking into account the dependence of waves frequency on the wave length: $\sigma = \frac{7.85}{\sqrt{\lambda_W}}$, 1/s, which is more discussed in detail below (item 12.4), the acceleration amplitude of the wave profile ordinate depending on the height of regular sea waves can be found as: $\ddot{r} = \frac{2.303}{r^{1/3}}$, m/s^2, and the dependence of the amplitude r of regular waves on their frequency σ has the form: $r = \frac{3.129}{\sigma^{3/2}}$, m (Fig. 7.9.7 a and b).



(a) (b)

Fig. 7.9.7

For the natural frequencies in the range from 0.5 to 0.75 1/s (item 12.3) of the heave of ship, the characteristics of which, given in the calculation, correspond to the prevailing frequencies of the spectrum in irregular waves (item 12.4), the acceleration of the wave profile points according to the graph corresponds appropriately close to: 2.5 m/s^2, which corresponds to strong storm waves and the order of dynamism coefficient values of about 1.25 (based on the ratio of the total accelerations experienced by the ship, including the acceleration of gravity 9.81 m/s^2 and the ship own motion to the acceleration of gravity). Taking into account the specified value of the dynamic coefficient, the calculated magnitude of bending moment in the section of the 6–th theoretical frame (approximately one third of the length from the bow perpendicular), used to check the overall longitudinal strength of the ship in this cross–section of the ship's hull, is equal to: 2 231 250 kN×m, and the shear force is: 56 500 kN.

When analyzing a complex stress state in different directions in critical sections of the ship hull and at critical points of these sections to find equivalent stresses and check strength, it becomes necessary to redefine the points in the Microsoft EXCEL calculation table, which is shown as a fragment of the worksheet (Fig. 7.9.4), since this table calculates the normal stresses during general bending on the cross–sectional lines of the ship hull, passing through the centers of the area of each of the structural elements in the section.

At the same time, in accordance with the formula of D.I. Zhuravsky: $\tau = \frac{Q \times S_{y-}}{I_y \times b}$ in the section of each structural element, the greatest shear stresses arise on the internal boundary of the section of this structural element, which is closer to the neutral axis $y - y$ of the cross–section of the ship hull, passing horizontally through center of cross–sectional area. In this case, the largest value is taken by the static moment of the sectional area S_{y-}, located outside the internal boundary of the section of the structural element.

In addition, the values of the shear stress depend on the width of the cross–section along a line parallel to the neutral axis $y - y$ of the ship hull cross–section, i.e. on the width b which is equal to twice the thickness s of vertical sheets, taking into account the symmetry of the cross–section of the ship, or twice the oblique width of inclined sheets, the cross–section of which is not parallel to the centerline plane of the ship, as well as on the change in width when connecting vertical sheets or on the change in width in sections of vertical sheets, on which the horizontal sheets (for example, stringers to side sheathing) are attached to.

Points designated by numbers are specified by applicates z, in which the normal stresses are determined: $\sigma = \frac{M \times z}{I_y}$, the tangential stresses: $\tau = \frac{Q \times S_{y-}}{I_y \times b}$, the main stresses: $\sigma_{13} = \frac{\sigma}{2} \pm \frac{1}{2} \times \sqrt{\sigma^2 + 4 \times \tau^2}$ (at: $\sigma_1 > \sigma_2 > \sigma_3$) and the equivalent stresses: $\sigma_E = \sigma_1 - \sigma_3$ (according to the hypothesis of the highest tangential stresses) are shown in the Microsoft

EXCEL calculation table on a fragment of the worksheet (Fig. 7.9.8). In addition, the found values are shown in the diagrams: σ, τ and σ_E (Fig. 7.9.5).

	J	K	L	M	N	O	P	Q	R
75		z , m	σ, MPa	b , mm	S_{y}, mm^2×m	τ , MPa	σ_1 , MPa	σ_3, MPa	σ_E , MPa
76	Point # 1.1	0.000	-122.1	16 800	0	0.0	0.0	-122.1	122.1
77	Point # 1.2	0.033	-121.3	66	3 888 108	32.4	2.1	-123.5	125.6
78	Point # 2	1.600	-87.3	66	7 462 270	62.3	10.0	-97.2	107.2
79	Point # 3.1	3.180	-52.9	66	7 798 272	65.1	15.5	-68.4	83.9
80	Point # 3.2	3.213	-52.2	66	7 942 740	66.3	16.1	-68.3	84.3
81	Point # 4	5.612	0.0	66	10 160 107	**84.8**	42.4	-42.4	84.8
82	Point # 5.1	6.240	13.7	66	9 914 529	82.7	48.8	-35.1	83.8
83	Point # 5.2	6.273	14.4	66	9 543 873	79.6	47.6	-33.3	80.9
84	Point # 6.1	9.300	80.2	66	7 974 644	66.5	92.2	-12.0	104.2
85	Point # 6.2	9.333	80.9	66	7 422 224	61.9	91.4	-10.5	101.9
86	Point # 7.1	10.600	108.5	59	6 568 514	60.9	116.4	-8.0	124.4
87	Point # 7.2	11.000	117.2	73	5 613 724	42.4	120.9	-3.7	124.6
88	Point # 8.1	13.770	177.4	73	3 820 254	28.8	178.6	-1.2	**179.8**
89	Point # 8.2	13.800	**178.1**	12 800	0	0.0	178.1	0.0	178.1

Fig. 7.9.8

Considering the safety factor, which, to prevent fatigue failure under high-cycle loading, can be taken equal to: $[n] = 2$, and considering the yield strength (the temporary stress) of carbon structural steel used in shipbuilding, equals from 390 to 490 MPa, in average: $\sigma_y = 440$ MPa, the permissible normal stresses can be determined as: $[\sigma] = \frac{\sigma_y}{[n]} = \frac{440}{2}$= 220, MPa, in this case, the permissible tangential stresses will be equal to: $[\tau] \approx 0.6 \times [\sigma] = 0.6 \times 220 = 132$, MPa.

The fragment of the worksheet (Fig. 7.9.8) and the diagrams: σ, τ and σ_E (Fig. 7.9.5) show the values of the highest normal stress ($\sigma = 178.1$ MPa), the highest shear stress ($\tau = 84.8$ MPa) and the equivalent stress ($\sigma_E = 179.8$ MPa according to the hypothesis of the highest tangential stresses) in the section of 6–th theoretical frame (approximately one third of the length from the bow perpendicular, Fig. 7.8.3) of the total bending moment and the shear force, taking into account the applied load dynamic coefficient 1.25.

Analysis of the calculated stresses at critical points in the section of the 6–th theoretical frame (approximately one third of the length from the bow perpendicular) allows one to draw the following conclusion: firstly, the highest stresses in this cross–section of ship hull are significantly less than the permissible normal stress of 220 MPa and the permissible shear stress of 132 MPa, and secondly, the equivalent stress, found in accordance with the hypothesis of the highest tangential stresses, acting in the upper structural elements of the cross–section of the ship is directly dependent on the highest normal stress arising in the hull of the ship under the influence of the general bending moment, and the highest equivalent stress practically does not depend on the shear stress in the ship hull under the action of shear (the transverse) force under, that are in the section of the 6–th

theoretical frame (approximately one third of the length from the bow perpendicular) act on the ship, the characteristics of which are assigned, having a relatively shallow draught and side depth, which affects the mechanical characteristics of ship hull experiencing the overall load.

The obtained conclusion gives grounds to change the conditions for checking the overall longitudinal strength and in the Microsoft EXCEL table to use data on the total load acting on the ship hull in the cross–section located closer to the midship frame of the ship (on the 9–th theoretical frame in which the found value of the total bending moment on calm water is the largest (2 197 851 kN×m, Fig. 7.8.3 d), that is considering the dynamic factor 1.25 the working bending moment is set 2 747 314 kN×m, although the shear force in this cross–section is practically zero (Fig. 7.8.3 c).

It would be mentioned that the factor of shear (the transverse) force and the tangential stress could be of greater importance when checking the overall longitudinal strength of a higher–sided ship (in terms of the ratio of side height and the ship length) or a ship of smaller dimensions (the length, displacement, etc.).

Taking into account the load dynamic coefficient of 1.25, the found mechanical stresses are: the normal σ, the tangential τ, the main σ_{13} and the equivalent σ_E (according to the hypothesis of the highest tangential stresses) are shown in the Microsoft EXCEL calculation table on a fragment of the worksheet (Fig. 7.9.9) and on diagrams: σ and σ_E (Fig. 7.9.10 a and b).

	J	K	L	M	N	O	P	Q	R
75		z , m	σ, MPa	b , mm	S_{y-}, mm^2×m	τ, MPa	σ_1 , MPa	σ_3, MPa	σ_E, MPa
76	Point # 1.1	0.000	-150.3	16 800	0	0.0	0.0	-150.3	150.3
77	Point # 1.2	0.033	-149.4	66	3 888 108	0.0	0.0	-149.4	149.4
78	Point # 2	1.600	-107.4	66	7 462 270	0.0	0.0	-107.4	107.4
79	Point # 3.1	3.180	-65.1	66	7 798 272	0.0	0.0	-65.1	65.1
80	Point # 3.2	3.213	-64.2	66	7 942 740	0.0	0.0	-64.2	64.2
81	Point # 4	5.612	0.0	66	10 160 107	0.0	0.0	0.0	0.0
82	Point # 5.1	6.240	16.8	66	9 914 529	0.0	16.8	0.0	16.8
83	Point # 5.2	6.273	17.7	66	9 543 873	0.0	17.7	0.0	17.7
84	Point # 6.1	9.300	98.8	66	7 974 644	0.0	98.8	0.0	98.8
85	Point # 6.2	9.333	99.6	66	7 422 224	0.0	99.6	0.0	99.6
86	Point # 7.1	10.600	133.6	59	6 568 514	0.0	133.6	0.0	133.6
87	Point # 7.2	11.000	144.3	73	5 613 724	0.0	144.3	0.0	144.3
88	Point # 8.1	13.770	218.5	73	3 820 254	0.0	218.5	0.0	218.5
89	Point # 8.2	13.800	**219.3**	12 800	0	0.0	219.3	0.0	**219.3**

Fig. 7.9.9

A comparison of the obtained and permissible stress magnitudes suggests that the ship, when loading only one cargo tank (in the stern) and ballast in the bow, has sufficient overall longitudinal strength, and the critical cross–section for overall bending of the ship at the 9–th theoretical frame, that is the section almost by the midship, and the critical points are located in the section of the cargo space trunk deck, in the stiffeners of this deck

and in the upper part of the carlines, in the upper part of the deck inclined plates and stiffeners and in this part of the inclined sheets. The remaining cross–sections of the ship hull and the remaining structural elements in the specified section along the midship frame have one or another safety margin in relation to the sections of the specified structural elements at their critical points.

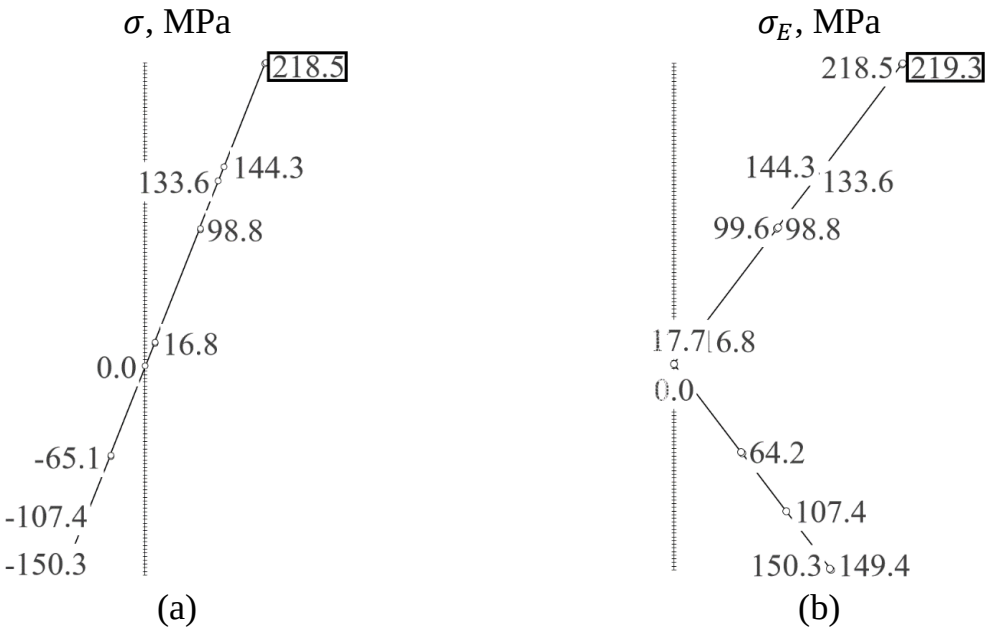


Fig. 7.9.10

One can see that the neutral axis $y - y$ passing through the center of the cross-sectional area of the ship, at the points of which the normal stresses during the overall bending of the ship are equal to zero, is located closer to the bottom of the ship than to the deck, since the longitudinal structural elements of the bottom, including the vertical keel, stringers and the chine streaks, as well as the stiffening ribs installed on them, which increase the sustainability when compressed in the plane of the plating sheets, are more massive than flooring decks, the carlines and stiffening ribs on the deck, which are subject to tension when the ship is bent.

It should also be mentioned that the overall longitudinal strength of the ship, which has been assessed, experiences a bend in calm water, in which the deck sheets are stretched and the bottom sheets are compressed, and this is quite typical for seagoing ships in the middle part of the hull, which, the Archimedes forces predominate due to the completeness of the shape of the submerged hull, and at the extremities weight forces predominate due to the smaller submerged volume and deadrise of the hull shape in the bow and stern parts of the ship, which leads to the characteristic bending of the ship in calm water.

The rules of maritime classification societies, united in the International Association of Classification Societies (IACS), usually contain multiple recommendations that allow taking into account the sustainability of sheet elements of a ship structures when

they are compressed in their plane, as well as recommendations for assessing the strength of individual plates located as part of the hull or flooring between the longitudinal and transverse elements of the hull set.

Checking the strength of the ship's hull must be done minus the expected corrosion or other wear, that is, the ship has to be sufficiently strong according to mechanical criteria, being subject to wear. In other words, the thickness of the ship plate structures is actually slightly greater than that used as data in the strength calculation. For the practical purposes, it is of interest to assess the wear of sheets, that is, to determine how much greater the thickness of the structural elements of a ship during its construction or repair, when worn sheets are replaced with equal strength with new sheets or geometrically similar to new sheets and made of the same material (the carbon structural steel of the same grade), as well as the structural elements of the hull used in the construction of the ship.

The following recommendations are known for assessing the corrosion rate of structural elements made of carbon steel during the operation of sea ships: the corrosion rate of deck flooring – 0.05 mm/year; of the structures exposed to petroleum products and crude oil – from 0.13 to 0.20 mm/year; of the outer surface of the side plating – 0.06 mm/year; of the side framing – 0.04 mm/year; the surfaces of sheets as part of structures of double sides and the bulkheads – from 0.04 to 0.08 (for double bottom) mm/year; on the surfaces of the internal structures of dry cargo ships, along which the loading equipment moves and on which the cargo is located – 0.13 mm/year; the chine sheathing streaks – 0.08 mm/year. It can be assumed that the rate of corrosive wear of the outer surfaces of carbon steel structures is about 0.05 mm/year, and in areas of more intense wear (on the outer sections of decks and on the side plating in the area of the variable waterline) – up to 0.15 mm/year.

In accordance with the established practice of technical supervision of ships in operation, carried out by maritime classification societies, the technical examination of the ship and its hull for the assignment of a certain category and seaworthiness class is usually carried out at intervals of five years, and intermediate surveys are also carried out during this five–year period. In addition to assess the corrosive wear of the hull, during inspections of the technical condition, dents of the plating and deformations of the frame in the areas of dents of the plating are also examined, as well as holes, leaks (the tightness of the ship's hull is checked) and other types of wear of structures, and based on the results of these surveys, an order for necessary replacement of worn sheets.

That is, assuming that the sheets are subject to wear from 5 to 10 years before replacement (one– and two–time period between classification surveys), and the ship retains its strong performances during this period of time, that is since the ship hull is quite strong, being subject to wear of the mentioned and other types, the thickness of the sheets during initial construction of the ship and when replaced during repairs is greater than the

thickness data used in assessing the strength of the ship, by an amount of 0.25 to 1.5 mm. And if the average thickness of external sheet structures is one to one and a half inches, then before replacement during repair, the reduction in strength (that is, permissible load) due to wear can be estimated at up to 5 %.

Questions and comments:

7.9.1. The use of spherical or membrane tanks on the ships for sea transportation of liquefied natural gas, double–hull design of these ships, including an outer extended hull and internal hulls – cargo tanks construct into the outer hull. Requirements of maritime classification societies related to the transport of liquefied natural gas.

7.9.2. The comparative technological, mechanical advantages, as well as the thermal insulation performances per unit of cargo using the spherical or membrane cargo tanks when transporting liquefied natural gas.

7.9.3. The structural elements in cross–section of the ship hull, experiencing the overall loads: the vertical and the horizontal keel, bottom, side and chine plating, flooring decks and inclined deck plates, stringers and the carlines, stiffening ribs of plating sheets and flooring.

7.9.4. Finding in the Microsoft EXCEL the geometric characteristics of cross–section of the ship hull, taking into account the characteristics of its structural elements experiencing load under the general bending of ship: the cross–sectional area of the ship hull, the static moment of the cross–sectional area relative to the comparison axis located in the plane of the bottom, applicates of the horizontal neutral axis of the cross–section, passing through it area center, the total proper moments of inertia of the areas of structural sections and transfer moments – relative to the neutral axis.

7.9.5. The assumptions when finding geometric characteristics of the cross–section of a ship hull when determining moments of inertia: neglect of the thicknesses of structural elements whose sections are parallel to the comparison axis (and to the neutral axis of the section, i.e. assuming these thicknesses as zero size).

7.9.6. Symmetrical consideration of the geometric characteristics of sections on both sides of ship, the permitted combination of the centers of the cross–sectional areas of groups of stiffeners with the sheets on which they are installed, and neglect of the own moments of inertia of the ribs, which increase the rigidity of individual sheet structural elements and their sustainability under the plane compression.

7.9.7. Performing verification calculations of strength of the ship hull within the framework of the assumption of corrosion and other wear of its structural elements, i.e. for cross–sectional dimensions of structural elements subject to wear.

7.9.8. The use in shipbuilding of metal parts having a standard half bulb plate profile as stiffening ribs that increase the sustainability of sheet structural elements of a ship when compressed in the plane of the sheets.

7.9.9. Determination of normal stress distribution diagrams under general bending of the ship's hull and tangential stresses under the action of shearing (transverse) forces on the ship's hull, formula D.I. Zhuravsky for finding tangential stresses.

7.9.10. The dynamic loads acting on the ship hull associated with the movement of waves, avoiding slamming (impacts by the bottom of the ship on the surface of the waves) by receiving ballast and leveling the trim of a partially loaded ship.

7.9.11. The empirical determination of dynamic coefficient, the correlation of loads acting on ship in sea conditions with additional accelerations on waves (that is, with the acceleration of the ordinates of the wave profile) and with the total acceleration, including additional accelerations and gravity acceleration.

7.9.12. The redefinition of coordinates (applicate relative to the neutral axis of the cross–section of the ship), in which the calculation of normal and tangential stresses are carried out under the action of the overall bending moment and shear (the transverse) force on the ship, taking into account the fact that in each structural element experiencing the overall loads the greatest shear stresses arise at the internal points of the cross–section of this structural element – located closer to the neutral axis of cross–section of the ship hull, and in the points of change the width of elements in their connections is also important.

7.9.13. The analysis of normal tangential and equivalent stresses in cross–sections and finding critical loaded points in critical sections. Characteristic of a fairly large and relatively low–sided sea ship is the predominance of normal stresses in the structures of the upper decks and their tensile stiffeners due to the overall bending of ship hull.

7.9.14. The bending of a sea ship in calm water, associated with the predominance of the Archimedes' forces in middle part of the ship due to completeness of shape in the middle part of submerged hull, and with predominance of the weight forces at the ends of the ship due to the smaller submerged volume and deadrise of the hull shape.

7.9.15. The need for amendments to the stresses that arise under compression of sheets in the bottom area due to overall bending, taking into account the mechanical sustainability of these sheets and the associated regulation by maritime classification societies of amendments to the forces acting under compression of sheets.

7.9.16. The intensity of corrosive wear of the structural elements of sea ship and the five–year frequency of the ship classification survey by the maritime classification societies with more frequent check of technical condition of hull, that is the influence of indicated periodicity on the deviations of thickness of structural elements used as data when calculating strength.

Chapter 8. Methods for specifying the motion of point and kinematics of elements in mechanical systems, the coordinate systems, curvature of motion trajectory, instantaneous centers of velocities and accelerations

8.1. A natural way to specify the motion of a material point

A natural way to assign motion in a moving coordinate system, that is in a coordinate system that is associated with a moving point, is shown on the diagram (Fig. 8.1.1).

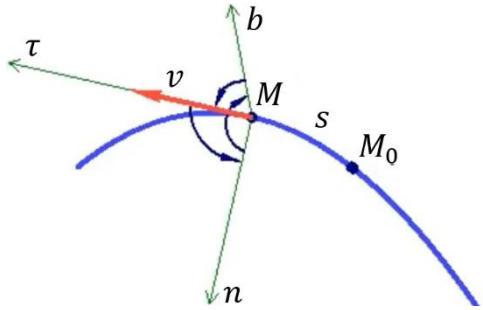


Fig. 8.1.1

The origin of coordinates at each moment of time coincides with the position of point M, moving along a given trajectory, i.e. the coordinate axes move. One of the axes $M\tau$ is directed tangent to the trajectory towards positive reference of the arc coordinate s.

The second axis Mn – normal one to the tangent is directed perpendicular to the tangent towards the concavity of the trajectory and lies in the plane in contact with the trajectory line s. If the curve s is flat, then the normal Mn is located in the plane of the curve itself and is then denominated the main normal.

The third axis, the binormal one Mb, is perpendicular to indicated two axes ($M\tau$ and Mn) and forms a right–handed coordinate system with them.

The movement of the point is specified in natural way under follows conditions:

1. The trajectory of the point motion, in the general case, the curvilinear one, is specified or can be found.

2. On the trajectory, the origin M_0 of the curvilinear coordinate is determined.

3. The positive direction of the coordinate reference is specified s.

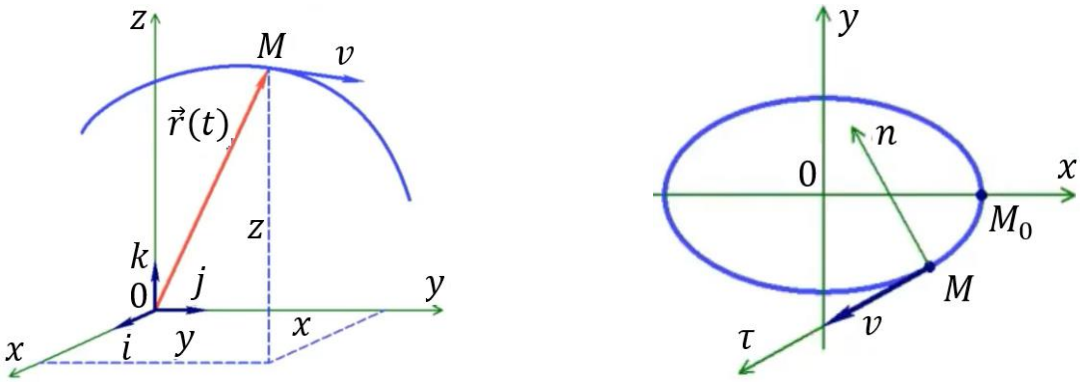
4. The law of change of the curvilinear coordinate s, is specified as a functional dependence on time: $s = M_0M = s(t)$.

Basic kinematic characteristics when moving of the point M along the trajectory M_0M are: the point velocity: $v = \frac{ds}{dt}$ and it complete acceleration: $a = \sqrt{a_\tau^2 + a_n^2}$.

The tangent component of the point M complete acceleration: $a_\tau = \frac{dv}{dt}$.

Normal component of the point complete acceleration: $a_n = \frac{v^2}{\rho}$, where ρ – radius of curvature of the trajectory of motion of the point.

In addition to the natural method of movement specifying, the following can be used: the vector method of assigning the movement (Fig. 8.1.2 a): $\vec{r} = \vec{r}(t)$ and the coordinate method (Fig. 8.1.2 b): $x = x(t)$, $y = y(t)$ and $z = z(t)$, which are formally interconnected: $\vec{r} = x \times \vec{\imath} + y \times \vec{\jmath} + z \times \vec{k}$.



(a) (b)

Fig. 8.1.2

Exercise. The movement of the point is determined in a vector way:
$\vec{r} = (1 - t) \times \vec{\imath} + (t - 1) \times \vec{\jmath}$.

The definition of the pattern of point motion along the trajectory: $s = s(t)$ is assigned, counting the arc coordinate s from the initial position of the point M_0.

From the given vector equation of motion of a point one obtains the law of motion in coordinate form: $x = (1 - t)$ and $y = (t - 1)$.

By replacing the time parameter to eliminate it in the equation: $t = 1 - x$, one can find the equation of the trajectory line in the form: $y = -x$.

The origin of the arc coordinate s (at $t = 0$) at point M_0 of the trajectory with coordinates: $x_0 = 1$ and $y_0 = -1$.

In the present position the point M has coordinates: x and y and its motion along the trajectory from the starting point M_0 is equal (Fig. 8.1.3):
$s = M_0M = \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(1 - t - 1)^2 + (t - 1 + 1)^2} =$
$= \sqrt{2 \times t^2} = \sqrt{2} \times t$.

The solution option by coordinate method of specifying movement of the point M is: $v = \frac{ds}{dt}$, $s = \int_0^t v \times dt$, $v = \sqrt{v_x^2 + v_y^2}$, $v_x = \frac{dx}{dt} = -1$, $v_y = \frac{dy}{dt} = 1$
and $v = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, then: $s = \int_0^t \sqrt{2} \times dt = \sqrt{2} \times t$.

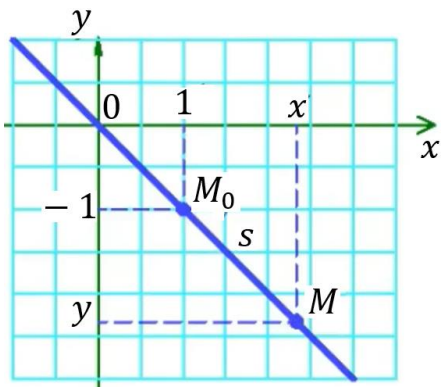


Fig. 8.1.3

Questions and comments:

8.1.1. The specifying of the motion of point in a moving coordinate system, in which the origin of coordinates is located at this point and moves with it, the abscissa axis is directed in the direction of movement and is tangent to the trajectory of movement, the ordinate axis is directed to the center of curvature of the trajectory, and the applicate axis is perpendicular to them and forms with them the right–handed coordinate system.

8.1.2. Elements of the natural way of specifying the movement of point: the curvilinearity of movement trajectory (in the general case), the origin of the curvilinear coordinate, the positive direction of coordinate, and the pattern of change of the curvilinear coordinate in time.

8.1.3. The vector method for specifying the movement of the point, when it is located at the end of a vector moving in fixed coordinates, and a coordinate way for specifying point motion, formally similar to the vector method.

8.1.4. The general kinematic characteristics when a point moves along a trajectory in a natural manner: the velocity of the point (the derivative of the trajectory with respect to time), the total acceleration of the point (the second derivative), the tangential and the normal components of the total acceleration.

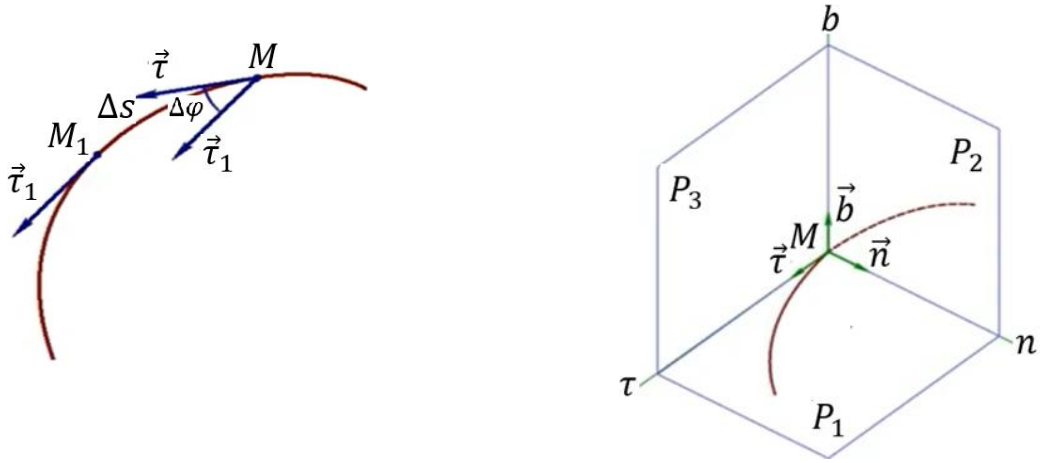
8.1.5. Sequence of transformation from the vector form of specifying motion to the coordinate form and to the natural form.

8.1.6. Finding the trajectory line of a point by eliminating the time parameter. Determination of the velocity and acceleration of point when moving along straight path depending on time.

8.2. The coordinate planes and axes associated with the trajectory of a material point motion

The touching plane. For a spatial curvilinear trajectory (Fig. 8.2.1 a) of the point M motion one can create the unit vector $\vec{\tau}$ of the tangent axis τ. At a distance Δs along a

curved path at a nearby point M_1 to create the unit vector $\vec{\tau}_1$ of the tangent axis τ. The vector $\vec{\tau}_1$ one can move parallel to the point M and draw the plane through two vectors: $\vec{\tau}$ and $\vec{\tau}_1$, applied to point M. In the limit at the tendency of the point M_1 to point M this plane will take a certain position P_2.



(a) (b)

Fig. 8.2.1

The plane P_2 got by this way (Fig. 8.2.1 b) is an osculating plane (with the trajectory of the point M). In the case if the curvilinear trajectory of motion of the point M is flat, then this trajectory is completely located in the touching plane.

The plane passing through the point M perpendicular to the tangent τ is denoted the normal plane P_1 (Fig. 8.2.1 b).

The intersection line of the touching P_2 and normal P_1 planes determines the main normal to the curvilinear trajectory of motion of the point M.

The plane passing through the point M perpendicular to the main normal n, is denoted the rectifying plane P_3. The intersection line of the rectifying P_3 and normal P_1 planes determines third axis of the natural coordinate system – the binormal b.

With a curvilinear trajectory moving of point M in the natural coordinate system, three mutually perpendicular axes are associated: the tangent axis τ, the main normal n and the binormal b. These axes have unit vectors (the orts, Fig. 8.2.1 b): $\vec{\tau}$, $\vec{n}$ and $\vec{b}$.

The unit vector of the tangent $\vec{\tau}$ to the axis trajectory is directed towards the reference of point coordinates on the trajectory. The ort of the main normal $\vec{n}$ is directed towards the concavity of the point's trajectory M.

The ort of the binormal $\vec{b}$ is directed in such a way that, looking towards it, the rotation of the ort of tangent $\vec{\tau}$ until it aligns with the main normal $\vec{n}$ would be counterclockwise, i.e. the unit vector of the binormal is the vector product of the unit vector of the tangent axis to the trajectory of the point and the unit vector of the normal axis.

Questions and comments:

8.2.1. A coordinate plane in contact with the trajectory of motion of a point – the limiting position of the plane in which the tangents to the trajectory of the point's movement in its different positions, which tend to each other along by the trajectory.

8.2.2. The flat trajectory of motion of a material point completely located in a contacting plane.

8.2.3. The normal coordinate plane – passing through a point perpendicular to the tangent line to the trajectory of point.

8.2.4. The main normal to the trajectory of a point's motion is the line of intersection of the normal and tangent coordinate planes.

8.2.5. Rectifying coordinate plane – passing through a moving point perpendicular to the main normal.

8.2.6. The axes of a moving coordinate system associated with a point when the motion of the point is naturally specified, which are the intersection of coordinate planes: osculating, normal and rectifying.

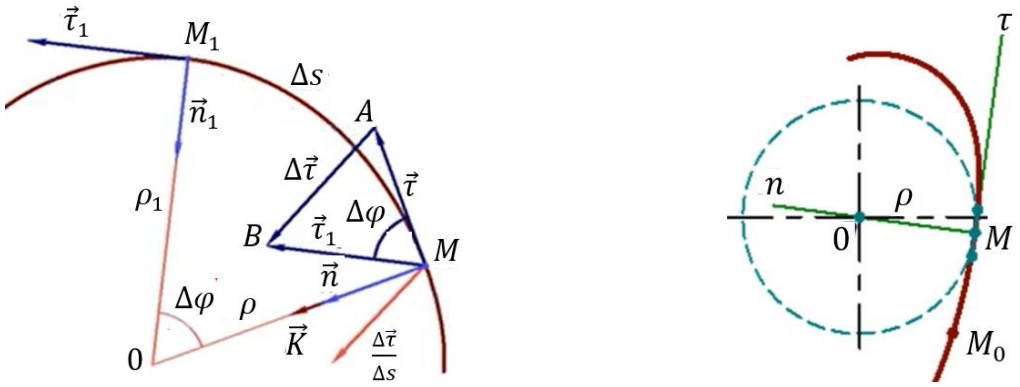
8.2.7. The direction of the unit vectors of the coordinate axes with a natural specification of the point's motion: the unit tangent to the trajectory of the axis directed towards the reference of coordinate on the trajectory, the unit of the main normal in the direction of the concavity of motion trajectory of the point, and the unit of the binormal in the direction of the vector product of the unit vectors of the tangent and the main normal.

8.3. Evaluation of trajectory curvature using the natural method of specifying the motion of a material point

The curvature vector $\vec{K}$ (Fig. 8.3.1 a) of trajectory of the point M is equal to the derivative when the tangent vector $\vec{\tau}$ changes on the arc Δs: $\vec{K} = \lim\limits_{\Delta s \to 0} \frac{\Delta \vec{\tau}}{\Delta s} = \frac{d\vec{\tau}}{ds}$, where $\Delta\vec{\tau} = \vec{\tau}_1 - \vec{\tau}$, and $\vec{\tau}_1$, $\vec{\tau}$ – the unit vectors of the tangent to the curvilinear trajectory of the point M motion.

The increment module $\Delta\tau$ (the $\Delta\vec{\tau}$) one can find by the angle of adjacency $\Delta\varphi$ of the triangle ABM: $\Delta\tau = \tau \times \Delta\varphi = 1 \times \Delta\varphi = \Delta\varphi$, and the modulus of the curvature vector is equal: $K = \frac{d\tau}{ds} = \frac{d\varphi}{ds}$.

Circle of curvature of motion trajectory of the point M, the center of curvature. If through the point M and two points close to it (Fig. 8.3.1 b) draw a circle, then as these close points tend to the point M in the limit, a circle will be obtained, which is denoted the circle of curvature of the point M trajectory.



(a) (b)

Fig. 8.3.1

The circle of curvature lies in the plane touching the trajectory of motion of the point M. The radius of this circle ρ is equal to the radius of curvature of the point M trajectory. The center 0 of the circle of curvature of the motion trajectory of the point M lies on the main normal n and is denoted the center of curvature.

Curvature of the trajectory of motion of the point M and radius of curvature. From the points: M and M_1 (Fig. 8.3.1 a) to the motion trajectory of the point one can draw normals in the directions: $\vec{n}$ and $\vec{n}_1$, which intersect at the point 0 (at the center of curvature of the arc MM_1).

Radii of curvature at points: M and M_1 are equal accordingly: ρ and ρ_1. As the arc length approaches zero MM_1: $\Delta s \to 0$ the radius of it curvature tends to the value ρ at the point M: $\rho_1 \to \rho$.

The increment ds of the arc MM_1 of the motion trajectory of the point M is related to the radius ρ and the increment $d\varphi$ of the direction rotation angle: $ds = \rho \times d\varphi$.

The magnitude of the curvature of the motion trajectory of the point M modulo equals: $K = \frac{d\varphi}{ds} = \frac{d\varphi}{\rho \times d\varphi} = \frac{1}{\rho}$. The curvature K of the trajectory of motion of the point M is a geometric indicator inverse to the radius ρ of curvature of the trajectory.

The direction of the curvature vector: $\vec{K} = \frac{1}{\rho} \times \vec{n}$ coincides with the direction of the main normal $\vec{n}$ to the trajectory of the point M.

The radius of curvature ρ of the curvilinear trajectory of the motion of point M is the magnitude inverse to the curvature index K: $\rho = \frac{1}{K}$. In particular, the radius of curvature of a rectilinear trajectory is equal to infinity, and the curvature of a rectilinear trajectory wherein is equal to zero. The radius of curvature of the trajectory of motion of the point M along the circle is equal to the radius of the circle (Fig. 8.3.1 b).

The normal acceleration of the point depends on the radius of curvature ρ: $a_n = \frac{v^2}{\rho}$, accordingly, the radius of curvature of the trajectory of the point M is equal: $\rho = \frac{v^2}{a_n}$.

Questions and comments:

8.3.1. The curvature vector of point movement trajectory as the derivative of the unit vector of the tangent to the variable – the movement along the arc of trajectory in its natural way of being specified by a coordinate system associated with the moving point.

8.3.2. Finding the vector of increment of ort of tangent to the trajectory of point motion by changing the direction of the trajectory, i.e. by the increment of the angle of adjacency between the unit vectors of the tangent in different places of the trajectory, the distance between which is equal to the increment of the point motion.

8.3.3. The radius of curvature of the point motion trajectory is equal to the limit of the ratio of tangent unit vector increment, taken modulo, to the increment of the point motion tending to zero, i.e. the derivative of the angle of adjacency (between the unit vectors of the tangents at different places on the trajectory) with respect to motion of point on the trajectory, which is a variable on which the angle depends as function.

8.3.4. The direction of the curvature vector of the point motion trajectory, coinciding with the direction of the main normal to the trajectory in the natural way of specifying the point motion.

8.3.5. The curvature vector of the trajectory equals to division of the unit vector of the main normal to the point trajectory by the magnitude of curvature radius. Infinite radius of curvature of a rectilinear trajectory. The radius of a circle, which is the radius of curvature of the trajectory of movement along a circle.

8.3.6. The normal acceleration of point, depending on the square of the point motion velocity along a trajectory and on the radius of curvature (inversely proportional) when the point motion is naturally specified in the coordinate system associated with the point.

8.4. Determination of instantaneous center of acceleration in plane motion

The acceleration $\vec{a}_M$ of the point M in the plane motion of a body (Fig. 8.4.1) and rotating around the pole is equal to the vector sum of the acceleration $\vec{a}_A$ of the pole A and the acceleration of the point $\vec{a}_{MA}$ in the rotation of the body around the pole:
$\vec{a}_M = \vec{a}_A + \vec{a}_{MA}$ and $\vec{a}_M = \vec{a}_A + \vec{a}^{\tau}_{MA} + \vec{a}^{n}_{MA}$,
where: $\vec{a}^{\tau}_{MA} = \varepsilon \times MA$ and $\vec{a}^{n}_{MA} = \omega^2 \times MA$.

The magnitude of acceleration $\vec{a}_{MA}$ of the point M in the rotation of the body around the pole A equals: $a_{MA} = \sqrt{(a^{\tau}_{MA})^2 + (a^{n}_{MA})^2} = MA \times \sqrt{\varepsilon^2 + \omega^4}$.

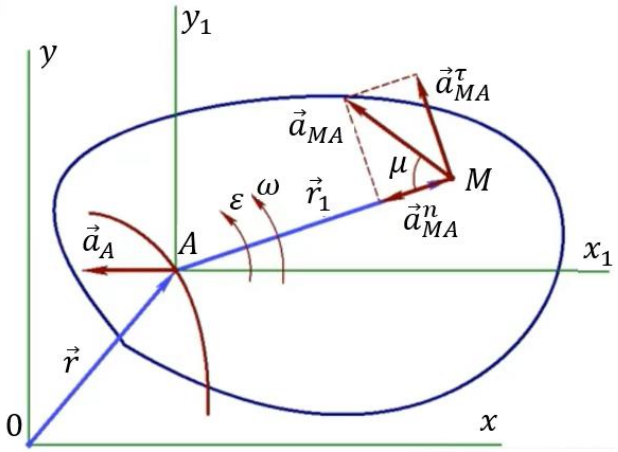


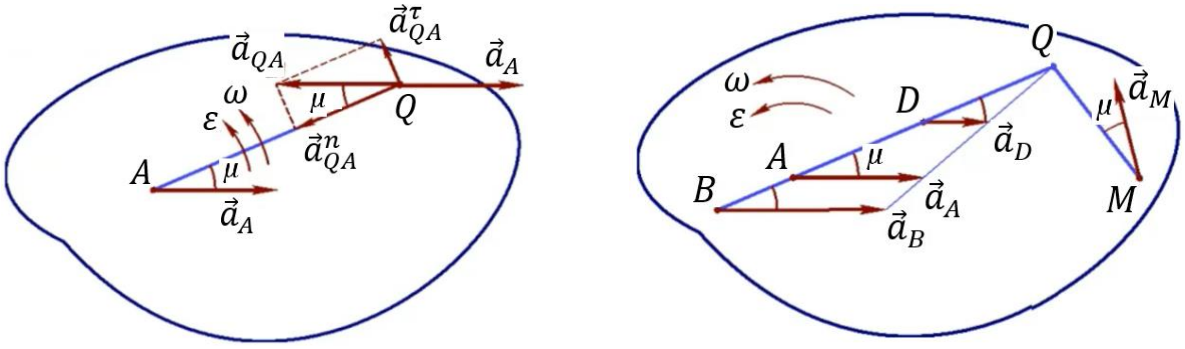
Fig. 8.4.1

The angle between the acceleration vector $\vec{a}_{MA}$ of the point M of a moving body and rotating around a pole and radius vector: $\vec{r}_1 = \overrightarrow{AM}$ is: $\mu = arctg\ \frac{a^{\tau}_{MA}}{a^{n}_{MA}} = arctg\ \frac{\varepsilon}{\omega^2}$.

Position of the instantaneous acceleration center. For a body performing plane motion at each moment of time in the coordinate plane there is a point at which the acceleration is equal to zero, this point is denoted the instantaneous center of acceleration.

Assuming the known data: $\vec{a}_A$, ω and ε it is required to find the position of the instantaneous center of acceleration (the point Q, whose acceleration is zero, Fig. 8.4.2 a and b): $\vec{a}_Q = \vec{a}_A + \vec{a}_{QA} = 0$, that is $\vec{a}_A = -\vec{a}_{QA}$, $\vec{a}_Q = \vec{a}^{\tau}_{QA} + \vec{a}^{n}_{QA}$

and $a_{QA} = \sqrt{(a^{\tau}_{QA})^2 + (a^{n}_{QA})^2}$.



(a) (b)

Fig. 8.4.2

Since: $\vec{a}_Q = 0$, then $a_A = a_{QA} = AQ \times \sqrt{\varepsilon^2 + \omega^4}$.

Therefore, length of the vector $\overrightarrow{AQ}$ from the pole of rotation A to the instantaneous center of acceleration Q equals: $AQ = \frac{a_A}{\sqrt{\varepsilon^2+\omega^4}}$, and direction of the vector $\overrightarrow{AQ}$ from the

pole of rotation A to the instantaneous center Q of acceleration is determined by angle μ according to the resulting formula: $\mu = arctg\ \frac{\varepsilon}{\omega^2}$.

To find the instantaneous center of acceleration Q one has to rotate vector $\vec{a}_D$ of pole A acceleration an acute angle μ in the direction of body rotation, if the rotation is accelerated or in opposite direction, if the rotation is slow. In the resulting direction one sets aside a segment AQ.

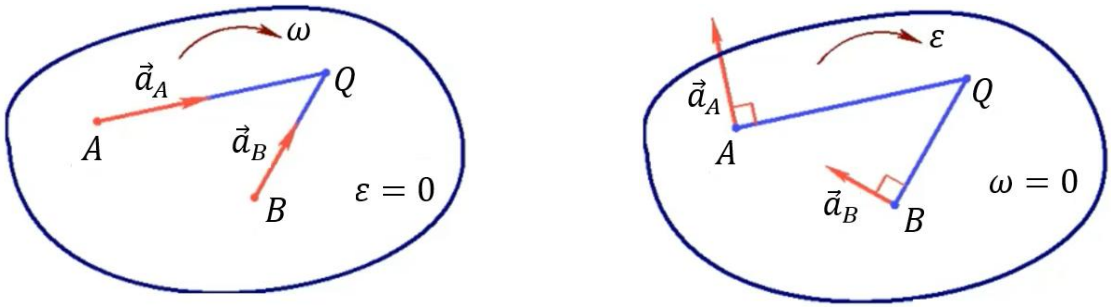
Determination of acceleration of the point M relative to the instantaneous center of acceleration Q. If the position of instantaneous acceleration center Q is established, then acceleration vector $\vec{a}_M$ of any point M of the body (Fig. 8.4.2 b) would be applied at this point and directed at an angle μ to the segment MQ, connecting this point with instantaneous acceleration center in such a way that, to coincide with the segment MQ the vector $\vec{a}_M$ is rotated in the direction of the body rotation ω, if the rotation of the body is accelerated: $\varepsilon > 0$ and the body is rotated in the opposite direction with respect to the rotation of the body ω, if the rotation is slow: $\varepsilon > 0$.

The magnitude of translational acceleration of the point M equals:
$a_M = MQ \times \sqrt{\varepsilon^2 + \omega^4}$.

The acceleration of the point M is proportional to the distance MQ from this point to the instantaneous center of accelerations Q (Fig. 8.4.2 b): $a_A/a_B/a_D = AQ/BQ/DQ$.

Position of the instantaneous center of acceleration Q in special cases:

1. If the angular acceleration of the body is zero: $\varepsilon = 0$, and the angular velocity of rotation is not zero: $\omega \neq 0$ (Fig. 8.4.3 a), then the angle between direction of acceleration and segment connecting the point (A or B) and the instantaneous center of acceleration Q is equal to zero: $\mu = 0$, that is $tg\ \mu = \frac{\varepsilon}{\omega^2} = 0$.



(a) (b)

Fig. 8.4.3

Vector of acceleration $\vec{a}_M$ of any point M of the body passes through the instantaneous center of acceleration of the body Q. That's why one can find the instantaneous center of acceleration as the point of intersection of the acceleration directions of any two points during body rotation.

2. If the angular velocity of rotation is zero: $\omega = 0$ (Fig. 8.4.3 b), and the angular acceleration is not zero: $\varepsilon \neq 0$, then the vector acceleration $\vec{a}_M$ of any point M of the body is perpendicular to the line connecting the instantaneous center accelerations with specified point: $tg\ \mu = \frac{\varepsilon}{\omega^2} = \infty$ and $\mu = \pi/2$ (that is 90°). The instantaneous center of acceleration of the body Q one can find as the intersection of perpendiculars to the acceleration vectors at two points.

Questions and comments:

8.4.1. Acceleration of point during plane motion of a body in the sum of the acceleration of the pole and the relative acceleration of point belonging to moving and rotating body under it rotation around the pole.

8.4.2. The magnitude of point acceleration in the rotational motion relative to the pole of rotation of the body to which the point belongs.

8.4.3. The instantaneous center of body acceleration performing plane motion is the point at which the acceleration is zero.

8.4.4. Relativity of acceleration (the opposite in direction and equal in magnitude) of the instantaneous center of acceleration and the point of body moving with acceleration.

8.4.5. Proportional to the angular acceleration of a body and the distance between each point of body and its instantaneous center of acceleration, the tangential relative acceleration of its points and the instantaneous center of acceleration relative to these points.

8.4.6. Proportional to the square of angular velocity and the distance between each point of the body and it instantaneous center of acceleration, the normal acceleration relative to its points and the instantaneous center of acceleration relative to these points.

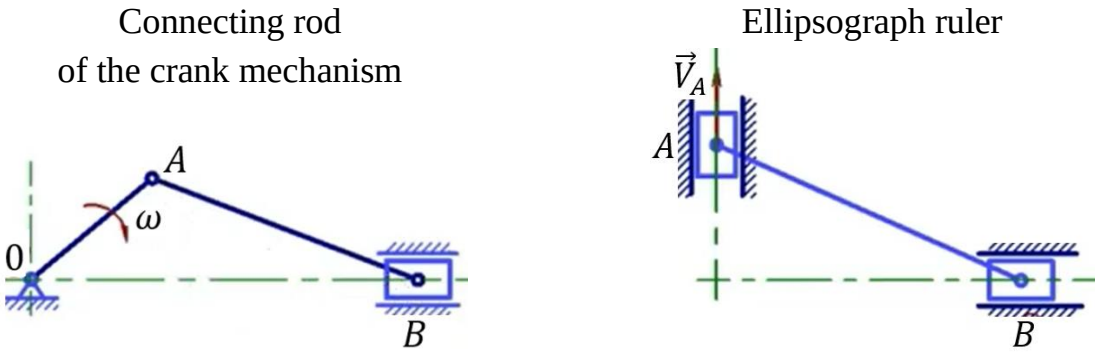
8.4.7. Finding the distance to the instantaneous center of velocities from given point of body motion with acceleration and the direction to the instantaneous center of accelerations in relation to the direction of acceleration of this point, i.e. the direction angle, depending on the ratio of tangential and normal relative accelerations of the point and the instantaneous center of acceleration.

8.4.8. The direction to the instantaneous center of acceleration lies on the line of acceleration of the point if the angular acceleration of moving body to which the point belongs is equal to zero. In this case, the instantaneous center of acceleration lies at the intersection of the acceleration directions of any two points of the body.

8.4.9. If the angular velocity of rotation of the body to which the points belong is equal to zero, and the angular acceleration is not equal to zero, then the direction to the instantaneous center of acceleration is perpendicular to the direction of acceleration of the point, in this case the instantaneous center of acceleration lies at the intersection of perpendiculars to the directions of acceleration of any two points.

8.5. Determining the instantaneous center of velocities and performing kinematic calculations for plane motion of a rigid body

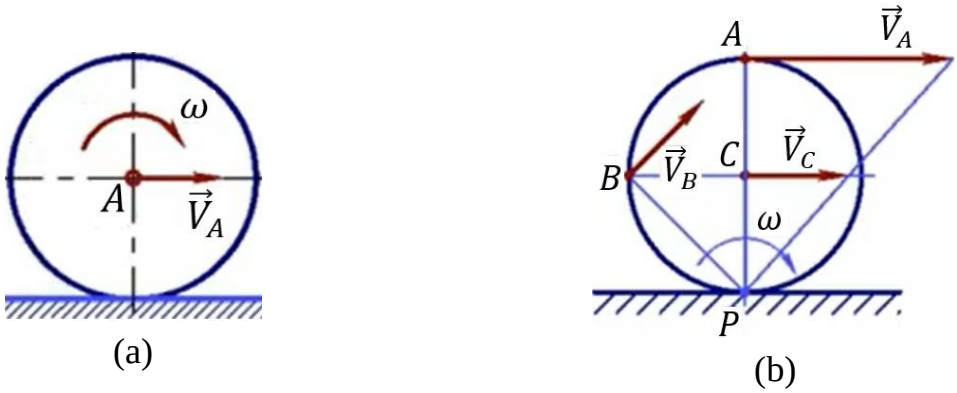
Plane motion (the plane–parallel) is the motion of a rigid body in which all it points move parallel to some fixed plane. Examples of plane motion of the mechanical system are shown on the diagrams (Fig. 8.5.1 a and b).



(a) (b)

Fig. 8.5.1

The rolling elements (the plane motion, ω and $\vec{V}_A$) of a wheel on a straight section of the road are shown on the diagram (Fig. 8.5.2 a). At each moment of time in the plane motion of body, if it angular velocity is not equal to zero: $\omega \neq 0$, there is a single point P in the plane of body motion, the speed of which is zero, this point is denoted the instantaneous center of velocities. The example: the location of the instantaneous velocity center for a moving wheel P is shown on the diagram (Fig. 8.5.2 b).



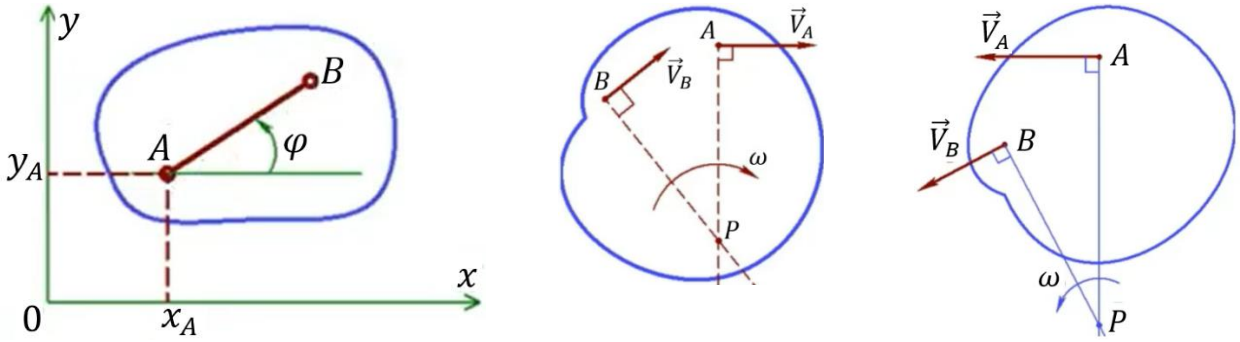
(a) (b)

Fig. 8.5.2

The elements of wheel motion, for which the following are known: the angular velocity ω and translational velocities of points: V_A and V_B are: $\omega = \frac{V_C}{CP}$, $V_A = \omega \times AP$, $\vec{V}_A \perp AP$ and $V_B = \omega \times BP$, $\vec{V}_B \perp BP$.

The plane motion of body can be defined in coordinate form by parametric equations of motion of any of its points A (Fig. 8.5.3 a): $x_A = x_A(t)$ and $y_A = y_A(t)$ and of the body rotation relative to this point A: $\varphi = \varphi(t)$.

The velocity of some other point B in plane motion of the body one can find, as: $\vec{V}_B = \vec{V}_A + \vec{V}_{BA}, V_{BA} = \omega \times AB$ and $\vec{V}_{BA} \perp AB, V_{Ax} = \dot{x}_A, V_{Ay} = \dot{y}_A$ and $V_A = \sqrt{V_{Ax}^2 + V_{Ay}^2}$.



(a) (b) (c)

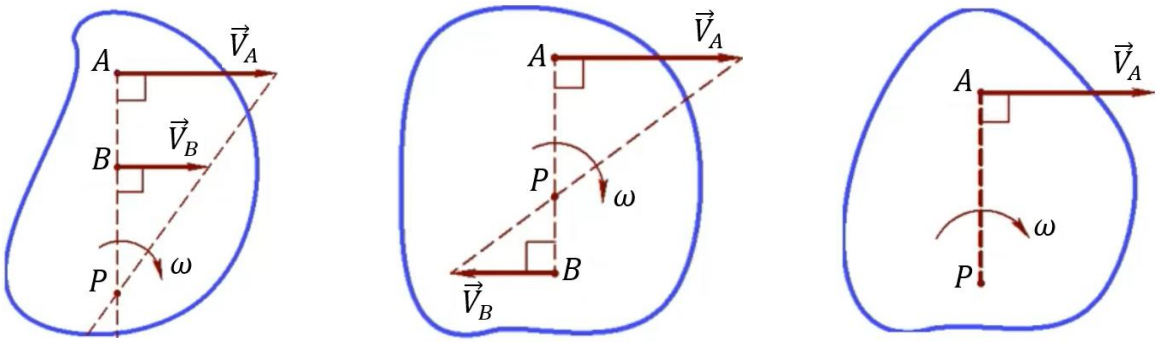
Fig. 8.5.3

General case of determining the position of the instantaneous velocity center. The directions of velocity vectors of two arbitrary points are specified: $\vec{V}_A$ and $\vec{V}_B$. The instantaneous center of velocities (the point P) is located at the intersection of perpendiculars to the velocity vectors at these points (A and B): $\vec{V}_A \perp AP$ and $\vec{V}_B \perp BP$. Wherein angular velocity equals: $\omega = \frac{V_A}{AP} = \frac{V_B}{BP}$, where AP and BP – the instantaneous radii (Fig. 8.5.3 b and c). The instantaneous velocity center P can be located both within the body contour and outside the body contour.

Methods for finding the instantaneous velocity center in special cases:

1. If both of the points of the rotating body (A and B) lie on a common perpendicular to the velocity vectors ($\vec{V}_A$ and $\vec{V}_B$, Fig. 8.5.4 a and b), then the position of the instantaneous velocity center P with respect to these points corresponds to the proportion: $\frac{V_A}{V_B} = \frac{AP}{BP}$. The intersection of a perpendicular drawn through these points with a straight line passing through the ends of the velocity vectors of these points determines the position of the instantaneous velocity center P.

2. The location of the instantaneous velocity center P one can find by the velocity V_A of one point A of the rigid body (Fig. 8.5.4 c) and by angular velocity ω of this body: $AP = V_A/\omega$.



(a) (b) (c)

Fig. 8.5.4

The instantaneous center of velocities does not exist (located at infinity) in the instantaneous translational motion of a body, when the velocities of two points (A and B), lying on a common perpendicular (Fig. 8.5.5 a) to the velocity vectors (or not lying on a common perpendicular, Fig. 8.5.5 b), are equal in magnitude and coincide in direction: $\vec{V}_A = \vec{V}_B$.

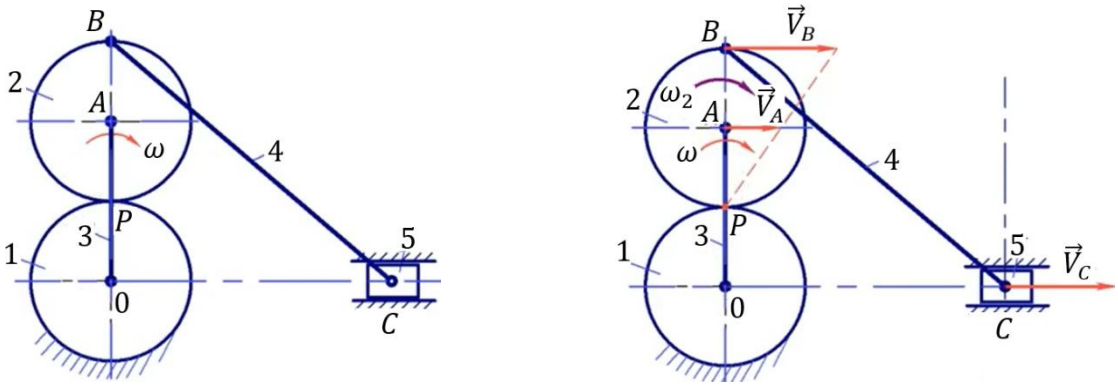
(a) (b)

Fig. 8.5.5

Exercise 1. The epicyclic mechanism (Fig. 8.5.6 a) consists of two wheels of the same radius R: fixed 1 and movable 2, driven by crank $0A$, which rotates with angular velocity ω. The connecting rod BC is pivotally connected to wheel 2, wherein the slider C moves in horizontal guides. The elements of the mechanism are numbered on the diagram: the fixed wheel – 1, the moving wheel – 2, the crank – $0A$ – 3, the connecting rod – BC – 4, the slider C – 5. In addition to the stationary wheel 1 in a given mechanism, the crank 1 rotates, the slider 5 moves translationally, and the wheel 2 and connecting rod 4 make a plane movement. It is required to determine velocity of the slider C in the position indicated on the diagram.

The velocity of the point A is determined as: $V_A = \omega \times 0A = 2 \times \omega \times R$.

The instantaneous velocity center of moving wheel 2 is located at the point P, that's why the angular velocity of wheel 2 is equal: $\omega_2 = \frac{V_A}{AP} = \frac{2 \times \omega \times R}{AP} = 2 \times \omega$.



(a) (b)

Fig. 8.5.6

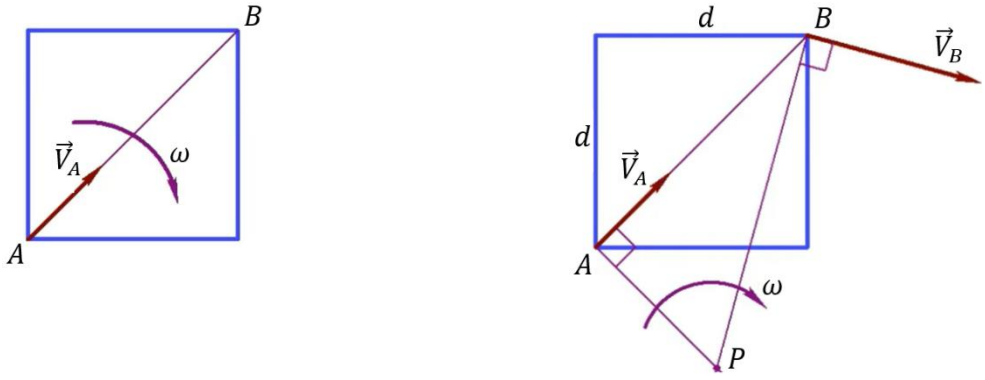
The velocity of the point B of moving wheel 2 equals:

$V_B = \omega_2 \times BP = 4 \times \omega \times R.$

The instantaneous center of velocities of link 4 lies at infinity, that is this link (connecting rod BC) performs instantaneous translational motion, that's why the velocity of the point C in the mechanism position indicated on the diagram is equal:

$V_C = V_B = 4 \times \omega \times R.$

Exercise 2. A square body with side d undergoes plane motion. In the position shown on the scheme (Fig. 8.5.7 a), the body has angular velocity ω and translational velocity $\vec{V}_A$ of the point A, diagonally directed AB (Fig. 8.5.7 b). The exercise assigns one to find velocity of the point B.



(a) (b)

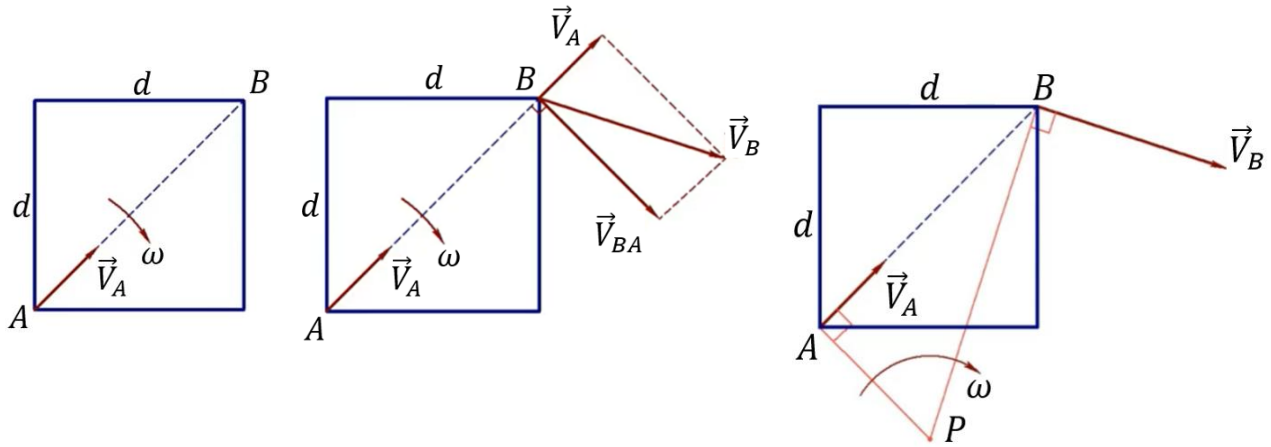
Fig. 8.5.7

The position of the instantaneous velocity center P one can find by the velocity $\vec{V}_A$ of the point A and by the angular velocity of rotation of the body ω, as: $AP = \frac{V_A}{\omega}$. Accordingly, the velocity $\vec{V}_B$ of the point B equals in value to: $V_B = \omega \times BP$. The instantaneous radius BP of curvilinear motion of the point B is determined as: $BP = \sqrt{AP^2 + AP^2}$.

Diagonal of a square equals: $AB = d \times \sqrt{2}$ and $BP = \sqrt{2 \times d^2 + V_A^2/\omega^2}$.

The velocity magnitude V_B of the point B equals: $V_B = \sqrt{2 \times d^2 \times \omega^2 + V_A^2}$.

Exercise 3. A square body (Fig. 8.5.8 a) with side: d =4 m makes a plane motion. At the calculated moment of time it angular velocity is equal: ω =1 s^{-1}, and the velocity vector $\vec{V}_A$ of the vertex A of the square is directed along it diagonal AB and is equal in magnitude: V_A =3 m/s. The exercise assigns one to determine the velocity $\vec{V}_B$ of the vertice of the square B.



(a) (b) (c)

Fig. 8.5.8

First solution. To solve the issue one can use the velocities addition theorem (Fig. 8.5.8 b): $\vec{V}_B = \vec{V}_A + \vec{V}_{BA}$, $AB = d \times \sqrt{2} = 4 \times \sqrt{2} = 5.66$ m,
$V_{BA} = AB \times \omega = 5.66 \times 1 = 5.66$ m/s and $V_B = \sqrt{V_A^2 + V_{BA}^2} = \sqrt{3^2 + 5.66^2} = 6.4$ m/s.

Second solution. The use of found distance BP to the instantaneous velocity center to find the velocity $\vec{V}_B$ (Fig. 8.5.8 c): $AP = V_A/\omega = 3/1 = 3$ m,
$BP = \sqrt{AB^2 + AP^2} = \sqrt{5.66^2 + 3^2} = 6.4$ m and $V_B = BP \times \omega = 6.4 \times 1 = 6.4$ m/s.

Third solution. Using the theorem on the projection of the velocities of two points onto a segment connecting these points. In the triangle ABP the angle α is sought between the velocity vector $\vec{V}_B$ and the segment vector AP: $\cos\alpha = \frac{AP}{BP} = \frac{3}{6.4} = 0.469$,
$V_A = V_B \times \cos\alpha$ and $V_B = V_A/\cos\alpha = 3/0.469 = 6.4$ m/s.

Questions and comments:

8.5.1. Plane motion of a rigid body, in which all points of the body move parallel to a certain fixed plane.

8.5.2. The instantaneous center of velocities in the plane motion of rigid body is a point whose velocity is zero in the plane of motion.

8.5.3. Elements of wheel rolling (the flat motion): the angular velocity of the wheel rotation, translational velocity of the wheel axis and it other points.

8.5.4. The coordinate assignment of plane motion of body by specification of the motion velocity of any point of body and the velocity of body rotation.

8.5.5. In the general case, finding the instantaneous center of velocities for plane motion of body at the intersection of lines drawn to two points of the body perpendicular to the velocities of these points.

8.5.6. If points of the body move in parallel at different velocities, then the instantaneous center of the velocities is located on a line passing through these points and which is perpendicular to the velocity vectors at these points, when it intersects with the line passing by the ends of their velocity vectors.

8.5.7. The instantaneous center of velocities is located on a line perpendicular to the velocity vector of point in plane motion of a body at the distance equal to the translational velocity of the point divided by the angular velocity of the body, if this distance is set aside from the direction of the velocity when turning at a right angle in the direction of the angular velocity.

8.5.8. In the instantaneous translational motion of body, when the velocities of two points of the body, lying on a common perpendicular to their velocity vectors, coincide, or when different perpendiculars (to the different points) are parallel to each other, the instantaneous center of velocities is located at infinity.

8.6. Finding elements of the trajectory of a material point using the Mathcad computer algebra system

Exercise 1. Find the trajectory of motion of the point, of its speed, acceleration and radius of curvature of the trajectory for a certain point in time: $t_1 = 2$, if the law of motion is specified parametrically in coordinate form in the Mathcad (Fig. 8.6.1).

$$x(t) := 6 \cdot \cos\left(\frac{\pi \cdot t}{6}\right) - 2 \qquad y(t) := 3 \cdot \cos\left(\frac{\pi \cdot t}{3}\right) + 4$$

$$t_1 := 2 \qquad x(2) = 1 \qquad y(2) = 2.5$$

Fig. 8.6.1

From the given coordinate forms, it can be seen that the range of definition along the x–axis is: from –8 to 4, and the range of values along the ordinate axis y: from 1 to 7.

To create a trajectory of motion of the point, it is advisable to perform mathematical transformations and to define the ordinate y by the abscissa x. For this conversion one can apply the double angle cosine formula: $\cos(2 \times \alpha) = 2 \times (\cos \alpha)^2 - 1$

and take into account that: $\frac{\pi \times t}{3} = 2 \times \frac{\pi \times t}{6}$.

By the first equation: $x = 6 \times \cos\frac{\pi \times t}{6} - 2$ one can find, that: $\cos\frac{\pi \times t}{6} = \frac{x+2}{6}$.

Thus, the trajectory equation in the form of the dependence of trajectory ordinate y on the abscissa x has the form: $y(x) = \frac{1}{6} \times (x+2)^2 + 1$.

Determining the shape of the trajectory of point is shown on the diagram of page fragment in the Mathcad (Fig. 8.6.2).

$$x = \begin{bmatrix} -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad y(x) := \frac{1}{6} \cdot (x+2)^2 + 1 \qquad y(x) = \begin{bmatrix} 1.667 \\ 1.167 \\ 1 \\ 1.167 \\ 1.667 \\ 2.5 \\ 3.667 \\ 5.167 \\ 7 \end{bmatrix}$$

Fig. 8.6.2

The resulting trajectory of the point motion has the shape shown on the graph (in the Mathcad, Fig. 8.6.3).

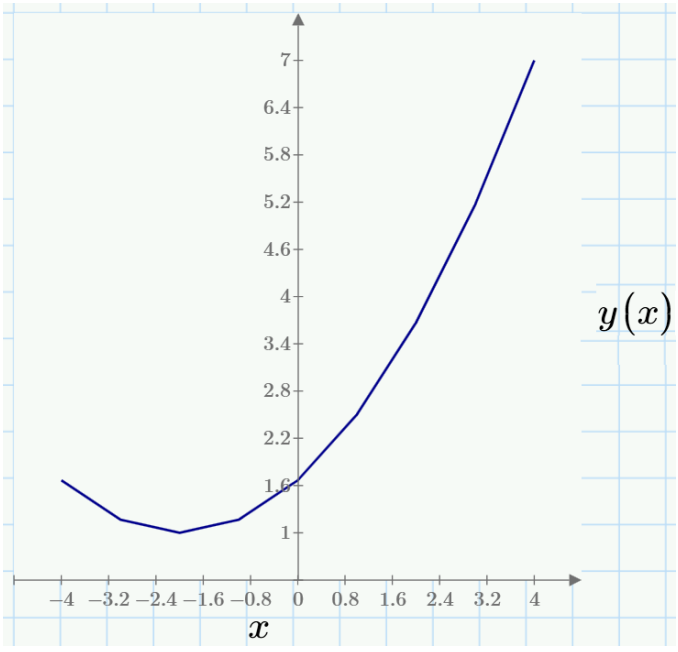


Fig. 8.6.3

Determining of the point velocity v_1 for the moment of time: $t_1 = 2$ one can perform numerically using the initially defined the pattern of motion in parametric coordinate form in the Mathcad (Fig.8.6.4).

$$v_x(t) := \frac{d}{dt}x(t) \qquad v_x(2) = -2.721 \qquad v_y(t) := \frac{d}{dt}y(t) \qquad v_y(2) = -2.721$$

$$v_{1x} := v_x(2) \qquad v_{1y} := v_y(2) \qquad v_1 := \sqrt{v_{1x}^{\,2} + v_{1y}^{\,2}} \qquad v_1 = 3.848$$

Fig. 8.6.4

Acceleration of the point a_1 at the moment of time: $t_1 = 2$ one can find numerically, also using the initially defined the parametrical law of motion in coordinate form in the Mathcad (see fragment of the page, Fig. 8.6.5).

$$a_x(t) := \frac{d}{dt}v_x(t) \qquad a_x(2) = -0.822 \qquad a_y(t) := \frac{d}{dt}v_y(t) \qquad a_y(2) = 1.645$$

$$a_{1x} := a_x(2) \qquad a_{1y} := a_y(2) \qquad a_1 := \sqrt{a_{1x}^{\,2} + a_{1y}^{\,2}} \qquad a_1 = 1.839$$

Fig. 8.6.5

Having found the velocity v_1 and acceleration of the point a_1 for the motion of time: $t_1 = 2$ one can determine the radius of curvature ρ_1 of it trajectory, which is equal: $\rho_1 = v_1^2/a_{1n}$, where a_{1n} – the normal acceleration directed radially to the center of curvature of the trajectory, which one can find as: $a_{1n} = \sqrt{a_1^2 - a_{1\tau}^2}$, and the tangential acceleration of the point $a_{1\tau}$ one can find as the projection of total acceleration a_1 onto the direction of velocity v_1: $a_{1\tau} = \frac{a_{1x} \times v_{1x} + a_{1y} \times v_{1y}}{v_1}$. The radius of curvature is determined using the Mathcad computer algebra system (see fragment of the page, Fig. 8.6.6).

$$a_{1\tau} := \frac{a_{1x} \cdot v_{1x} + a_{1y} \cdot v_{1y}}{v_1} \qquad a_{1\tau} = -0.582 \qquad a_{1n} := \sqrt{a_1^{\,2} - a_{1\tau}^{\,2}} \qquad a_{1n} = 1.745$$

$$\rho_1 := \frac{v_1^{\,2}}{a_{1n}} \qquad \rho_1 = 8.485$$

Fig. 8.6.6

The results of determining velocities and accelerations for the moment of time: $t_1 = 2$ one can indicate on the trajectory graph (Fig. 8.6.7).

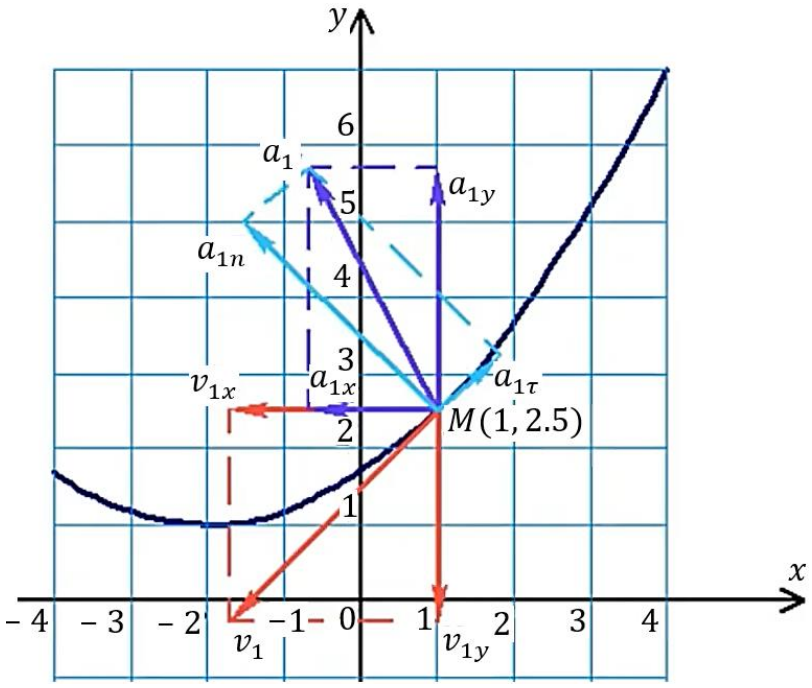


Fig. 8.6.7

Exercise 2. In coordinate form, the pattern of point motion on the plane $0xy$ is parametrically specified as: $x = 5 \times \sin(\pi \times t/3)$ and $y = 3 + 5 \times \cos(\pi \times t/3)$, m.

The exercise assigns one to determine the trajectory of the point M motion, the position of the point on the trajectory at the initial moment and direction of it motion, to find it velocity and acceleration, as well as the radius of curvature of the trajectory of the point M one second after the start of motion.

Finding the trajectory, the initial position and direction of the point M motion. To obtain the shape of the trajectory in explicit form, it is necessary to exclude the time parameter t from the equations of motion. The dependencies are used:

$(\sin\alpha)^2 + (\cos\alpha)^2 = 1$ and $\frac{x}{5} = \sin\left(\frac{\pi \times t}{3}\right)$, $\frac{y-3}{5} = \cos\left(\frac{\pi \times t}{3}\right)$, that is $\frac{x^2}{5^2} + \frac{(y-3)^2}{5^2} = 1$.

One can see that the trajectory of the point M is a circle (Fig. 8.6.8) with center at point with the coordinates (0, 3).

The intersection points with the coordinate axes of circular trajectory of the point M are: at $x = 0$, $(y-3)^2 = 5^2$, m, that is ordinate is: $y = 3 \pm 5$, m
and at $y = 0$, $x^2 + 3^2 = 5^2$, that is $x = \pm 4$, m.

Initial position of the point M_0 on the trajectory is determined by its coordinates: at $t = 0$, $x_0 = 5 \times \sin(\pi \times 0/3) = 0$ and $y_0 = 3 + 5 \times \cos(\pi \times 0/3) = 8$, m.

The position of the point M_1 on the trajectory one second after the motion start is: $t_1 = 1$ s, $x_1 = 5 \times \sin(\pi \times 1/3) = 4.33$ and $y_1 = 3 + 5 \times \cos(\pi \times 1/3) = 5.5$, m.

Determining of the point velocity. The projections of velocity of the point M onto the plane $0xy$ are: $V_x = \dot{x} = 5 \times \frac{\pi}{3} \times \cos\left(\frac{\pi \times t}{3}\right)$ and $V_y = \dot{y} = -5 \times \frac{\pi}{3} \times \sin\left(\frac{\pi \times t}{3}\right)$, m/s.

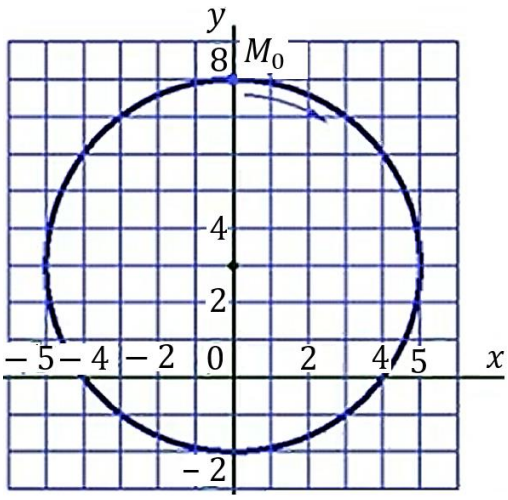


Fig. 8.6.8

One second after the motion start the velocity projections of the point M are equal: $t_1 = 1$, s, $V_{1x} = 5 \times \frac{\pi}{3} \times \cos\left(\frac{\pi \times 1}{3}\right) = 2.62$ and $V_{1y} = -5 \times \frac{\pi}{3} \times \sin\left(\frac{\pi \times 1}{3}\right) = -4.53$, m/s.

Vector of velocity $\vec{V}_1$ of the point M_1 one second after the motion start: $t_1 = 1$ s and the projections of velocity vector ($\vec{V}_{1x}$ and $\vec{V}_{1y}$) are indicated on the trajectory graph (Fig. 8.6.9).

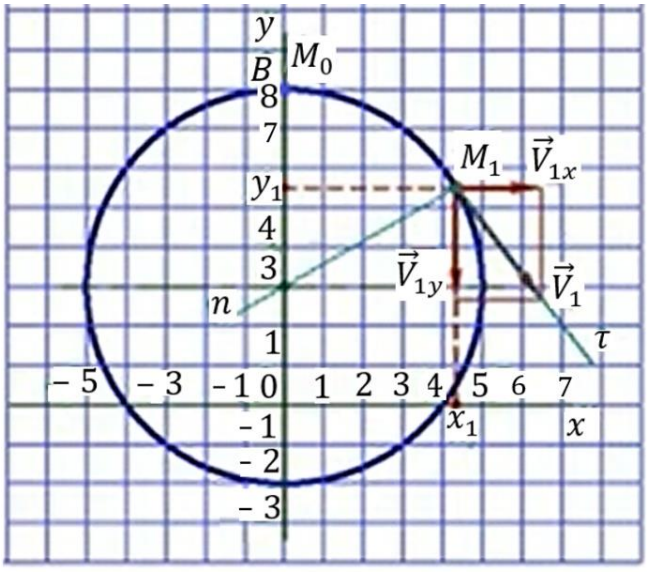


Fig. 8.6.9

The magnitude of the velocity vector of the point M_1 one second: $t_1 = 1$ s after start of the motion equals: $V_1 = \sqrt{V_{1x}^2 + V_{1y}^2} = \sqrt{2.62^2 + 4.53^2}$=5.24, m/s.

Projections of the point M acceleration on the plane $0xy$ coordinate axes are: $a_x = \dot{V}_x = -5 \times \left(\frac{\pi}{3}\right)^2 \times \sin\left(\frac{\pi \times t}{3}\right)$ and $a_y = \dot{V}_y = -5 \times \left(\frac{\pi}{3}\right)^2 \times \cos\left(\frac{\pi \times t}{3}\right)$, m/s^2.

One second after start of the motion: $t_1 = 1$ s the projections of acceleration of the point M on the coordinate axes are equal: $a_{1x} = -5 \times \left(\frac{\pi}{3}\right)^2 \times \sin\left(\frac{\pi \times 1}{3}\right) = -4.75$ and $a_{1y} = -5 \times \left(\frac{\pi}{3}\right)^2 \times \cos\left(\frac{\pi \times 1}{3}\right) = -2.74$, m/s^2.

The magnitude of the acceleration vector of the point M_1 one second after start of the motion: $t_1 = 1$ s equals: $a_1 = \sqrt{a_{1x}^2 + a_{1y}^2} = \sqrt{(-4.75)^2 + (-2.74)^2}$=5.48, m/s^2.

Acceleration vector $\vec{a}_1$ of the point M_1 one second after start of motion: $t_1 = 1$ s and projections of the acceleration vector ($\vec{a}_{1x}$ and $\vec{a}_{1y}$) are indicated on the graph (Fig. 8.6.10).

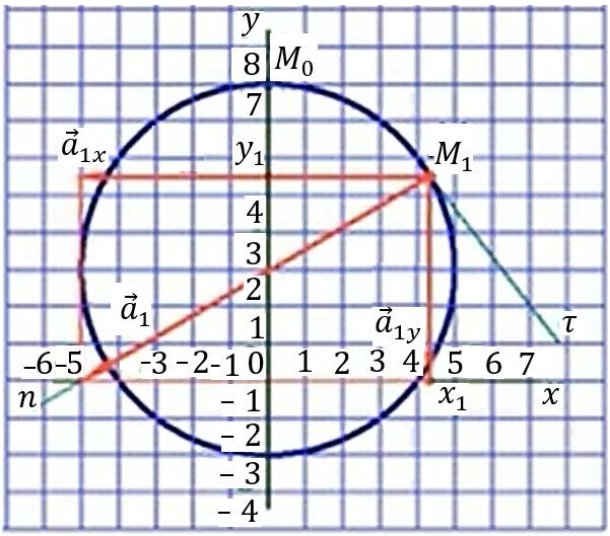


Fig. 8.6.10

The sequence of finding tangential and normal accelerations of the point M and the radius of curvature of trajectory is as follows. The tangential acceleration of the point M is associated with the change of it velocity and can be found using the scalar product of the point velocity $\vec{V}$ and it total acceleration $\vec{a}$, which allows one to determine the projection of acceleration onto the direction of velocity as: $a_\tau = \frac{dV}{dt} = \frac{a_x \times V_x + a_y \times V_y}{V}$.

One second after start of motion: $t_1 = 1$ s the magnitude of tangential acceleration of the point M_1 is: $a_{1\tau} = \frac{a_{1x} \times v_{1x} + a_{1y} \times v_{1y}}{v_1} = \frac{2.62 \times (-4.75) + (-4.53) \times (-2.74)}{5.24} = -0.006 \cong 0$.

The magnitude of normal acceleration of the point M: $a_n = \frac{V^2}{\rho}$, which is equal, on the other hand: $a_n = \sqrt{a^2 - a_\tau^2}$ coincides with value of the total acceleration a, when the tangential acceleration is zero ($a_\tau = 0$). Then one second after start of motion: $t_1 = 1$ s the magnitude of normal acceleration of the point M_1 equals: $a_{1n} = a_1$ =5.48, m/s^2.

Radius of curvature of the point M_1 trajectory one second after start of the motion: $t_1 = 1$ s is equal: $\rho_1 = \frac{V_1^2}{a_{1n}} = \frac{5.24^2}{5.48} = 5.01 \approx 5$, m.

According to the results of graphs determining, the velocity vector is indeed directed tangentially to the trajectory of motion of the point M. The tangential acceleration of the point M is zero: $a_\tau = 0$, since the velocity of the point does not depend on time and is constant in magnitude:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{\left(\frac{5\times\pi}{3} \times \cos\left(\frac{\pi\times t}{3}\right)\right)^2 + \left(-\frac{5\times\pi}{3} \times \cos\left(\frac{\pi\times t}{3}\right)\right)^2} = \frac{5\times\pi}{3} = 5.24, \text{ m/s},$$

that is: $a_\tau = \frac{dV}{dt} = 0$, wherein the radius of curvature of the trajectory ρ has a constant value, and the total acceleration $\vec{a}$ of the point M is directed along the normal n to the trajectory of it motion.

Questions and comments:

8.6.1. Specifying in the Mathcad computer algebra system the trajectory of a point on plane in parametric form. Finding the area for determining the coordinates of trajectory along the abscissa and ordinate axis. Mathematical transformation of a given law of motion into coordinate form with the exception of the time parameter.

8.6.2. Determining the coordinates of the trajectory of point in the Mathcad in the form of vector arrays (the symmetrical values along the abscissa axis within the definition area) and the graphical plotting the trajectory.

8.6.3. The determination by numerical taking derivatives in the Mathcad computer algebra system of projections of the velocity and acceleration of a moving point onto the coordinate axes and the values of velocity and acceleration from their projections.

8.6.4. Using velocity projections found in the Mathcad when specifying the point's motion parametrically to determine the sines and cosines of the direction angles of point's trajectory line. The use of scalar product of vectors specified by coordinates: the velocity of a point and its total acceleration.

8.6.5. Determination of the projection of acceleration onto the direction of the motion of point, that are the tangential acceleration, and projections onto the normal perpendicular to the direction of motion of the point, i.e., the projections of acceleration in the natural way of specifying motion, when the origin of coordinates is associated with the point, the tangent axis coincides with the direction of motion, the normal axis is directed along the radius of curvature, and the binormal axis is their vector product.

8.6.6. The analytical determination in the parametric form of specifying the motion of projections of point velocity and acceleration by taking derivatives of the trajectory, as well as finding the final values of velocity and acceleration.

8.6.7. The determining of acceleration projections on the tangent axis to the trajectory on the normal axis when the natural method of specifying the motion of a point is used with the velocity projections found when specifying the motion parametrically.

8.6.8. Determining of radius of curvature of the point trajectory by found the velocity and by the magnitude of the acceleration projection onto the normal axis.

8.7. Kinematic calculation of a flat mechanism

Exercise. The characteristics of the three–element mechanism shown on the diagram (Fig. 8.7.1) are given (see Table 8.7.1). The size of link 2 is twice the maximum size of link 1: $BC = 2 \times R_1$. It is assigned to characterize the type of motion of each element and to determine the angular velocities and angular accelerations of links: 1 and 2, as well as to find the linear velocities and accelerations of points: A, B and C, belonging to the corresponding element of the mechanism.

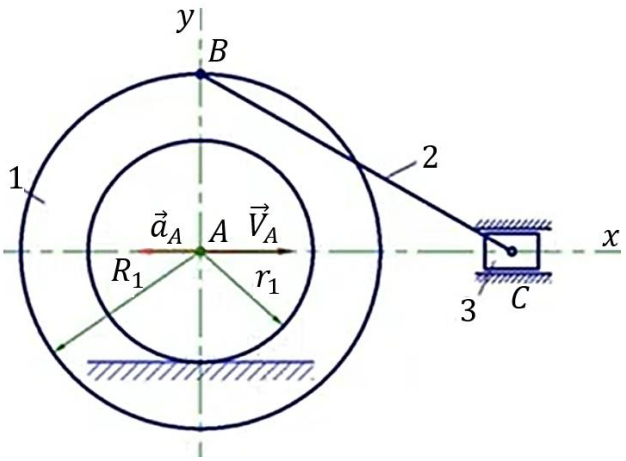


Fig. 8.7.1

Table 8.7.1

V_A, m/s	a_A, m/s^2	r_1, m	R_1, m	$BC = l_2$, m
1.4	1.8	0.5	0.8	1.6

Determining the velocities of the points: A, B and C and the angular velocities of the elements: 1 and 2. The coil 1 and the rod 2 make a plane motion, the slider 3 – the forward motion. The point P is the instantaneous center of element 1 velocity (Fig. 8.7.2).

The angular velocity of element 1equals: $\omega_1 = \frac{V_A}{AP} = \frac{V_A}{r_1} = \frac{1.4}{0.5} = 2.8$, s^{-1},

$V_B = BP \times \omega_1 = (R_1 + r_1) \times \omega_1 = (0.8 + 0.5) \times 2.8 = 3.64$, m/s.

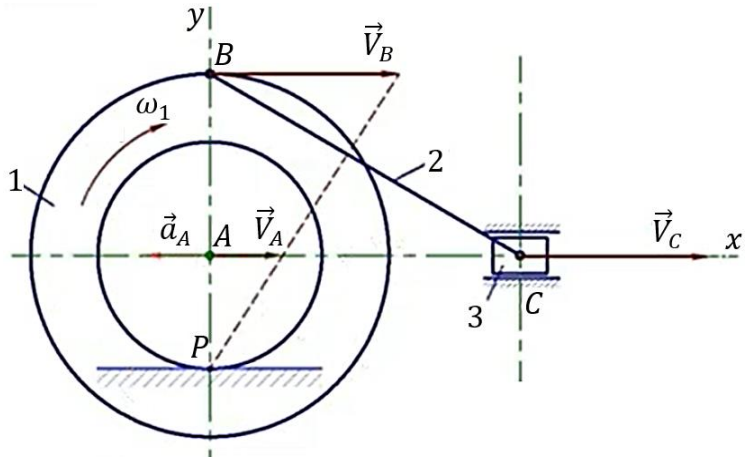


Fig. 8.7.2

Angular velocity of element 2 is zero, because the extreme points: B and C of this element move in parallel: $\omega_2 = 0$.

The translational velocities of points: B and C are equal: $V_B = V_C = 3.64$, m/s.

The translational velocity of the slider 3 equals: $V_C = 3.64$, m/s.

To find the accelerations of the points: A, B and C and angular accelerations of the links: 1 and 2 one uses the diagram (Fig. 8.7.3).

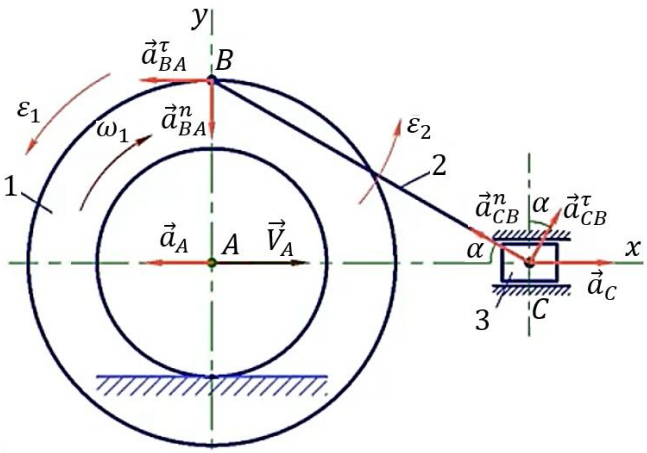


Fig. 8.7.3

The determination of angular acceleration of element 1 and translational accelerations of points: A and B is as follows.

The angular acceleration of element 1 equals: $\varepsilon_1 = \frac{1}{r_1} \times \frac{dV_A}{dt} = \frac{a_A}{r_1} = \frac{1.8}{0.5} = 3.6$, s^{-2}.

Tangential acceleration of the point B during rotation of element 1 relative to the point A equals: $a^{\tau}_{BA} = BA \times \varepsilon_1 = R_1 \times \varepsilon_1 = 0.8 \times 3.6 = 2.88$, m/s^2.

The normal acceleration of the point B under rotation of element 1 relative to the point A equals: $a^{n}_{BA} = BA \times \omega_1^2 = R_1 \times \omega_1^2 = 0.8 \times 2.8^2 = 6.272$, m/s^2.

To find the acceleration projections of the point B the vector of sum is used:

$\vec{a}_B = \vec{a}_A + \vec{a}^{\tau}_{BA} + \vec{a}^{n}_{BA}$, taking into account which the projections of acceleration of the point B onto the coordinate axes on $0xy$ plane are equal:

$a_{Bx} = -a_A - a^{\tau}_{BA} = -1.8 - 2.88 = -4.68$, m/s^2 and $a_{By} = -a^{n}_{BA} = -6.272$, m/s^2.

The magnitude of the resulting acceleration of the point B is:

$a_B = \sqrt{a^2_{Bx} + a^2_{By}} = \sqrt{4.68^2 + 6.272^2}$=7.83, m/s^2.

Translational acceleration of the point C of the link 2 (the slider) is determined considering the vector sum: $\vec{a}_C = \vec{a}_B + \vec{a}^{\tau}_{CB} + \vec{a}^{n}_{CB}$. The angular velocity of element 2 is zero, because the outermost points: B and C of this element move in parallel: $\omega_2 = 0$. That's why the normal acceleration of the point C during zero rotation of link 2 relative to point B is zero: $a^{n}_{CB} = 0$.

Because the size of element 2 is twice the maximum size of link 1: $BC = 2 \times R_1$, then the angle between element 2 and the direction of movement of the slider (of the point C) is at the angle: $\alpha = 30°$. The slider C moves along the axis x. That's why the projection of the acceleration of the point C onto the y axis is zero: $a_{By} - a^{\tau}_{CB} \times \cos\alpha = 0$

and $a^{\tau}_{CB} = -\frac{a_{By}}{\cos\alpha} = -\frac{-6.272}{\cos 30°} = 7.240$, m/s^2, and the total acceleration magnitude of the point C of the element 2 (the slider) equals:

$a_C = a_{Bx} + a^{\tau}_{CB} \times \sin\alpha = -4.68 + 7.24 \times \sin 30° = -1.06$, m/s^2.

The angular acceleration of element 2 equals: $\varepsilon_2 = \frac{a^{\tau}_{CB}}{BC} = \frac{a^{\tau}_{CB}}{l_2} = \frac{7.24}{1.6} = 4.53$, s^{-2}.

Questions and comments:

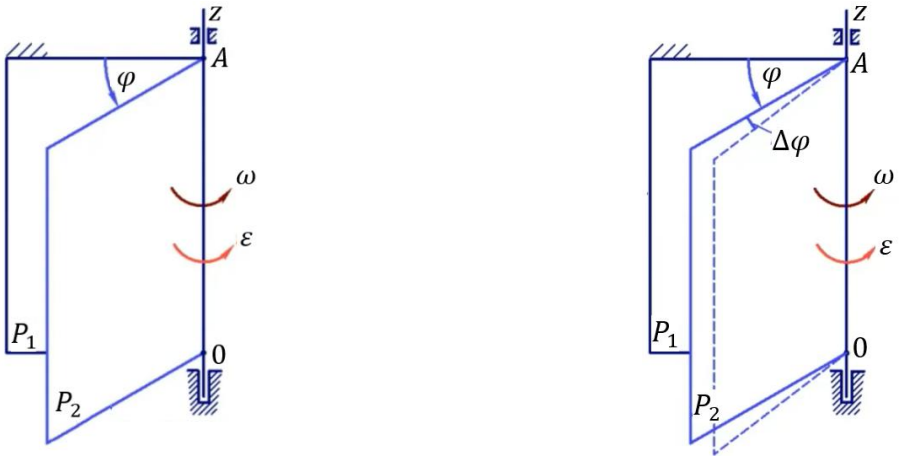
8.7.1. Plane movement of the mechanism and its element (the elements of mechanical system). The translational movement. The instantaneous velocity center. The angular velocities of elements. Translational velocities of points.

8.7.2. Tangential acceleration, nominal acceleration, angular acceleration. Acceleration of a point moving relative to another point (the pole), corresponding to the sum of the acceleration of the pole and the acceleration of point (the normal and tangential), when the point moves relative to the pole.

Chapter 9. Forces acting on elements during the movement of mechanical system (the mechanism), the laws of motion under influence of forces

9.1. Characteristics of body motion when rotating around a fixed axis

Rotation of a rigid body around fixed axis is a movement at which there is continuously a geometric locus of fixed points of the body on a straight line, which is the axis of rotation. A body rotating around a fixed axis (Fig. 9.1.1 a) has one degree of freedom. The turn angle of the body φ (the angular coordinate) is positive if the rotation is counterclockwise, looking towards from the positive direction of the $0z$ axis.



(a) (b)

Fig. 9.1.1

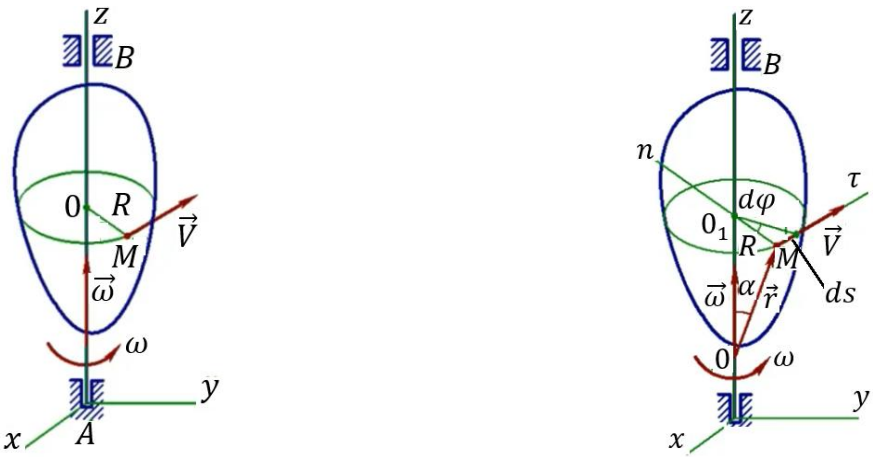
The equation of rotational motion of body around fixed axis has the form: $\varphi = f(t)$.

Average angular velocity equals: $\frac{\Delta\varphi}{\Delta t}$. That's why the angular velocity one can determine as the derivative with respect to time of the function of angle of rotation of the body: $\omega = \lim\limits_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt} = \dot{\varphi}$.

The units of angular velocity ω are: radian/s, that is 1/s, s^{-1}. Also, the angular frequency can be a characteristic of rotation speed n: the revolutions per second, which is equal to the angular velocity ω, multiplied by $(2 \times \pi)$, or the revolutions per minute: rpm.

Vector of angular velocity $\vec{\omega}$ is directed along the axis of rotation in the direction from which the rotation of body is visible counterclockwise (Fig. 9.1.2 a).

The linear velocity (the translational) of point M (Fig. 9.1.2 b) when moving along a circular path s, equals to: $V = \frac{ds}{dt}$, is associated with the turn and rotation of body: $V = R \times \frac{d\varphi}{dt} = R \times \omega$ because: $ds = R \times d\varphi$, then.



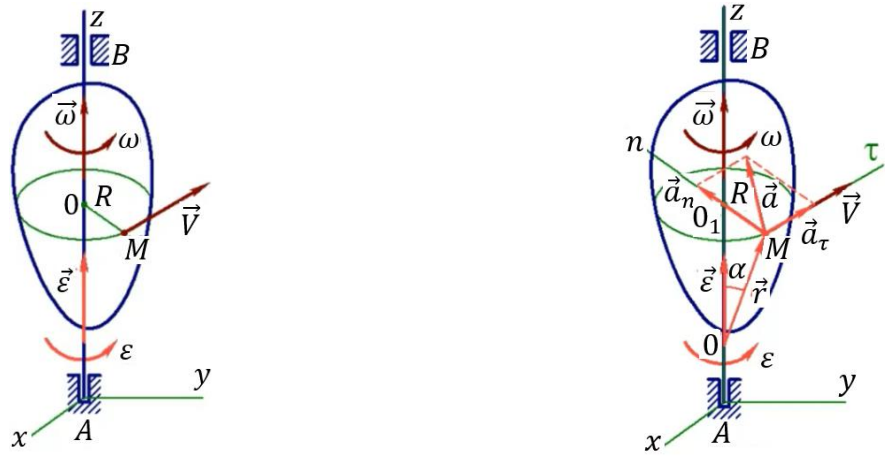
(a) (b)

Fig. 9.1.2

The vector linear velocity $\vec{V}$ of the point M of a rotating body at motion of the point along a circular path one can find by the vector product: $\vec{V} = \vec{\omega} \times \vec{r}$, and the magnitude of velocity of the point M equals: $V = \omega \times r \times \sin\alpha = \omega \times R$.

The angular acceleration of a rotating body (Fig. 9.1.3 a) equals:

$\varepsilon = \lim\limits_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \dot{\omega}$.



(a) (b)

Fig. 9.1.3

The angular acceleration ε of a rotating body is the second derivative $\ddot{\varphi}$ with respect to time from the angular displacement: $\varepsilon = \frac{d\omega}{dt} = \frac{d^2\omega}{dt^2} = \dot{\omega} = \ddot{\varphi}$. Dimension of angular acceleration ε is: radian/s^2, 1/s^2, s^{-2}.

The vector of angular acceleration $\vec{\varepsilon}$ during accelerated rotation of the body coincides in direction with the vector of angular velocity $\vec{\omega}$ (Fig. 9.1.3 a) and is directed in the opposite direction when rotation slows down.

Linear (the translational) acceleration $\vec{a}$ of point M of a rotating body equals:

$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(\vec{\omega}\times\vec{r})}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\varepsilon} \times \vec{r} + \vec{\omega} \times \vec{V}.$

Tangent $\vec{a}_\tau$ and normal $\vec{a}_n$ components of the linear acceleration $\vec{a}_C$ of point C (Fig. 9.1.4) and the total acceleration of this point are equal:

$\vec{a}_C = \vec{a}_\tau + \vec{a}_n$, $\vec{a}_\tau = \vec{\varepsilon} \times \vec{r}$ and $\vec{a}_n = \vec{\omega} \times \vec{V}$,

$a_\tau = \frac{dV}{dt} = \varepsilon \times R$, $a_n = \frac{V^2}{R} = \omega^2 \times R$ and $a = \sqrt{a_\tau^2 + a_n^2} = R \times \sqrt{\varepsilon^2 + \omega^4}$.

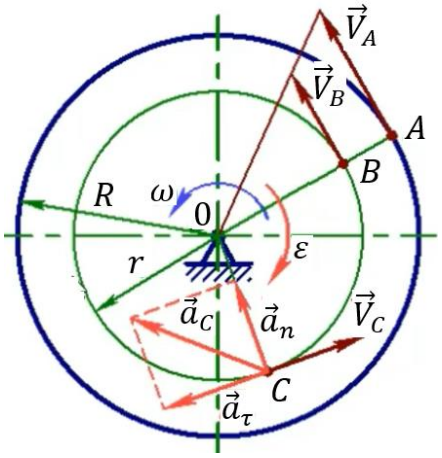


Fig. 9.1.4

Questions and comments:

9.1.1. The rotation movement of a body in which there is a locus of its fixed points on straight line (the axis). Body rotating around fixed axis having one degree of freedom.

9.1.2. The positive direction of body rotational motion, if, looking towards from the positive direction of the axis, the rotation occurs counterclockwise.

9.1.3. Time dependence of the angle of rotation of a body, which is the equation of its rotational motion.

9.1.4. The time derivative of a rotation angle, which is the angular velocity of rotation, measured in radians per second (International System of Units SI) as opposed to the rotational velocity, measured in revolutions per second or per minute (rpm).

9.1.5. The linear velocity vector of point during body rotation equal to the vector product of the vectors: the angular velocity and the radius–vector from coordinates origin to the point.

9.1.6. The magnitude of linear (the translational) velocity of point equals to the product of the radius from the axis to this point and the angular velocity of rotation.

9.1.7. The angular acceleration of a rotating body equals to the time derivative of the angular velocity or the second derivative of the rotation angle.

9.1.8. The direction of angular acceleration vector, coinciding with the direction of angular velocity during accelerated rotation and opposite to the velocity when rotation slows down.

9.1.9. Determining the angular acceleration vector by multiplying magnitude of the angular acceleration during rotation of a body by the vector of its angular velocity and dividing the resulting vector by magnitude of the angular velocity.

9.1.10. The vector of linear (the translational) acceleration of a point during rotation of body is equal to the derivative of the translational velocity vector with respect to time or the sum of the products: the time derivative of the angular velocity vector by the radius–vector and the angular velocity vector with time derivative of the radius–vector, taking into account the correspondence of the translational velocity to the vector the product of the angular velocity by the radius–vector of the point in accordance with the rules for finding the derivative of the product.

9.1.11. Tangential acceleration vector of point during rotation of a rigid body equals to vector product of the angular acceleration vectors of the rotating body (i.e., time derivative of the vector of its angular velocity) and the radius–vector of the body point relative to the origin, while the magnitude of the tangential acceleration vector is equal to the product of angular acceleration by radius of the circular trajectory of point relative to the axis of rotation.

9.1.12. Normal acceleration vector of point during rotation of a rigid body is equal to vector product of the angular velocity vectors of the rotating body (i.e., the derivative of the angle of rotation with respect to time) and the translational velocity vector of the body point, while the magnitude of the normal acceleration vector equals to product of the square of angular velocity and the radius of the circular trajectory points relative to the axis of rotation.

9.2. Kinematic elements of mechanism motion

The translational motion of a rigid body (without rotation) and the motion of a material point one can specify in three–dimensional coordinates depending on the time parameter t as: $x = f_1(t)$, $y = f_2(t)$ and $z = f_3(t)$.

The law of rotation of a rigid body around a fixed axis, wherein, has the form: $\varphi = f_4(t)$, the angular velocity and angular acceleration depend on the angular displacement of the body are determined as: $\omega = \frac{d\varphi}{dt} = \dot{\varphi}$ and $\varepsilon = \frac{d\omega}{dt} = \frac{d^2\omega}{dt^2} = \dot{\omega} = \ddot{\varphi}$, and the magnitude of rotational velocity of point (the linear translational velocity of the point during the rotation of the body) is: $V = \omega \times R$.

Wherein vector of acceleration $\vec{a}$, it magnitude a and the projections: onto the direction of motion of point a_τ and onto the normal to the trajectory of it curvilinear motion a_n are equal: $\vec{a} = \vec{a}_\tau + \vec{a}_n$, $\vec{a}_\tau = \vec{\varepsilon} \times \vec{r}$, $a_\tau = \varepsilon \times R$,
$\vec{a}_n = \vec{\omega} \times \vec{V}$, $a_n = \frac{V^2}{R} = \omega^2 \times R$ and $a = \sqrt{a_\tau^2 + a_n^2}$.

The considered issues are below that are solved using the specified dependencies.

Exercise 1. Two wheels are in external engagement (Fig. 9.2.1). Radii of the wheels are: R_1 =20 cm and R_2 =60 cm, as well as the angular velocity is assigned: ω_1 =3 s^{-1} and the angular acceleration of the first wheel is: ε_1 =12 s^{-2}. The exercise assigns one to find the angular velocity ω_2 and angular acceleration ε_2 of the second wheel, as well as linear (the translational) acceleration a_A of the point A on the second wheel.

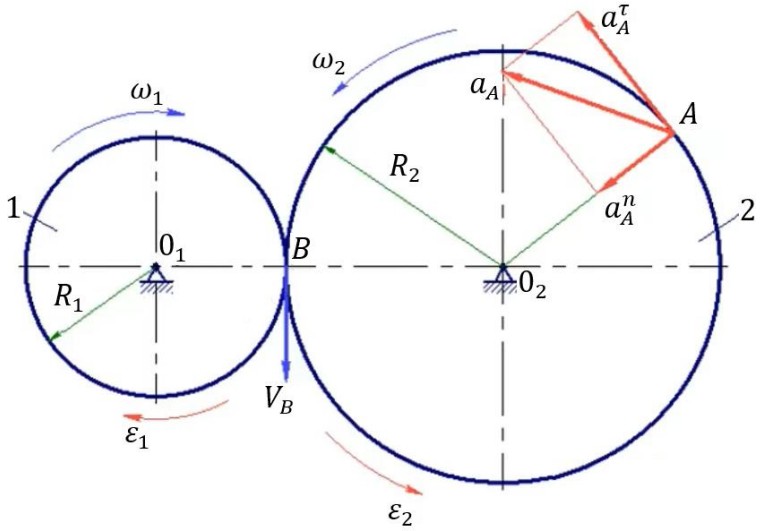


Fig. 9.2.1

The velocity of the point B of engagement of wheels: 1 and 2 is determined considering the angular velocity ω_1 of wheel 1 and its radius R_1, and the velocity equals:
$V_B = R_1 \times \omega_1 = 0.2 \times 3 = 0.6$, m/s.

The angular velocity of wheel 2 is determined depending on speed of the point B of engagement of wheels: 1 and 2 and on the radius R_2 of wheel 2 and equals:
$\omega_2 = \frac{V_B}{R_2} = \frac{0.6}{0.6} = 1.0$, s^{-1}.

The angular acceleration ε_2 of wheel 2 is to be find taking into account the angular acceleration ε_1 of wheel 1 and the ratio of the radii of wheels: 1 and 2 (R_1 and R_2) as:
$\varepsilon_2 = \frac{R_1 \times \varepsilon_1}{R_2} = \frac{0.2 \times 12}{0.6} = 4$, s^{-2}.

Vector $\vec{a}_A$ of the point A acceleration is the sum of its projections onto the tangent line $\vec{a}_A^\tau$ to the motion trajectory of the point A and to the normal to the trajectory $\vec{a}_A^n$ and is equal: $\vec{a}_A = \vec{a}_A^\tau + \vec{a}_A^n$. The projections ($a_A^\tau$ and a_A^n) of linear (the translational) acceleration a_A of the point A, located on wheel 2, are equal: $a_A^\tau = R_2 \times \varepsilon_2 = 0.6 \times 4 = 2.4$, m/s^2

and $a_A^n = R_2 \times \omega_2^2 = 0.6 \times 1^2 = 0.6$, m/s^2. The magnitude of total acceleration of the point A equals: $a_A = \sqrt{(a_A^\tau)^2 + (a_A^n)^2} = \sqrt{2.4^2 + 0.6^2} = 2.47$, m/s^2.

Exercise 2. Two wheels engage internally (Fig. 9.2.2).

The radii of the wheels are given: R_1 =20 cm and R_2 =100 cm, as well as angular velocity: ω_1 =2 s^{-1} and angular acceleration: $\varepsilon_1 = -4$ s^{-2} of the wheel 1. The exercise assigns one to find the motion velocity V_A, as well as the translational acceleration a_A of the point A on the second wheel.

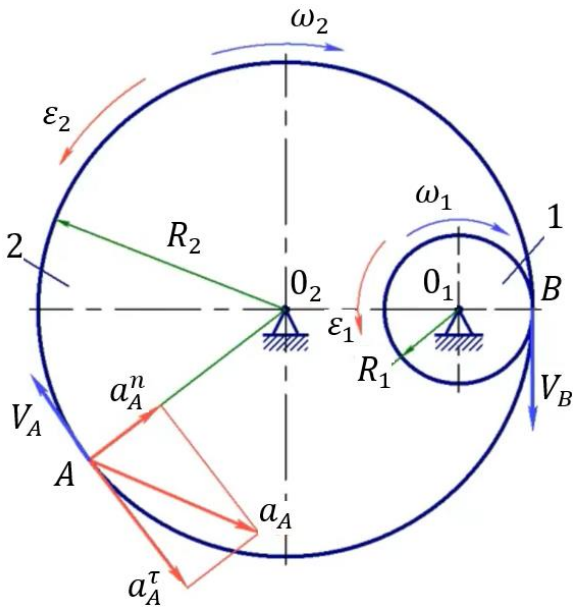


Fig. 9.2.2

Velocity of the point B of engagement of wheels: 1 and 2 is determined taking into account the angular velocity ω_1 of wheel 1 and its radius R_1 and equals:
$V_B = R_1 \times \omega_1 = 0.2 \times 2 = 0.4$, m/s.

The magnitude of velocity of the point A of wheel 2 is equal by magnitude of the velocity of the point B (both of points are located on the outer surface of wheel 2): $V_B = V_A = 0.4$, m/s, that's why the angular velocity of wheel 2, taking into account the ratio of wheels' radii: 1 and 2 (R_1 and R_2), equals: $\omega_2 = \frac{R_1 \times \omega_1}{R_2} = \frac{V_B}{R_2} = \frac{0.4}{1} = 0.4$, s^{-1}, and the angular acceleration of the wheel 2 is equal: $\varepsilon_2 = \frac{R_1 \times \varepsilon_1}{R_2} = \frac{0.2 \times (-4)}{1} = -0.8$, s^{-2}.

Vector $\vec{a}_A$ of the translational acceleration of the point A is the sum of its projections: $\vec{a}_A^\tau$ and $\vec{a}_A^n$ is: $\vec{a}_A = \vec{a}_A^\tau + \vec{a}_A^n$. Projections ($a_A^\tau$ and a_A^n) of acceleration a_A of the point A, located on wheel 2, are equal: $a_A^\tau = R_2 \times \varepsilon_2 = 1 \times (-0.8) = -0.8$, m/s^2 and $a_A^n = R_2 \times \omega_2^2 = 1 \times 0.4^2 = 0.16$, m/s^2. The magnitude of the total acceleration of the point A equals: $a_A = \sqrt{(a_A^\tau)^2 + (a_A^n)^2} = \sqrt{(-0.8)^2 + 0.16^2} = 0.82$, m/s^2.

Exercise 3. The law of load 1 motion (Fig. 9.2.3) is specified:

$s = 3 \times t^3 + 15$, cm and characteristics of the belt drive elements are as follows, the wheels' radii are specified: r_2 =30 cm, R_2 =50 cm and R_3 =60 cm. The exercise assigns one to find the linear acceleration values of the points: A and B (the a_A and a_B) at t_1 =1 s, as well as the velocity V_B of the point B.

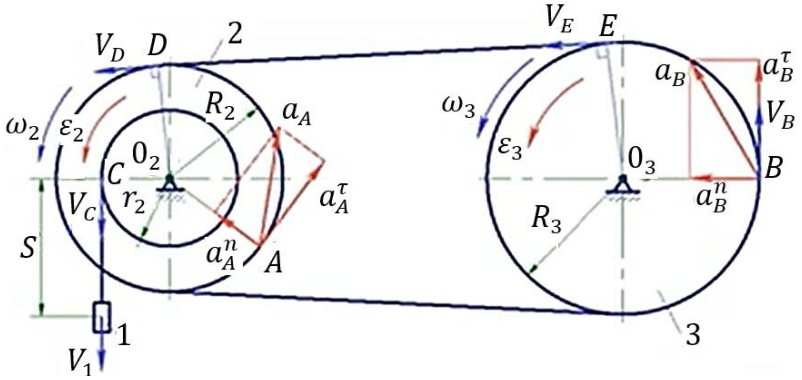


Fig. 9.2.3

To find the velocity V_1 of load 1, which is equal to the velocity V_C of the point C, one needs to find time t derivative of the displacement function s of load 1 as: $V_C = V_1 = \frac{ds}{dt} = 9 \times t^2$, cm/s.

That's why the angular velocity of wheel 2 (the first belt pulley) is equal: $\omega_2 = \frac{V_C}{r_2} = \frac{9 \times t^2}{30} = 0.3 \times t^2$, s^{-1}, and the angular acceleration of wheel 2 is:
$\varepsilon_2 = \frac{d\omega_2}{dt} = 0.6 \times t$, s^{-2}.

For the assigned moment of time: t_1 =1 s the values of angular velocity ω_2, angular acceleration ε_2 and the magnitude of translational acceleration a_A, determined by the sum of projections – the tangent a_A^τ and the normal accelerations a_A^n of the point A, are equal:
$\omega_2 = 0.3$, s^{-1}, $\varepsilon_2 = 0.6$, s^{-2}, $a_A^\tau = R_2 \times \varepsilon_2 = 50 \times 0.6 = 3.0$, cm/s^2,
$a_A^n = R_2 \times \omega_2^2 = 50 \times 0.3^2 = 4.5$, cm/s^2
and $a_A = \sqrt{(a_A^\tau)^2 + (a_A^n)^2} = \sqrt{3^2 + 4.5^2} = 5.41$, cm/s^2.

The belt driven velocities of points: B, E and D are equal:
$V_D = V_E = V_B = R_2 \times \omega_2 = R_3 \times \omega_3$.

Wherein angular velocity ω_3 of wheel 3 and translational velocity V_B of the point B are: $\omega_3 = \frac{R_2 \times \omega_2}{R_3} = \frac{50 \times 0.3}{60} = 0.25$, s^{-1} and $V_B = R_3 \times \omega_3 = 60 \times 0.25$=15, cm/s, and the angular acceleration of wheel 3, taking into account the ratio of the radii (R_2 and R_3) of the wheels: 2 and 3, connected by the belt drive, is: $\varepsilon_3 = \frac{R_2 \times \varepsilon_2}{R_3} = \frac{50 \times 0.6}{60} = 0.5$, s^{-2}.

For a given moment of time: t_1 =1 s the magnitude of translational acceleration a_B, determined by the sum of projections – the tangent a_B^τ and normal acceleration a_B^n of the point B, are equal: $a_B^\tau = R_3 \times \varepsilon_3 = 60 \times 0.5 = 30$, cm/s^2,

$a_B^n = R_3 \times \omega_3^2 = 60 \times 0.25^2 = 3.75$, cm/s^2
and $a_B = \sqrt{(a_B^\tau)^2 + (a_B^n)^2} = \sqrt{30^2 + 3.75^2} = 30.23$, cm/s^2.

Questions and comments:

9.2.1. Specifying the translational motion (without rotation) of a rigid body and the motion of a material point in three–dimensional coordinates.

9.2.2. The law of rotation of a rigid body and the differential dependences of angular velocity and angular acceleration on the angle of rotation of a rigid body.

9.2.3. Finding the translational acceleration of a material point in projections onto moving axes, associated with the point (by natural specification of motion of the point): onto the axis tangent to the trajectory of motion of the point, on the normal axis in the direction of the trajectory curvature vector.

9.2.4. The dependences of tangent and normal translational accelerations of point on its velocity, on the angular velocity of rotation of the body to which the point belongs and on the radius of curvature of trajectory of the point.

9.2.5. The ratio of angular velocity is equal to the inverse ratio of the radii of the wheels in mesh, or in a belt drive (or in a chain drive).

9.3. The action of sliding friction on solids and mechanical elements

At the tendency to move a stationary body and at its motion along a rough surface in the contact site, the force appears of sliding friction $\vec{F}_{fr}$ (the mesh force), preventing the motion. The total reaction of rough (dry) surface $\vec{R}$ includes both the normal component $\vec{N}$ (i.e. the pressure force) and the sliding friction force $\vec{F}_{fr}$, directed against the motion (Fig. 9.3.1 a).

(a) (b)

Fig. 9.3.1

The magnitude of the surface reaction consists of the sliding friction force: $F_{fr} = \sum_{k=1}^{n} F_{k\tau}$ and the normal component: $N = \sum_{k=1}^{n} F_{kn}$, equal total magnitude of the projections of all forces onto the normal ort to the supporting surface axis.

Magnitude of the surface reaction equals: $R = \sqrt{N^2 + F_{fr}^2}$.

The laws of dry friction were described by Charles Coulomb (1781). There is a linear relationship between limiting (the maximum) value of the friction force F_{fr}^{max}, corresponding to the beginning of movement, and the normal force N.

The body is in motionless balance as long as the condition is met (the Amonton–Coulomb law): $F_{fr} \leq F_{fr}^{max}$, where: $F_{fr}^{max} = f \times N$ – the greatest value of the force of sliding friction, f – the sliding friction coefficient, which does not depend on the area of contacting surfaces, but depends on the material and condition of the rubbing hard surfaces.

The sliding friction coefficient f corresponds to the friction angle φ, the tangent of which is equal to the friction coefficient. That is the friction angle φ occurs between the greatest sliding friction force F_{fr}^{max} of the rough (dry) surface and the direction of the normal reaction $\bar{N}$: $tg\ \varphi = \frac{F_{fr}^{max}}{N} = f$.

The friction cone (Fig. 9.3.2) is the surface of a conical figure described around a generatrix of a circle whose radius is equal to the total reaction F_{fr}^{max} of the rough surface, and the height corresponds to a normal reaction $\vec{N}$. If the friction coefficient f is the same in all directions, then the friction cone is circular.

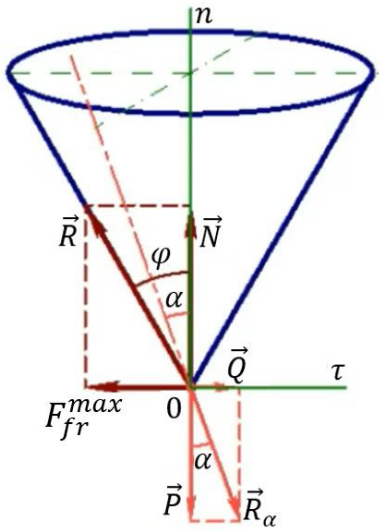


Fig. 9.3.2

For a motionless balance of body on a rough surface it is necessary and sufficient that the line of action of external forces $\vec{R}_\alpha$ passes inside the friction cone, or, in the limiting case, passes along its generatrix through the top of the cone: $\alpha \leq \varphi$ and $Q \leq F_{fr}$.

Questions and comments:

9.3.1. The sliding friction force that occurs on a dry, rough contact site between bodies and prevents their relative motion. The complete reaction of surface, including the component normal to the surface – the pressure force and a component tangential to the surface – the sliding friction force.

9.3.2. Linear proportional relationship between the maximum value of sliding friction force corresponding to the beginning of motion and the pressure force normal to the surface.

9.3.3. The friction angle, the tangent of which is equal to the ratio of the sliding friction force to the pressure force, i.e. sliding friction coefficient, which does not depend on area of the contacting surfaces but depends on the surfaces materials and conditions of rubbing of the surfaces.

9.3.4. A circular cone of friction with equal friction coefficients in all sliding directions, formed by the pressure force, as the height of the cone, and the greatest sliding friction force, as the radius of the cone generatrix.

9.3.5. The condition of stationary equilibrium of body on a rough surface, under the influence of friction and external force, which normal component is the pressure force, and the shear force is the component of the external force lying in the plane parallel to the supporting surface, which does not lead to movement of the body if the line of external force action passes inside the friction cone, including the surface of the friction cone, i.e. if the line of action is not outside the cone.

9.4. Determination of rolling friction force on a plane elastic surface

The force of rolling friction (the rolling resistance) occurs for a roller on the supporting surface due to crumple of the surface area of the roller and the site of the supporting surface on which the roller is located.

For a roller in a stationary position, to which an active horizontal force $\vec{Q}$ is applied (Fig. 9.4.1), causing a moment that turns the roller: $\sum M_C = -Q \times R$, when reaching a certain limit value of this force $\vec{Q}$ the roller begins to move.

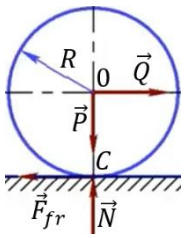
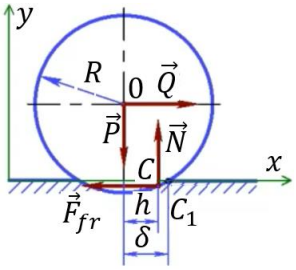
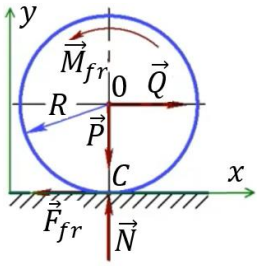


Fig. 9.4.1

The state of the motionless balance of the roller cannot be explained on the basis of a model of an absolutely rigid bodies (that is, the bodies of roller and\or the supporting surface). Contact of the roller with the supporting surface is along the crumple area both from the side of the roller and from the side of the supporting surface. The conditions for a stationary balance of roller on the surface (Fig. 9.4.2 a) in projections on the $0xy$ axes contain the equations: $\sum F_{kx} = 0,\ Q - F_{fr} = 0,\ \sum F_{ky} = 0,\ N - P = 0$ and $\sum M_C = 0,$ $P \times h - Q \times R = 0$, that are: $F_{fr} = Q, N = P, h = \frac{Q \times R}{P}$ and $h_{max} = \delta$, where δ – the coefficient of rolling friction (the rolling resistance), $\sum M_C$ – the sum of moments of forces relative to the point C of roller rotation.



(a)



(b)

Fig. 9.4.2

The condition of stationary balance of the roll has the form: $h \leq \delta$. The moment of rolling friction (the rolling resistance) is equal: $M_{fr} = \delta \times N$. In terms of the moment M_{fr} of rolling friction and acting forces – the force Q, bringing the roller out of its state of motionless equilibrium, and the force N, pressing the roller to the supporting surface, condition of stationary equilibrium of the roller has the form: $M_{fr} \leq \delta \times N$ or $Q \leq \frac{\delta}{R} \times N$.

Exercise 1. To the uniform roller (Fig. 9.4.3 a) weighing: $P = 2$ kN the horizontal force is applied $\bar{F}$. They are assigned: the rolling friction coefficient: δ =0.006 m, the sliding friction coefficient: f =0.2, the roller radius: R=0.6 m, the size $0A$=0.4 m (the elevation of the point of the application of the force $\bar{F}$ above the roller axis). The exercise assigns one to determine the largest value of the force $\bar{F}$, at which the roller does not slide and does not roll under the influence under this force.

The conditions of balance of roller on the surface in projections on the $0xy$ axes have the form: $\sum F_{kx} = 0$, $F - F_{fr} = 0$ and $\sum F_{ky} = 0$, $N - P = 0$,
as well as the total of moments relative to turning point D of the roller are:
$\sum M_D = 0$, $P \times h - F \times AD = 0$, that is $F_{fr} = Q$, $N = P$.

Sliding of the roller can occur when the action on the roller of the force is greater than: $F_1 = F_{fr} = f \times P = 0.2 \times 2 = 0.4$, kN.

Fig. 9.4.3

Rolling can occur under the condition that the roller is acted upon by a force that is greater than: $F_2 = \frac{\delta}{AD} \times P = \frac{0.006}{1} \times 2 = 0.012$, kN.

Conclusion, the sliding and rolling of the roller is absent when: $F_{max} \leq 0.012$, kN.

Exercise 2. To a uniform roller (Fig. 9.4.3 b) weighing: P =700 N the force $\bar{F}$ is applied at an angle of 30°. The exercise assigns one to determine the smallest value of this force at which the roller begins to roll with sliding, if the radius of the roller is: R=1 m, the sliding and rolling friction coefficients are equal, accordingly: f =0.2 and δ =0.008 m.

Conditions for motionless equilibrium of the roller on the surface in projections onto $0xy$ axes look like: $\sum F_{kx} = 0$, $F \times \cos 30° - F_{fr} = 0$ and $\sum F_{ky} = 0$,
$N - P - F \times \sin 30° = 0$, as well as the total of moments relative to the point of rotation C of roller: $\sum M_C = 0$, $-N \times \delta + F \times \cos 30° \times OC = 0$, at $OC = R$.

Sliding of the roller will occur when the roller is subjected to a force greater than: $F_{fr} = f \times N = f \times (P + F \times \sin 30°)$.

Moreover, the force F, applied at an angle of 30° not only increases the pressure of the roller on the supporting surface by $(F \times \sin 30°)$, but also affects the roller in the direction of overcoming the force of friction, which at the condition of balance of the roller would be no less than: $F_{fr} = F \times \cos 30°$, then the condition of balance, at which the roller will not slide, has the form: $F \times \cos 30° \leq f \times (P + F \times \sin 30°)$, and the applied force would be no greater than: $F_1 \leq \frac{f \times P}{\cos 30° - f \times \sin 30°} = \frac{0.2 \times 700}{0.866 - 0.2 \times 0.5} = 183$, N.

As this is indicated above, the force F not only increases the pressure of roller on the supporting surface by ($F \times \sin 30°$), but also acts ($F \times \cos 30°$) on the roller in the direction of overcoming the force of friction (the rolling resistance). The rolling will occur under the condition, if: $F \times \cos 30° \times R \geq \delta \times (P + F \times \sin 30°)$, and if on the roller the force F acts, which is greater than: $F_2 \geq \frac{\delta \times P}{R \times \cos 30° - \delta \times \sin 30°} = \frac{0.008 \times 700}{1 \times 0.866 - 0.008 \times 0.5} = 6.5$, N.

The minimal magnitude of the force F, applied at an angle of 30°, at which the roller begins to roll with sliding, equals: $F_{min} = 183$ N.

Questions and comments:

9.4.1. The crumple of a section of surface of the roller and the crumple of section of the supporting surface, which is the cause of the occurrence of rolling friction force. The state of the motionless balance of the roller is inexplicable based on the model of an absolutely rigid body (the roller and\or the supporting surface).

9.4.2. The condition of motionless balance of roller in the form of a limiting displacement of the line of action of the reaction normal to the support surface from the projection of the axis of the roller onto the support surface when this surface and the roller too are deformed by pressure force equal to the normal support reaction.

9.4.3. Dependence of the rolling friction moment on the rolling friction coefficient, which depends on the rigidity of the supporting surface and the rigidity of the roller, and on the normal support reaction.

9.4.4. The sequence of assessing the stationary equilibrium of the roller, taking into account the conditions of its rolling and sliding.

9.5. Finding the Coriolis acceleration for complex motion of a material point

Coordinate systems and types of complex motion of the point. In a moving coordinate system, the movement of the point is relative. At the absence of relative motion, the movement of the point with moving coordinate system in relation to fixed system is transferring. The movement of the point relative to fixed (the main) coordinate system is absolute.

Theorem on velocities addition of point in the complex motion. In the complex motion, the absolute velocity $\vec{V}$ of point is equal to vector sum of the point transferring $\vec{V}_e$ and relative $\vec{V}_r$ velocities: $\vec{V} = \vec{V}_e + \vec{V}_r$.

Theorem on addition of accelerations of the point in complex motion (the Coriolis theorem). In complex motion, the absolute acceleration $\vec{a}$ of moving point is equal to the vector sum of transferring acceleration $\vec{a}_e$, the relative one $\vec{a}_r$ and the Coriolis acceleration $\vec{a}_k$: $\vec{a} = \vec{a}_e + \vec{a}_r + \vec{a}_k$.

Coriolis acceleration characterizes the influence of non–translational transferring motion on relative velocity of point and the relative motion on transferring velocity.

The Coriolis acceleration vector is determined by the vector product ($\vec{\omega}_e \times \vec{V}_r$) using the formula: $\vec{a}_k = 2 \times \vec{\omega}_e \times \vec{V}_r$.

The magnitude of Coriolis acceleration equals: $a_k = 2 \times \omega_e \times V_r \times \sin\alpha$, where α – the angle between vectors: the angular velocity transferring rotation $\vec{\omega}_e$ of the moving coordinate system relative to the stationary one and the translational velocity of the point in the moving coordinate system $\vec{V}_r$ (the relative velocity).

The Coriolis acceleration is zero under the conditions:

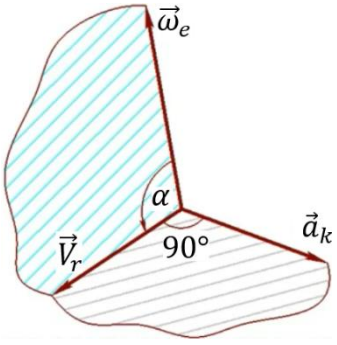
– at translational transferring motion, that is, when the moving coordinate system does not rotate relative to fixed coordinate system (the main): $\vec{\omega}_e = 0$;

– at the absence of relative motion, when: $\vec{V}_r = 0$, that is, when the point does not move in the moving coordinate system;

– when the vectors of the angular velocity of the transferring rotation of the moving coordinate system relative to the fixed one $\vec{\omega}_e$ and the translational relative velocity of the point in the moving coordinate system $\vec{V}_r$ are parallel: $\sin\alpha = 0$, and the angle α between them is equal zero or π (180°), in other words, when the rotational movement of the points of the moving coordinate system relative to the fixed coordinate system (the main) is perpendicular to the translational movement of points in the moving coordinate system.

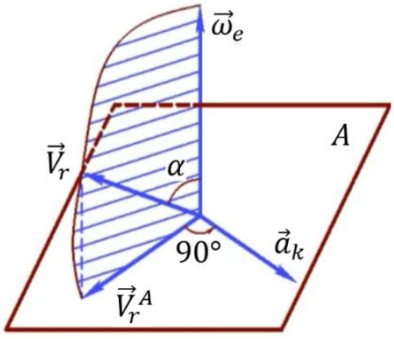
The direction of the Coriolis acceleration vector is determined by the vector product rule (Fig. 9.5.1 a), which is by the «right screw» rule for the product of vectors: $\vec{\omega}_e \times \vec{V}_r$, or by N.Ye. Zhukovsky's rule (Fig. 9.5.1 b): by finding the projection $\vec{V}_r^A$ of the relative velocity vector $\vec{V}_r$ onto the plane A perpendicular to the angular velocity vector $\vec{\omega}_e$ of transferring rotation of the moving coordinate system relative to the fixed system, and then rotating the projection of the relative velocity vector $\vec{V}_r^A$ by an angle $\pi/2$ (that is by 90°) in the direction of the moving coordinate system transferring rotation.

Vector product rule



(a)

Zhukovsky's rule



(b)

Fig. 9.5.1

Exercise 1. The angular velocity of the Earth's rotation is:

$\omega_e = \frac{2\times\pi}{24\times3600} = 7.272 \times 10^{-5}$, radian/s.

It is required to determine the Coriolis acceleration at the Earth's poles (Fig. 9.5.2) at the movement of the point M along the meridian and at the equator, as well as at the equator at the movement along the parallel.

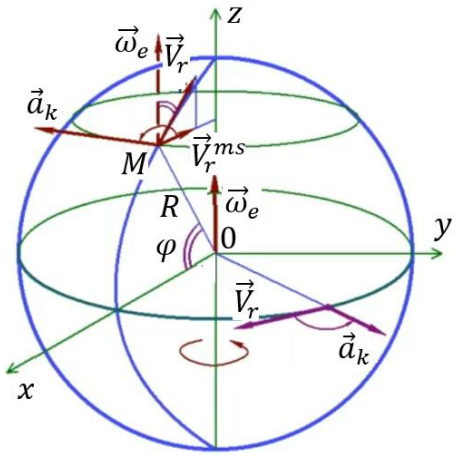


Fig. 9.5.2

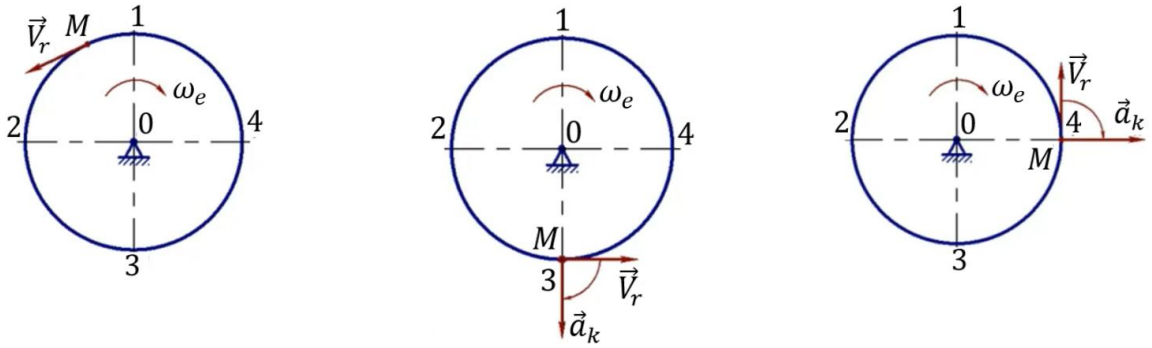
When moving along the Earth's surface, the Coriolis acceleration is determined using the formula: $a_k = 2 \times \omega_e \times V_r \times \sin\varphi$, where φ – the latitude, V_r – the relative velocity of the point M motion along the Earth's surface.

When moving of the point M along the meridian at the poles: $\sin\varphi = 1$, the Coriolis acceleration equals: $a_k = 2 \times 7.272 \times 10^{-5} \times V_r \times 1 = 0.000145 \times V_r$, m/s^2.

When moving of the point M in any direction tangential to the surface of the Earth at the equator: $\sin\varphi = 0$, the Coriolis acceleration is zero.

Exercise 2. The exercise assigns one to determine the direction of the Coriolis acceleration for point M in the positions 3 and 4 (Fig. 9.5.3 a), which moves along the rim of the disk with speed V_r, and the disk wherein rotates about the horizontal axis 0 with angular velocity ω.

Using the «right screw» rule for the product of vectors: $\vec{\omega}_e \times \vec{V}_r$, or Zhukovsky's rule one can find the direction of Coriolis acceleration: in position 3 – vertically down (Fig. 9.5.3 b), in position 4 – horizontally to the right (Fig. 9.5.3 c).



(a) (b) (c)

Fig. 9.5.3

Questions and comments:

9.5.1. Complex movement of a point, when there is a moving coordinate system, in which the point moves, and fixed coordinate system in which the moving coordinate system moves.

9.5.2. Relative motion of point in a moving coordinate system. The transferring motion of point relative to a fixed coordinate system, when there is no relative motion of point in the moving coordinate system and the point moves along with the moving coordinate system.

9.5.3. The absolute velocity vector of point is equal to the sum of vectors: the transferring velocity (i.e. the velocity of movement of the coordinate system) and the relative velocity of the point in the moving coordinate system.

9.5.4. The absolute acceleration vector during complex motion of point, equal to the sum of the acceleration vectors: the transferring, the relative and the Coriolis accelerations, which the last characterizes the influence of transferring motion on the relative velocity of point and the relative motion on the motion transferring velocity.

9.5.5. The Coriolis acceleration vector is equal to twice the vector product of the vectors: the angular velocity of the transferring rotation of the moving coordinate system in the stationary system and the translational velocity of the point in the moving coordinate system (i.e., the relative velocity of the point).

9.5.6. Absence of Coriolis acceleration during complex motion of point, when the moving coordinate system does not rotate in the stationary system, or when the point does not move in the moving coordinate system (if there is no the relative motion), or when the point in the moving coordinate system moves relatively along a line parallel to with the angular velocity vector of moving system rotation relative the stationary system.

9.5.7. The «right screw» rule when finding the direction of the Coriolis acceleration vector, taking into account the vector product of the vectors: the angular velocity of transferring rotation of the moving coordinate system in a stationary system and the vector of

the relative translational velocity of a point in the moving coordinate system: if one rotates the angular velocity vector in the direction vector of the translational velocity of the point, then the rotation axis is parallel to the Coriolis acceleration, and the direction of the Coriolis acceleration along this axis corresponds to the «right screw» rule when turning the angular velocity vector.

9.5.8. The direction of the Coriolis acceleration vector determining by the N.Ye. Zhukovsky's rule: if one finds a plane perpendicular to the transferring vector of the angular velocity of rotation of the moving coordinate system, and if he projects the relative velocity vector of point in the moving coordinate system onto this plane, then rotating the projection of the relative velocity vector by an angle $\pi/2$ (that is 90°) in the direction of rotation of the moving coordinate system, one gets the direction of the Coriolis vector.

9.5.9. Coriolis acceleration when moving along the Earth's surface, equal to zero at the equator and proportional to the velocity of movement on the Earth's poles with a proportionality coefficient equal to twice the angular velocity of the earth's rotation, equal to two π per day (considering day in seconds).

9.5.10. The centrifugal direction of Coriolis acceleration (from the axis of transferring rotation) when a material point moves along the rim of a rotating disk.

9.6. Equilibrium conditions taking into account active forces in accordance with d'Alembert's principle

Exercise. The weightless vertical shaft AK (Fig. 9.6.1 a), mounted in bearings: A and K, rotates at a constant angular velocity: ω=10 s^{-1}. Rigidly attached to the shaft are: the weightless rod 1 at the point E at an angle of 90^0, of length: l_1=0.4 m and mass: m_1=8 kg at the end and the uniform rod 2 of length: l_2=0.6 m at the point B at an angle: β=60^0 and mass: m_2=6 kg. Both rods lie in the same plane. The length of each of identical sections of shaft AK equals: $AB = BD = DE = EK = b = 0.4$ m. The exercise assigns one to determine the reaction of thrust bearing A and the bearing K.

The coordinate system Axy which is movable (the rotating) together with the shaft AK is selected so that the rods: 1 and 2 are in the coordinate plane Axy.

The elements of mechanical system (the rotating shaft) are affected by active gravity forces: $\vec{P}_1$ and $\vec{P}_2$, shown on the scheme (Fig. 9.6.1 b), as well as the reactions of the thrust bearing: $\vec{R}_A = \vec{X}_A + \vec{Y}_A$ and the bearing: $\vec{R}_k$.

In addition, the inertial forces act on elements of the rotating shaft ($\vec{F}_1^I$ and $\vec{F}_2^I$): $\vec{F}_1^I = -m_1 \times \vec{a}_1^I$, because $h_1 = l_1 = 0.4$ m, then the magnitude of force of inertia F_1^I equals: $F_1^I = m_1 \times l_1 \times \omega^2 = 8 \times 0.4 \times 10^2 = 320$, N,

$\vec{F}_2^I = -m_2 \times \vec{a}_2^I$, and the magnitude of force of inertia F_2^I equals: $F_2^I = m_2 \times \frac{l_2}{2} \times \sin\beta \times \omega^2 = 6 \times \frac{0.6}{2} \times \sin 60° \times 10^2 = 156$, N.

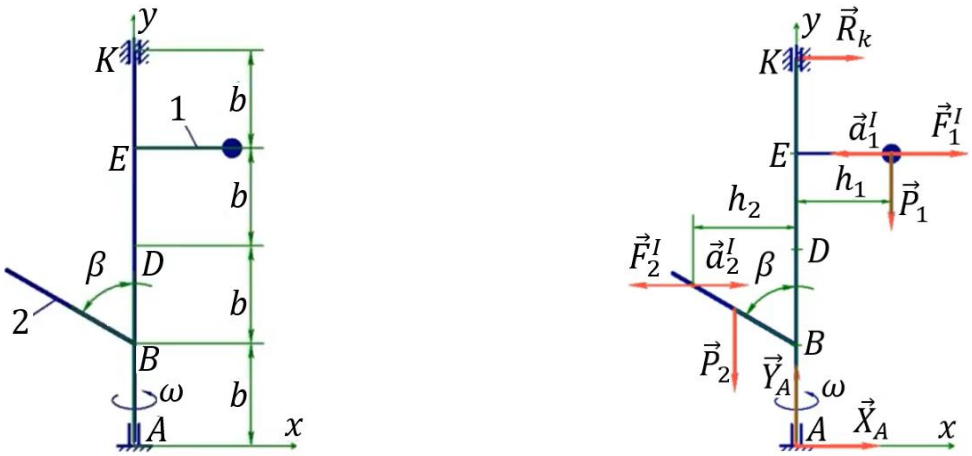


Fig. 9.6.1

Diagram of inertial forces acting on a homogeneous rod 2 with length: l_2=0.6 m is a triangle. The resultant of inertia forces F_2^I would be applied to the rod horizontally so that the line of action of the force passes through the geometric center of the diagram, which is at a distance: $\frac{2}{3} \times l_2$ of the point B of the attachment of the rod to the shaft (wherein: $h_2 = \frac{2}{3} \times l_2 \times \sin\beta$, Fig. 9.6.2).

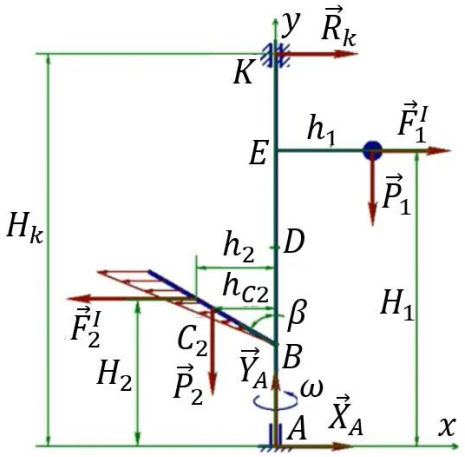


Fig. 9.6.2

The equilibrium conditions are as follow. In accordance with d'Alembert's principle, the active forces, reactions of connections and supports and force of inertia form the balanced system: $\sum F_{kx} = 0$, $X_A + R_k + F_1^I - F_2^I = 0$, $\sum F_{ky} = 0$, $Y_A - P_1 - P_2 = 0$ and $\sum M_A = 0$, $P_2 \times h_{C2} + F_2^I \times H_2 - P_1 \times h_1 - F_1^I \times H_1 - R_k \times H_k = 0$.

When specifying the equilibrium conditions, the dimensions of the elements of the mechanical system (the rotating shaft) are used:

$H_2 = b + \frac{2}{3} \times l_2 \times \cos\beta = 0.4 + \frac{2}{3} \times 0.6 \times \cos 60° = 0.6$, m,

$H_1 = 3 \times b = 3 \times 0.4 = 1.2$, m, $H_k = 4 \times b = 4 \times 0.4 = 1.6$, m.

The gravity forces of elements of the rotating shaft are:

$P_1 = m_1 \times g = 8 \times 9.81 = 78.48$, N and $P_2 = m_2 \times g = 6 \times 9.81 = 58.86$, N.

The thrust and bearing reactions are equal to:

$X_A = -R_k - F_1^I + F_2^I = 191.56 - 320 + 156 = 27.56$, N,

$Y_A = P_1 + P_2 = 78.48 + 58.86 = 137.34$, N,

$R_k = \frac{1}{H_k} \times (P_2 \times h_{C2} + F_2^I \times H_2 - P_1 \times h_1 - F_1^I \times H_1) =$

$= \frac{1}{1.6} \times (58.86 \times 0.26 + 156 \times 0.6 - 78.48 \times 0.4 - 320 \times 1.2) = -191.56$, N.

Verification (the condition of equality to zero of moments relative to the point D):

$\sum M_D = X_A \times 2 \times b + P_2 \times h_{C2} - F_2^I \times (2 \times b - H_2) - P_1 \times h_1 -$

$-F_1^I \times b - R_k \times 2 \times b = 27.56 \times 2 \times 0.4 + 58.86 \times 0.26 - 156 \times (2 \times 0.4 - 0.6) -$

$-78.48 \times 0.4 - 320 \times 0.4 - (-191.56) \times 2 \times 0.4 \approx 0.$

Questions and comments:

9.6.1. Dynamic equilibrium of mechanical system under the influence of active forces, reactions of supports and joints and the inertial forces in accordance with d'Alembert's principle.

9.6.2. The action of d'Alembert's principle regardless of the choice of coordinate system – both in a fixed coordinate system and in moving one associated with a mechanical system or its elements.

9.6.3. The action of d'Alembert's principle, relating to efforts of any kind – both to forces acting on a mechanical system and its elements, and to the moments of forces.

9.6.4. Composition of terms and form of dynamic equilibrium equations considering the d'Alembert's principle.

9.7. Direct issue of dynamics, determination of acting forces by analyzing the elements of motion of a mechanical system

Exercise 1. Material point M (Fig. 9.7.1) with mass: m =1 kg moves according to the law: $\vec{r} = \vec{\imath} \times e^{2\times t} + \vec{\jmath} \times \frac{\cos(2\times t)}{2} - \vec{k} \times t^2$, m, where $\vec{r}$ – radius–vector of the point in the inertial coordinate system $0xyz$ (which is in a stationary system or moving uniformly and rectilinearly). The exercise assigns one to find the value of the force of inertia acting on the material point M at initial moment of time: $t_0 = 0$ s.

Individual coordinates of the point M change according to parametric laws in time: $x = e^{2\times t}$, $y = \frac{\cos(2\times t)}{2}$ and $z = -t^2$, m.

Acceleration projections of the point M depending on the time parameter, change as: $\ddot{x} = 4 \times e^{2\times t}$, $\ddot{y} = -2 \times \cos(2 \times t)$ and $\ddot{z} = -2$, m/s^2.

The projections ($F_x = m \times a_x$, $F_y = m \times a_y$ and $F_z = m \times a_z$) of the force of inertia: $\vec{F} = m \times \vec{a}$, acting on a material point are equal: $F_x = 4 \times e^{2\times t}$, $F_y = -2 \times \cos(2 \times t)$ and $F_z = -2$, N.

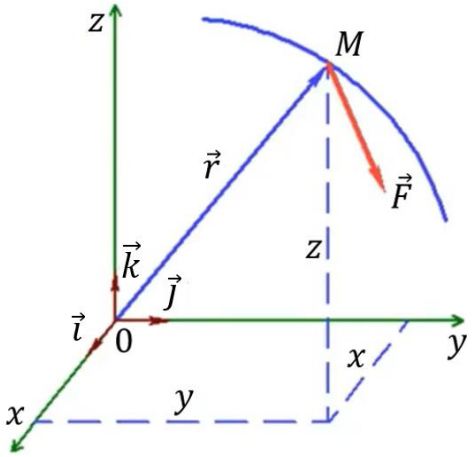
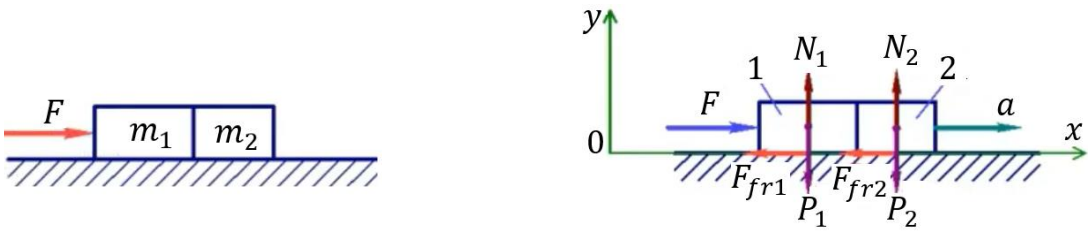


Fig. 9.7.1

At the initial moment of time: $t_0 = 0$ s the projections of force of inertia F_0, acting on the material point M, are equal: $F_{x0} = 4$, $F_{y0} = -2$ and $F_{z0} = -2$, N and magnitude of the force of inertia is: $F_0 = \sqrt{F_{x0}^2 + F_{y0}^2 + F_{z0}^2} = \sqrt{4^2 + (-2)^2 + (-2)^2} = 2 \times \sqrt{6}$, N.

Exercise 2. Two contacting parts of the block with masses: m_1 =6 kg and m_2 =4 kg, located on a rough horizontal surface (Fig. 9.7.2 a), are set in motion by horizontal constant force: F =25 N.



(a) (b)

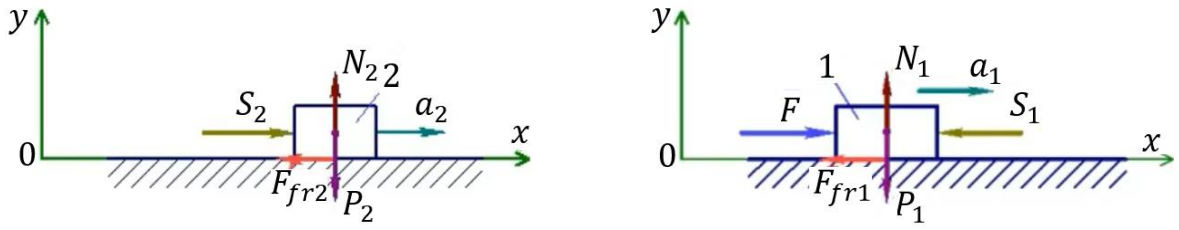
Fig. 9.7.2

The exercise assigns one to determine the acceleration of parts of the block, if the coefficient of sliding friction at the point of their contact with the supporting surface is equal: f =0.2 and also to find the force S of the interaction of parts of the block.

The total mass of both parts of the block equals: $m = m_1 + m_2 = 6 + 4 = 10$, kg.

The equation of motion of two parts of the block is condition of the balance of forces according to d'Alembert's principle, taking into account both frictional and inertial forces, and applied forces: $(m_1 + m_2) \times \vec{a} = \sum \vec{F}_k$, and the acceleration of the block as it moves along the surface one can find using the projections of forces onto the axis $0x$: $m \times a = F - f \times (N_1 + N_2)$, that's why the acceleration of two parts of block equals: $a = \frac{F}{m} - f \times g = \frac{25}{10} - 0.2 \times 9.81 = 0{,}54$, m/s^2.

The action of the interaction force of the bars is shown in the diagram (Fig. 9.7.3 a and b): $S_1 = S_2 = S$.



(a) (b)

Fig. 9.7.3

When determining the force S of the interaction of parts of the block one can consider the condition of the balance of forces acting on the block 1, taking into account frictional forces and the inertial forces in addition to the applied ones:

$S = F - m_1 \times (a + f \times g) = 25 - 6 \times (0.54 + 0.2 \times 9.81) = 10$, N.

The same result one can find by the balance of forces acting on the block 2, including, in addition to the applied forces, also the frictional and inertial forces:

$S = m_2 \times (a + f \times g) = 4 \times (0.54 + 0.2 \times 9.81) = 10$, N.

Questions and comments:

9.7.1. The inertial reference coordinate system: the stationary or moving uniformly and rectilinearly.

9.7.2. Finding by double time differentiation the projections of the acceleration of a point for specified parametric the law of motion (in dependence on the time parameter) or for the assigned vector definition of the point motion. Determination of projections on the axis of the force vector acting on the body and the magnitude of this force.

9.7.3. The drawing up of equation of forces equilibrium and to finding the elements of motion of mechanical system under the influence of these forces and to find the forces acting between the elements of the system, using the condition of balance of projections of forces acting on elements of mechanical system – both the frictional forces and inertial ones, as well as the applied forces.

9.8. Inverse issue of dynamics, determination of the elements of motion of a mechanical system under the influence of efforts, applied to it

Exercise 1. A material point M of unit mass moves along the horizontal plane $0xy$ under the action of force: $\vec{F} = -(\vec{\imath} \times \sin t + \vec{\jmath} \times \cos t)$. The exercise assigns one to find the pattern of motion of the point M and the equation of it trajectory (Fig. 9.8.1 a), if at the initial moment of time: $t = 0$ this point was in position $M_0(0; 1)$ and had a velocity: $\vec{V}_0 = \vec{\imath}$.

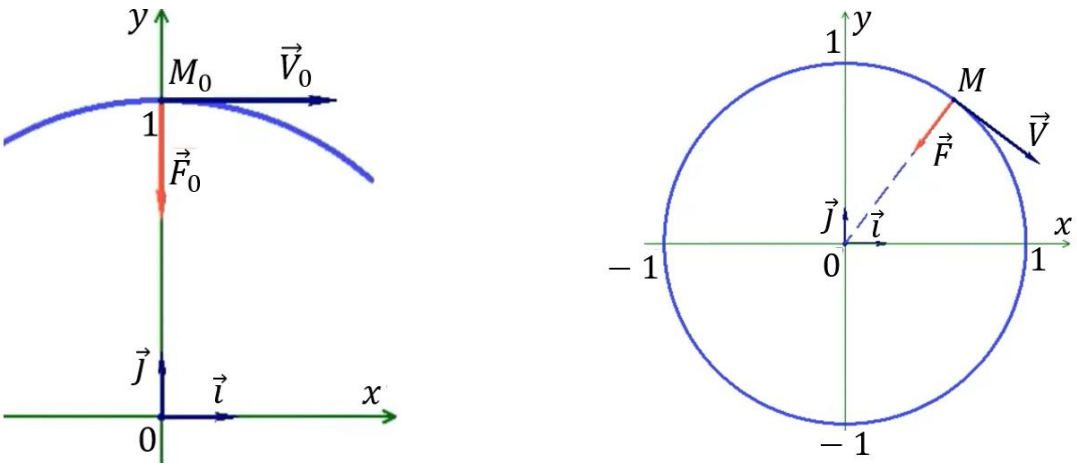


Fig. 9.8.1

At the initial moment of time, the force acts on the point $M_0(0; 1)$:
$\vec{F}_0 = -(\vec{\imath} \times \sin 0 + \vec{\jmath} \times \cos 0) = -\vec{\jmath}$.

The force acting on point M of unit mass imparts acceleration to this point, and displacement of the point M (i.e. the projection of this displacement) one can determine by finding the acceleration: $\frac{dV_x}{dt} = -\sin t$ and $\frac{dV_y}{dt} = -\cos t$ and applying time integration.

The velocity projections (V_x and V_y) of the point M correspond to the first integral of it acceleration: $V_x = -\int \sin t \times dt + C_1 = \cos t + C_1$
and $V_y = -\int \cos t \times dt + C_2 = -\sin t + C_2$.

The displacement projections (x and y) of the point M correspond to the iterated integral: $\frac{dx}{dt} = \cos t + C_1$, $x = \int \cos t \times dt + C_1 \times \int dt + C_3 = \sin t + C_1 \times t + C_3$ and $\frac{dy}{dt} = -\sin t + C_2$, $y = -\int \sin t \times dt + C_2 \times \int dt + C_4 = \cos t + C_2 \times t + C_4$.

When determining the integration constants (C_1, C_2, C_3 and C_4) the initial conditions are used: at $t = 0$, $x_0 = 0$, $y_0 = 1$, $V_{x0} = 1$ and $V_{y0} = 0$, that is: $C_1 = C_2 = C_3 = C_4 = 0$.

The velocity (V_x and V_y) and displacement (x and y) projection functions in coordinate parametric form are: $V_x = \cos t$, $V_y = -\sin t$ and $x = \sin t$, $y = \cos t$.

The shape of the motion trajectory of the point M is a circle of unit radius (Fig. 9.8.1 b): $x^2 + y^2 = 1$.

The force $\vec{F}$, acting on the material point M, is directed radially to center of the circle perpendicular to the velocity $\vec{V}$, i.e. the scalar product of force vector and the velocity vector is equal to zero: $\vec{F} \times \vec{V} = (-\vec{\imath} \times \sin t - \vec{\jmath} \times \cos t) \times (\vec{\imath} \times \cos t - \vec{\jmath} \times \sin t) =$ $= -\sin t \times \cos t + \sin t \times \cos t = 0$, that is vectors are perpendicular: $\vec{F} \perp \vec{V}$.

Exercise 2. A material point M with mass: $m = 1$ kg moves along the horizontal axis $0x$ under the influence of force: $\vec{F} = F(t) \times \vec{\imath}$. Change in time of the value of the force $F(t)$, i.e. the projection of force onto the $0x$ axis is shown on the graph (Fig. 9.8.2), and the amplitude value of the force corresponding to the initial moment of time: t_0 =0, equals: F_0 =10 N. Time intervals of movement of the material point M: 1–st time interval (t =0 … 4 s), 2–nd (t =4 … 8 s) and 3–rd (t =8 … 10 s). The exercise assigns one to determine the velocity of the point M at the moment of time: t =10 s, if the initial conditions of movement of the point M are zero, that is at t_0 =0, V_0 =0.

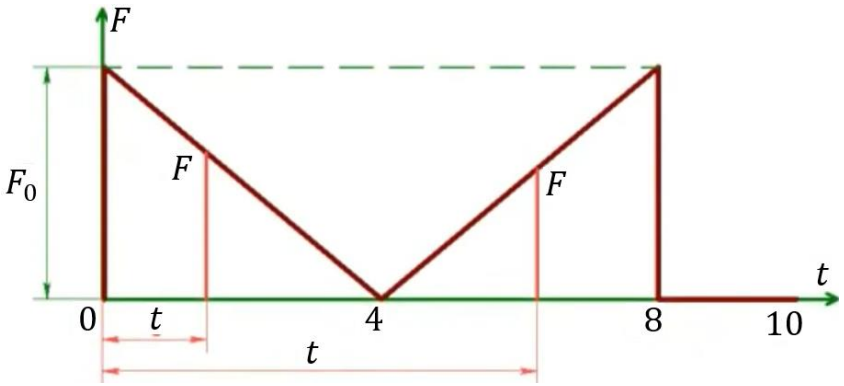


Fig. 9.8.2

In 1–st time interval (t =0 … 4 s) the force F acting on the material point M motion one can determine considering that: $\frac{F}{F_0} = \frac{4-t}{4}$.

Since the amplitude value of the force corresponding to the initial moment of time: t_0 =0, equals: F_0 =10 N, then: $F = F_0 - \frac{F_0}{4} \times t = 10 - \frac{5}{2} \times t$.

Since the mass is equal to: $m = 1$ kg, then the acceleration of the material point M in the 1–st time interval (t =0 … 4 s) equals: $\frac{dV}{dt} = m \times \frac{dV}{dt} = F = 10 - \frac{5}{2} \times t$, m/s^2, and the velocity is determined by integrating the acceleration function as:

$V = -\int \left(10 - \frac{5}{2} \times t\right) \times dt + C_1 = -\frac{5}{4} \times t^2 + 10 \times t + C_1$, m/s.

When finding the integration constant C_1 the initial condition is used:

at $t = 0$, $V_0 = 0$, then: C_1 =0.

At the end of first time interval (t =0 … 4 s) the velocity of the material point is equal: at $t = 4$, $V = -\frac{5}{4} \times 4^2 + 10 \times 4 = 20$, m/s.

In the second time interval of motion (t =4 … 8 s) the force F acting on the material point M one can determine considering that: $\frac{F}{F_0} = \frac{4-t}{8-4}$, with the mass of the point M equal to: $m = 1$ kg, its acceleration in the 2–nd time interval of motion (t =4 … 8 s) equals: $\frac{dV}{dt} = m \times \frac{dV}{dt} = F = \frac{F_0}{4} \times t - F_0 = \frac{5}{2} \times t - 10$, m/s^2, and the point velocity is determined by integrating the acceleration function as:

$V = -\int \left(\frac{5}{2} \times t - 10\right) \times dt + C_2 = \frac{5}{4} \times t^2 - 10 \times t + C_2$, m/s.

When finding the integration constant C_2 the initial condition is used: at $t = 4$ s, $V_{20} = 20$ and $C_2 = 40$, m/s, and the functional dependence on time of the velocity V of material point M in the 2–nd time interval of it motion (t =4 … 8 s) has the form:

$V = \frac{5}{4} \times t^2 - 10 \times t + 40$, m/s, that is at the end of second time interval the velocity of the point M equals: $V = \frac{5}{4} \times 8^2 - 10 \times 8 + 40 = 40$, m/s at $t = 8$.

The velocity V of material point M in the third motion interval (t =8 … 10 s) is constant, because the force acting on a point is zero: $V = 40$, m/s.

Questions and comments:

9.8.1. Finding projections of velocity and displacement of a point by integrating parametric functions (i.e. the dependencies on the time parameter) of projections onto the coordinate axes of force acting on the body. Substitution of initial conditions and determination of integration constants.

9.8.2. Determination of the coordinate form of trajectory of the point by excluding the time parameter from the found parametric functions of the point's coordinates depending on time.

9.8.3. The direction of the force acting on a material point (the angle of direction of the force relative to the tangent to the trajectory at the point), determined by the criterion that the scalar product of the force vector and the vector of velocity of the point is zero.

9.8.4. Finding time (the duration) for a point to move the specified distance under the influence of a force by integrating the acceleration function twice, taking into account the initial conditions.

9.8.5. The determination of the elements of motion of material point (the acceleration, velocity and displacement) when a variable over time intervals force acts on the point.

9.9. Solving the equation of motion of a load under the action of external forces, weight and friction

Exercise. Load (the material body) D with a certain mass m, having received an initial velocity $\vec{V}_0$ at the point A, moves in bent pipe ABC, located in a vertical plane and having an inclined section AB and horizontal section BC (Fig. 9.9.1).

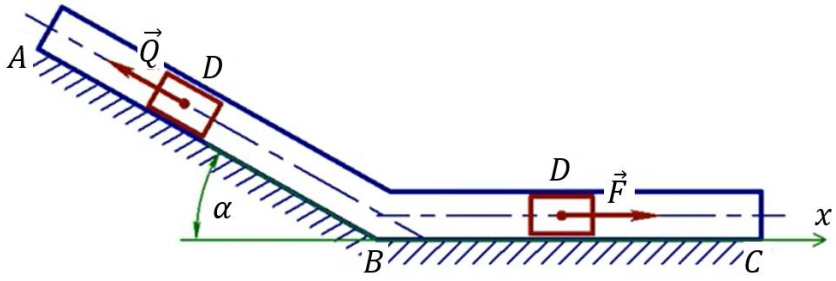


Fig. 9.9.1

In section AB in addition to its weight $\vec{P}$ (gravity), the moving load is subject to a constant force $\bar{Q}$ of a given direction and the force of the medium resistance $\bar{R}$, proportional to the square of the load velocity, and the resistance coefficient is equal: $\mu = 0.2$. At the point B the load, without changing the velocity, moves to the section BC of pipe, where, in addition to the force of gravity $\vec{P}$ act: the force of friction (coefficient of friction of the load sliding on the pipe is equal: $f = 0.25$) and the variable force of magnitude F_x, directed along the axis of the section BC. The characteristics of the mechanical system are given in the Table 9.9.1.

Table 9.9.1

m, kg	α	V_0, m/s	Q, N	R, N	l, m	t_1, s	F_x, N
4	30°	10	3	$0.2 \times V^2$	4	–	$5 \times t^3$

Considering the load as a material point D and knowing the length l of the section AB it is required to determine the pattern of the load D movement in the section BC: $x = x(t)$, where: $x = BD$. The friction of the load on the inner surface of pipe in section AB is not taken into account.

In section AB the z axis runs along the pipe and one can determine the magnitude of acceleration $\ddot{z}$ of the load D considering all the forces acting on it, as: $m \times \ddot{z} = \sum F_{kz}$, that is $m \times \frac{dV_z}{dt} = P \times \sin\alpha - Q - R$. Using the symbol substitute: $V_z = V$ and adopting the magnitudes of forces: $P = m \times g$ and $R = \mu \times V^2$ into the equation, where μ – the resistance coefficient for load D, the equation of the load motion along the pipe section AB has the form: $m \times \frac{dV}{dt} = m \times g \times \sin\alpha - Q - \mu \times V^2$.

Eliminating the time parameter t by substitution: $z = z(t)$ and $V = \frac{dz}{dt}$, that is: $\frac{dV}{dt} = \frac{dV}{dt} \times \frac{dz}{dz} = V \times \frac{dV}{dz} = \frac{1}{2} \times \frac{d(V^2)}{dz}$, the equation for the movement of the load D along the pipe section AB one can convert to the function sought of the load velocity square (V^2), depending on the coordinate z of the load displacement, instead of the function of its velocity V, which depends on the parameter t in the original equation:

$\frac{d(V^2)}{dz} = \frac{2}{m} \times (m \times g \times \sin\alpha - Q) - \frac{2\times\mu}{m} \times (V^2)$,

or $\frac{d(V^2)}{dz} = -\frac{2\times\mu}{m} \times \left[(V^2) - \frac{1}{\mu} \times (m \times g \times \sin\alpha - Q)\right]$.

The resulting equation is linear with respect to the function sought of squared velocity (V^2) of the load D motion, depending on coordinate of its movement z along the pipe section AB (Fig. 9.9.2), in contrast to the original equation, which was nonlinear with respect to motion velocity function of the load V, depending on time parameter t.

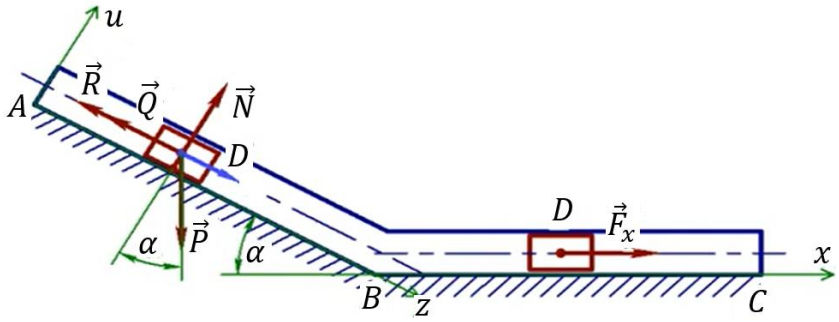


Fig. 9.9.2

The equation the designations of groups of the values included in the equation are used: $b = \frac{1}{\mu} \times (m \times g \times \sin\alpha - Q)$ and $\beta = \frac{2\times\mu}{m}$. After substitution of the indicated groups designations the equation takes the form: $\frac{d(V^2-b)}{dz} = -\beta \times (V^2 - b)$

and $\frac{d(V^2-b)}{(V^2-b)} = -\beta \times dz$, $\int \frac{d(V^2-b)}{(V^2-b)} = -\int \beta \times dz + C_1$ and $\ln(V^2 - b) = -\beta \times z + C_1$.

When finding the integration constant C_1 the initial condition is used: $z_0 = 0$ m and $V_0 = 10$ m/s at $t = 0$ s: $C_1 = \ln(V_0^2 - b)$, then:

$\ln(V^2 - b) = -\beta \times z + \ln(V_0^2 - b)$, $\ln\frac{(V^2-b)}{(V_0^2-b)} = -\beta \times z$,

$(V^2 - b) = (V_0^2 - b) \times e^{-\beta \times z}$ and $V = \sqrt{(V_0^2 - b) \times e^{-\beta \times z} + b}$.

Magnitudes of the designated groups are: $b = \frac{1}{0.2} \times (4 \times 9.81 \times \sin 30° - 3) = 83$ and $\beta = \frac{2 \times 0.2}{4} = 0.1$. After substituting the determined generalized values into the resulting solution the velocity of motion of the load D along section AB depending on coordinate of its displacement z along the pipe one can find as: $V = \sqrt{17 \times e^{-0.1 \times z} + 83}$, m/s.

At the point B between the sections AB and BC, that is at: $z = l = 4$ m the velocity of the load D equals: $V = \sqrt{17 \times e^{-0.1 \times 4} + 83} = 9.72$, m/s.

In section BC (Fig. 9.9.3) one can determine the magnitude of the acceleration $\ddot{x}$ of the load D, taking into account all the forces acting on it, as: $m \times \ddot{x} = \sum F_{kx}$, that is: $m \times \ddot{x} = m \times \frac{dV_x}{dt} = F_x - F_{fr} = 5 \times t^3 - f \times m \times g$, that's why: $\frac{dV_x}{dt} = \frac{5}{m} \times t^3 - f \times g$ because: $F_{fr} = f \times P = f \times m \times g$.

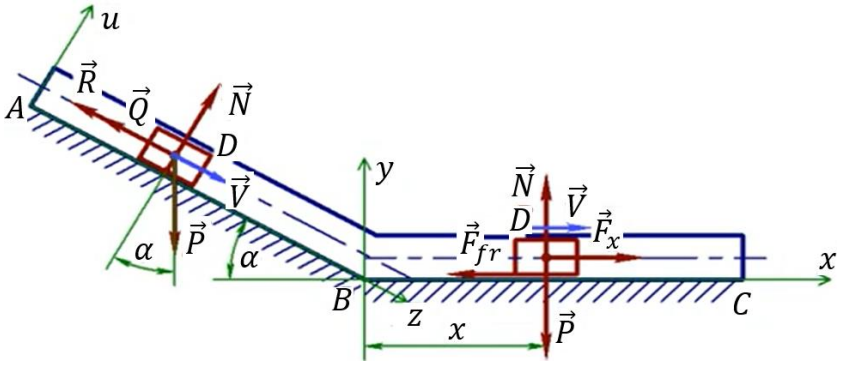


Fig. 9.9.3

The velocity V_x of motion of the load D along pipe in the section BC is determined by integrating the equation: $dV_x = \frac{5}{m} \times t^3 \times dt - f \times g \times dt$, $m = 4$ kg, $f = 0.25$, $g = 9.81$ m/s^2, $V_x = 1.25 \times \int t^3 \times dt - 2.45 \times \int dt$ and $V_x = 0.31 \times t^4 - 2.45 \times t + C_2$.

When finding the integration constant C_2 the initial condition is used: $x_{20} = 0$ and $V_{20} = 9.72$ m/s at $t_{20} = 0$, then: $C_2 = V_{20} = 9.72$ m/s and $V_x = 0.31 \times t^4 - 2.45 \times t + 9.72$.

Displacement x of the load D in section BC is determined by integrating the equation: $\frac{dx}{dt} = V_x$, $dx = 0.31 \times t^4 \times dt - 2.45 \times t \times dt + 9.72 \times dt$, $x = 0.31 \times \int t^4 \times dt - 2.45 \times \int t \times dt + 9.72 \times \int dt + C_3$, $C_3 = 0$, because: $x_{20} = 0$ at $t_{20} = 0$.

That's why the pattern of motion of the load D in section BC is:

$x(t) = 0.062 \times t^5 - 1.225 \times t^2 + 9.72 \times t$, m.

Questions and comments:

9.9.1. The equilibrium of moving load (the material body) in straight inclined section under the influence of the forces of the weight, friction, support reaction and inertia.

9.9.2. Projection of forces acting on the load onto the coordinate axes to determine the acceleration of the load, its velocity and displacement.

9.9.3. Replacing the time variable with translational displacement variable in the equation of dynamic equilibrium of moving load, showing the dependence of its acceleration on time. Replacing the load velocity function with the squared velocity function.

9.9.4. The grouping in equations of multipliers corresponding to individual elements of motion of the load (the phase terms in the transformed balance equation of a moving load: the term in the equation proportional to the second time derivative of the load velocity and the term with the displacement of the load).

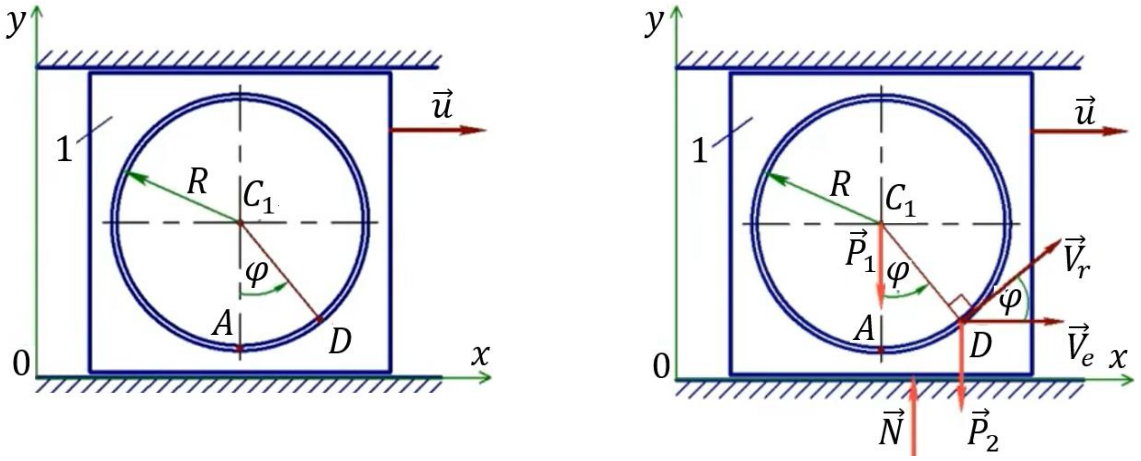
9.9.5. Finding the function of the load velocity square and the element of the load motion – the velocity depending on the load displacement, determining the integration constants. Determination of the final speed of the load at the extreme point of the section, serving as the initial condition of motion of the load on the next section.

9.9.6. The equation of dynamic equilibrium of horizontal motion of the load under the influence of the inertia and friction forces. Determining the velocity by integrating the equation of the load motion (its acceleration), taking into account the initial conditions, and determining the displacement by re–integrating the equation of the load motion.

Chapter 10. The amount of translational and rotational motion, the momentum and moment of momentum, the energy of mechanical system and the inertia characteristics of it elements

10.1. The use of mechanical system Momentum Change Theorem to determine elements of the system motion

Exercise. The mechanical system consists of a rectangular vertical plate (Fig. 10.1.1) with the mass: $m_1 = 20$ kg, moving along horizontal guides, and the load D with the mass: $m_2 = 6$ kg.



(a) (b)

Fig. 10.1.1

At the moment of time: $t_0 = 0$ s, when velocity of the plate equals: $u_0 = 1.5$ m/s, the load D under the influence of internal forces begins to move along the annular groove of the plate of radius: $R = 1.2$ m.

The central angle determining position of the load D relative to the point A of the plate: $\varphi = \angle AC_1D$, changes according to the pattern: $\varphi(t) = \frac{\pi}{6} \times (t^2 - 1)$, radian. Considering the load D as a material point, The exercise assigns one to find the pattern of the plate velocity change: $u = u(t)$.

The theorem on the change in momentum $\frac{dQ_x}{dt}$ of mechanical system in differential form in the projection onto the x–axis of forces F_{kx}^e and impulse Q_x (the indicator of momentum Q, equal to the product of moving mass and velocity) has the form: $\frac{dQ_x}{dt} = F_{kx}^e$.

If external forces are no applied to the body ($F_{kx}^e = 0$), then the momentum does not change: $Q_x = const$ and $\frac{dQ_x}{dt} = 0$.

The momentum of the assigned mechanical system (Fig. 10.1.1) is equal to the sum of the momentum Q_{1x} of plate 1 with mass: $m_1 = 20$ kg and the momentum Q_{2x} of load D with the mass: $m_2 = 6$ kg, which moves along the annular groove on this plate/

The total momentum in projection onto the x–axis equals: $Q_x = Q_{1x} + Q_{2x}$, kg×m/s, where: $Q_{1x} = m_1 \times V_{1x}$ – the momentum of plate motion in the projection onto the x–axis. Because he projection of the velocity of plate onto the x–axis V_{1x} is denominated: u, the projection of momentum of the plate onto x–axis is: $Q_{1x} = m_1 \times u$, kg×m/s.

Projection of momentum of the load onto the x–axis is:

$Q_{2x} = m_2 \times V_{rx} + m_2 \times V_{ex} = m_2 \times \left(\frac{1}{3} \times R \times \pi \times t \times \cos\varphi(t) + u\right)$, where V_{rx} – the relative velocity of the load (that is velocity of the load relative to the plate), $V_{ex} = V_e = u$ – the transferring velocity of the plate relative to fixed coordinate system, both velocities values (V_{rx} and V_{ex}) are considered in the projection onto the x–axis.

Momentum of the mechanical system equals:

$Q_x = (m_1 + m_2) \times u + m_2 \times \frac{1}{3} \times R \times \pi \times t \times \cos\left(\frac{\pi}{6} \times (t^2 - 1)\right)$.

The initial conditions are: at $t_0 = 0$, $Q_{0x} = (m_1 + m_2) \times u_0$.

Since external forces are not applied to the body ($F^e_{kx} = 0$) and the momentum does not change: $Q_x = const$, then $Q_x = Q_{0x}$

and $(m_1 + m_2) \times u + m_2 \times \frac{1}{3} \times R \times \pi \times t \times \cos\left(\frac{\pi}{6} \times (t^2 - 1)\right) = (m_1 + m_2) \times u_0$,

the pattern of the plate velocity is: $u = u(t) = u_0 - \frac{m_2 \times R \times \pi \times t}{3 \times (m_1 + m_2)} \times \cos\left(\frac{\pi}{6} \times (t^2 - 1)\right)$,

$u = 1.5 - \frac{6 \times 1.2 \times \pi \times t}{3 \times (20+6)} \times \cos\left(\frac{\pi}{6} \times (t^2 - 1)\right) = 1.5 - 0.289 \times t \times \cos\left(\frac{\pi}{6} \times (t^2 - 1)\right)$, m/s.

Questions and comments:

10.1.1. The index of momentum (the amount of the mechanic motion) equals to the product of body mass and its velocity. Momentum of mechanical system is equal to the sum of mechanical impulses (the momentums) of the system elements.

10.1.2. The Momentum Change Theorem: the time derivative of momentum of body is equal to the external force acting on the body.

10.1.3. The equalities of the projections of momentum on the axes and the products of masses and the projections of their velocities onto the coordinate axes.

10.1.4. Mechanical impulse during complex motion of a body, which projections are equal to the sum of the projections of the impulse of relative motion and the impulse of transferring motion (considering the signs of the velocity projections).

10.1.5. The concept of invariance of the amount of motion of the system, used when finding the elements of mechanical system motion (it acceleration, speed and displacement), in which the movement of any of its elements affects the movement of the

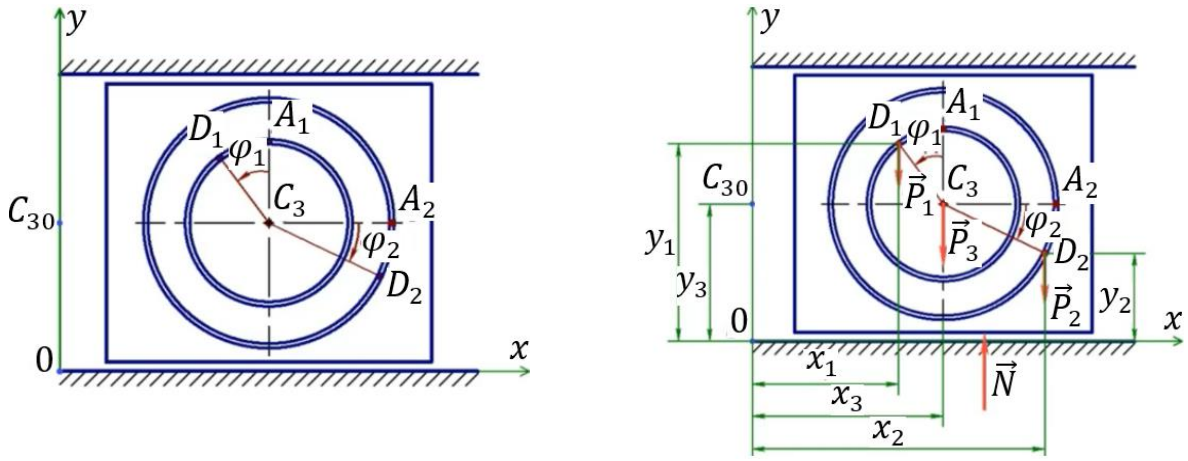
remaining elements in such a way that the momentum of the mechanical system does not change in total, if the mechanical system is not acted upon by external forces.

10.1.6. The variability of the sum of indicators of mechanical impulses of system elements (the vector sum or in projections onto coordinate axes) in accordance with the action of an external force, considering its direction, magnitude and the dependence of external force on time parameter.

10.2. Determination of patterns of changes in the coordinates of the center of mass and the support reaction of a mechanical system using the Theorem on the motion of the center of mass

In accordance with the Theorem on the motion of the center of mass of a mechanical system, the product of the mass of the system and the acceleration of its center of mass is equal to the main vector of all external forces.

Exercise. The mechanical system (Fig. 10.2.1 a) consists of the loads: D_1 with the mass: m_1=1 kg and D_2 with mass: m_2=2 kg and of the rectangular vertical plate with the mass: m_3=4 kg, which moves along the horizontal guides parallel to x–axis.



(a) (b)

Fig. 10.2.1

At the time moment when the system is stationary under the influence of internal forces, the loads begin to move along the gutters, which are circles of radii: r=0.5 m and R=1.0 m.

When the loads: D_1 and D_2 move, the corresponding coordinate central angles relative to moving system (the plate) change according to parametric dependences on time t, s: $\varphi_1 = \frac{\pi \times t}{6}$ and $\varphi_2 = \frac{\pi}{3} \times (2 - t)$, radian.

Considering the loads as material points and neglecting the resistance to their movement, The exercise assigns one to determine the patterns of changes in time of the

following values: the coordinate $x_3(t)$ of the center of mass C_3 of the plate (which is the pattern of motion of the plate) and the total normal reaction of the guides $N(t)$, which is total magnitude of the reactive forces acting from the guides on the vertical plate.

To find the pattern of changes of the coordinate $x_3(t)$ of the center of mass C_3 when the plate is moving in the absence of external influences on it (that is at equality to zero of the projections of external forces on the x–axis): $\sum F_{kx} = 0$.

It is taken into account that the center of mass of the rectangular plate and loads, i.e. the whole mechanical system does not experience the acceleration, therefore: $M \times \ddot{x}_C = 0$, and under the initial conditions of a stationary plate: $t = 0$, $V_C = 0$, $M \times \dot{x}_C = const = 0$, where: $M = m_1 + m_2 + m_3 = 1 + 2 + 4 = 7$, kg – the total mass of the mechanical system.

The total static moment of mass of the elements of mechanical system is equal:

$M \times x_C = m_1 \times x_1 + m_2 \times x_2 + m_3 \times x_3$, kg×m,

$x_1 = x_3 - r \times \sin\varphi_1 = x_3 - r \times \sin\left(\frac{\pi \times t}{6}\right)$, kg×m

and $x_2 = x_3 + R \times \cos\varphi_2 = x_3 + R \times \cos\left(\frac{\pi}{3} \times (2 - t)\right)$, kg×m,

$M \times x_C = m_1 \times \left(x_3 - r \times \sin\left(\frac{\pi \times t}{6}\right)\right) + m_2 \times \left(x_3 + R \times \cos\left(\frac{\pi}{3} \times (2 - t)\right)\right) +$

$+ m_3 \times x_3,$

$M \times x_C = (m_1 + m_2 + m_3) \times x_3 - m_1 \times r \times \sin\left(\frac{\pi \times t}{6}\right) + m_2 \times R \times \cos\left(\frac{\pi}{3} \times (2 - t)\right).$

Abscissa of the center of gravity of the mechanical system is: $x_{30} = 0$ at $t = 0$, and the static moment of mass of the mechanical system at the initial moment of time is determined as: and $M \times x_C = M \times x_{C0} = m_2 \times R \times \cos\left(\frac{2 \times \pi}{3}\right)$,

$(m_1 + m_2 + m_3) \times x_3 - m_1 \times r \times \sin\left(\frac{\pi \times t}{6}\right) + m_2 \times R \times \cos\left(\frac{\pi}{3} \times (2 - t)\right) =$

$= m_2 \times R \times \cos\left(\frac{2 \times \pi}{3}\right),$

$x_3 = \frac{1}{m_1 + m_2 + m_3} \times$

$\times \left[m_1 \times r \times \sin\left(\frac{\pi \times t}{6}\right) - m_2 \times R \times \cos\left(\frac{\pi}{3} \times (2 - t)\right) + m_2 \times R \times \cos\left(\frac{2 \times \pi}{3}\right)\right],$

the pattern of the coordinate $x_3(t)$ of the center of mass C_3 change when moving plate moves in the absence of external influences on the mechanical system is:

$x_3 = 0.07 \times \sin\left(\frac{\pi \times t}{6}\right) - 0.29 \times \cos\left(\frac{\pi}{3} \times (2 - t)\right) - 0.14$, m.

To determine the total normal reaction of the guides $N(t)$, which is the total magnitude of the reactive forces acting from the guides on the vertical plate, one is to consider that the projection onto the y–axis of inertia forces when moving elements of the mechanical system is equal to the sum of all external influences on the mechanical system,

including the weights of its elements ($\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$) and reactions from the guides $\vec{N}$:
$M \times \ddot{y}_C = \sum F_{ky} = N - P_1 - P_2 - P_3$,
$N = (m_1 + m_2 + m_3) \times g + M \times \ddot{y}_C$, $M \times y_C = m_1 \times y_1 + m_2 \times y_2 + m_3 \times y_3$, kg×m,
$y_3 = const$, $y_1 = y_3 + r \times \cos\varphi_1 = y_3 + r \times \cos\left(\frac{\pi \times t}{6}\right)$, kg×m
and $y_2 = y_3 - R \times \sin\varphi_2 = y_3 - R \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$, kg×m,
$M \times y_C = m_1 \times \left(y_3 + r \times \cos\left(\frac{\pi \times t}{6}\right)\right) + m_2 \times \left(y_3 - R \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)\right) +$
$+ m_3 \times y_3$,
$M \times y_C = (m_1 + m_2 + m_3) \times y_3 + m_1 \times r \times \cos\left(\frac{\pi \times t}{6}\right) - m_2 \times R \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$,
$M \times \dot{y}_C = -m_1 \times r \times \frac{\pi}{6} \times \sin\left(\frac{\pi \times t}{6}\right) + m_2 \times R \times \frac{\pi}{3} \times \cos\left(\frac{\pi}{3} \times (2 - t)\right)$,
$M \times \ddot{y}_C = -m_1 \times r \times \frac{\pi^2}{36} \times \cos\left(\frac{\pi \times t}{6}\right) + m_2 \times R \times \frac{\pi^2}{9} \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$,
$M \times \ddot{y}_C = -1 \times 0.5 \times \frac{\pi^2}{36} \times \cos\left(\frac{\pi \times t}{6}\right) + 2 \times 1.0 \times \frac{\pi^2}{9} \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$, N,
$M \times \ddot{y}_C = -0.137 \times \cos\left(\frac{\pi \times t}{6}\right) + 2.193 \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$, N,
$P_1 + P_2 + P_3 = M \times g = 7 \times 9.81 = 68.67$, N.

Full normal reaction of the guides $N(t)$ equals:
$N = 68.67 - 0.137 \times \cos\left(\frac{\pi \times t}{6}\right) + 2.193 \times \sin\left(\frac{\pi}{3} \times (2 - t)\right)$, N.

Questions and comments:

10.2.1. Acceleration of the center of mass of a mechanical system equals to the main vector of all external forces divided by the mass of the system.

10.2.2. The use of Theorem on the motion of the center of mass of a mechanical system to find accelerations (as well as velocities and displacements) of elements of a mechanical system when external forces act on its elements, including when their total action is zero.

10.2.3. Integration of the acceleration of the center of mass of a mechanical system, equal to zero in the absence of external forces acting on the system, leading to a decision about the immobility of the total center of mass of the mechanical system, i.e. to the invariance with respect to the initial position of the total static moment of the system elements relative to the fixed coordinate system.

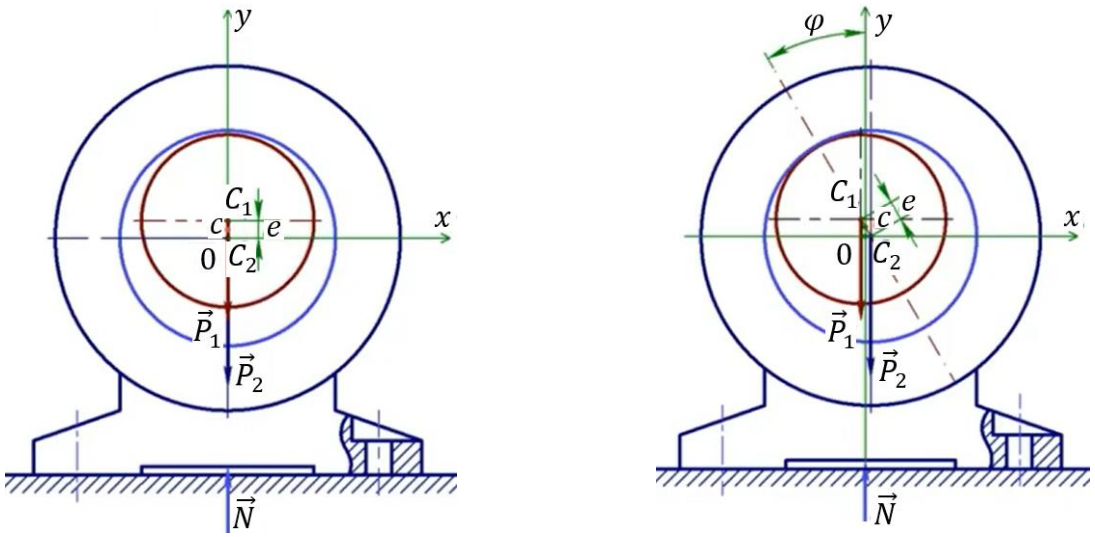
10.2.4. The forces between elements of a mechanical system moving with acceleration in the absence of external forces acting on the system.

10.2.5. The reaction appearance on the side of a fixed or guiding support, which does not allow all or some coordinate movements of the mechanical system (depending

on the mechanical elements of the support), when relative movement of system elements occurs.

10.3. Finding the forces acting on a rotor turning with eccentricity using the theorem on the motion of the center of mass

Exercise. The electric motor rotor of specified mass m_1, installed in the stator of mass m_1 with eccentricity e and rotates at constant angular velocity ω (Fig. 10.3.1).



(a) (b)

Fig. 10.3.1

The exercise assigns one to determine:

1. The equation of horizontal vibrations of electric motor assuming not bolted to the smooth foundation.

2. The time pattern of changes in the horizontal shear force acting on the electric motor mounting bolts.

3. The value of angular velocity of the rotor at which the joint of the electric motor mount opens.

The equation of horizontal vibrations of an electric motor not bolted to a smooth foundation is determined as follows. One would choose the coordinate system with the origin in a position where the displacements of the centers of mass of the mechanical system x_{C0}, rotor x_{10} and stator x_{20} are equal to zero: $x_{C0} = x_{10} = x_{20} = 0$ at t =0.

The mass of mechanical system (the electric motor) equals: $M = m_1 + m_2$.

If no external forces act on the mechanical system, that is: $\sum F_{kx} = 0$ (for the example of forces projections onto the axis x), then the inertial forces attributed to whole mechanical system are equal to zero in total: $M \times \ddot{x}_C = 0$ and $\ddot{x}_C = 0$, and the system's center of mass is stationary: $x_C = const$ (that is: $M \times x_C = const$), and in the absence of

external influence the position of the static moment of mass ($M \times x_C$) of mechanical system does not change, or the center of mass of the mechanical system moves uniformly and rectilinearly: $\dot{x}_C = const$ (that is $M \times \dot{x}_C = const$).

The static mass moment of the electric motor ($M \times x_C$) is equal to the sum of static mass moment of the rotor ($m_1 \times x_1$) and static moment of stator mass ($m_2 \times x_2$): $M \times x_C = m_1 \times x_1 + m_2 \times x_2$, if the origin of coordinates is combined with the center of mass of the mechanical system, then the static moment of mass of the system is equal to zero: $M \times x_{C0} = 0$.

When the rotor rotates with eccentricity e (Fig. 10.3.1 b) the abscissas of the centers of mass of the rotor x_1 and stator x_2 are related using the formula, which is also a dependence on the rotation angle φ of the rotor: $x_1 = -(e \times \sin\varphi - x_2)$.

In the absence of external influence, the static moment of mass of the electric motor is equal to zero when the origin of coordinates is aligned with the center of mass, which also depends on the eccentricity e on abscissas of the centers of mass of the rotor x_1 and stator x_2, on the rotation angle φ of the rotor, which is determined as:
$M \times x_C = -m_1 \times (e \times \sin\varphi - x_2) + m_2 \times x_2 = 0$.

At rotor turning with eccentricity e for the constancy of zero static mass moment of the mechanical system ($M \times x_C$) the horizontal displacement of the center of mass of the stator, not bolted to a smooth foundation, one can find as (Fig. 10.3.2 a):
$x_2 = \frac{m_1}{m_1+m_2} \times e \times \sin(\omega \times t)$.

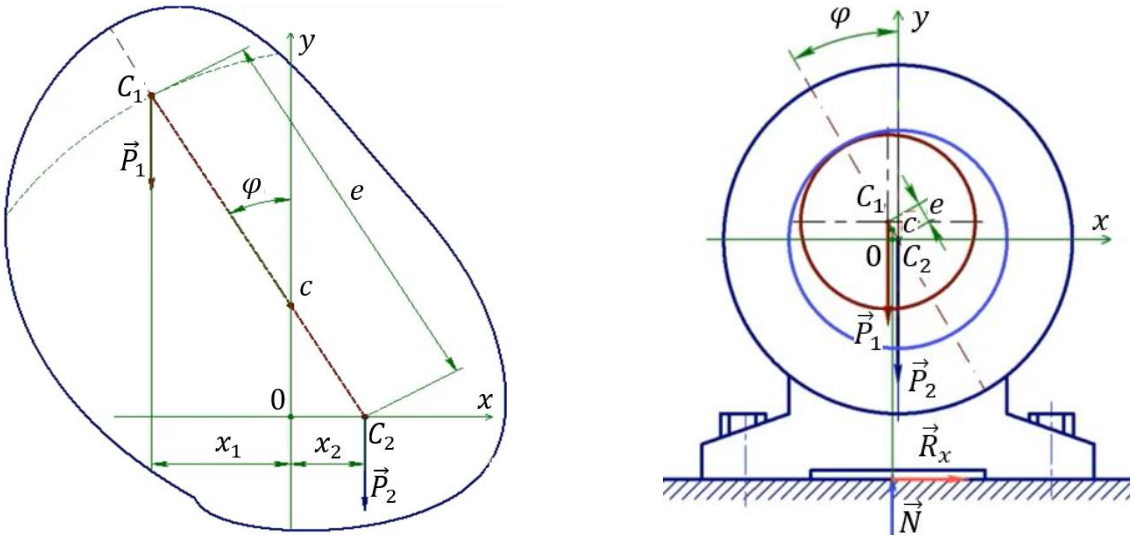


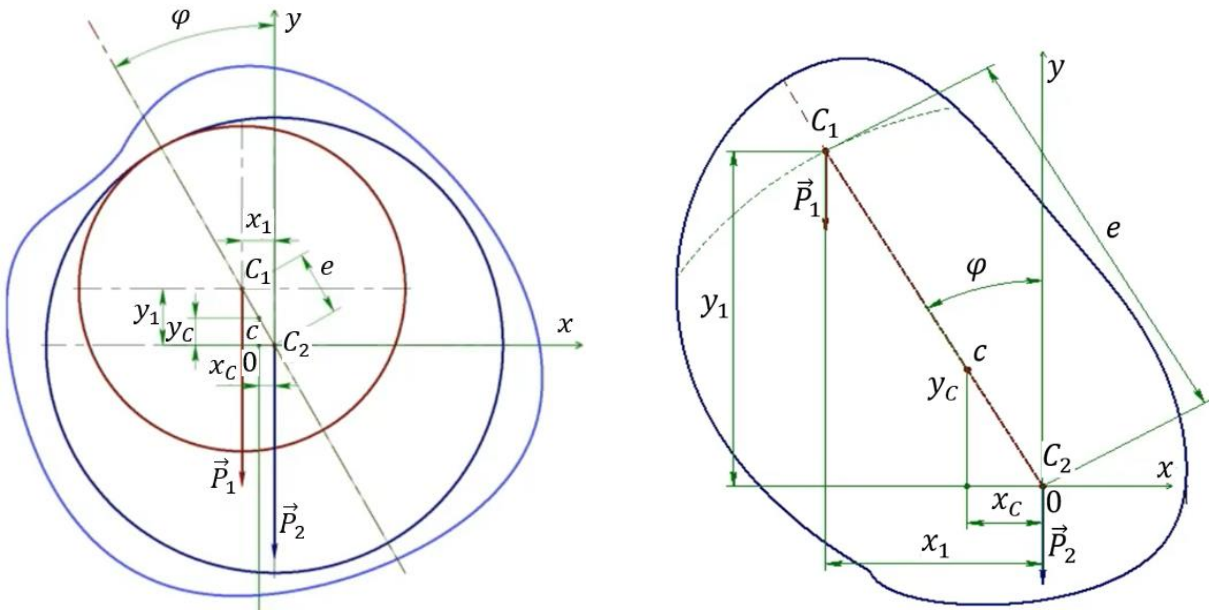
Fig. 10.3.2

If the stator is fixed (Fig. 10.3.2 b), then when the rotor rotates with eccentricity e the horizontal shear force will act on the stator mounting bolts $\vec{R}_x$.

The horizontal shear force magnitude R_x is equal, on one hand, to the horizontal projection of total inertia force acting on the mechanical system, and on other hand, to the sum of external forces counteracting the resulting inertia force: $M \times \ddot{x}_C = \sum F_{kx} = R_x$.

Since the static moment of mass of the electric motor ($M \times x_C$) is equal to the sum of the static moments of mass of the rotor ($m_1 \times x_1$) and stator ($m_2 \times x_2$), then when the stator is fixed (at $x_2 = 0$) the static moment of mass of the electric motor one can find as: $M \times x_C = m_1 \times x_1 + m_2 \times x_2 = m_1 \times e \times \sin\varphi + m_2 \times 0$.

At the movement of the center of mass of the rotor, taking into account the eccentricity e of its rotation (Fig. 10.3.3): $e \times \sin\varphi = e \times \sin(\omega \times t)$ the horizontal component of inertial force acting on the rotor and transmitted to the stator mounts equals:
$R_x = M \times \ddot{x}_C = -m_1 \times e \times \omega^2 \times \sin(\omega \times t)$,
and the amplitude of the horizontal component of inertial force is: $R_x^{max} = m_1 \times e \times \omega^2$.



(a) (b)

Fig. 10.3.3

Opening of the joint supporting a loose motor on the surface can occur when the amplitude of the force of inertia at the rotation of the rotor with eccentricity e exceeds the weight of the mechanical system (the electric motor, including rotor and stator).

Since the stator is stationary, then the amplitude of the force of inertia is equal: $m_1 \times e \times \omega^2$ in the upper phase position of turning rotor, taking into account the eccentricity e of its rotation.

Since the weight of the mechanical system in total is equal: $(m_1 + m_2) \times g$, then the condition for opening joint of the installed electric motor, which is held from horizontal displacement by fasteners and from vertical opening by its own weight, has the form: $m_1 \times e \times \omega^2 = (m_1 + m_2) \times g$.

The angular velocity of rotation with eccentricity e of the electric motor rotor, at which the joint opens, and the electric motor can rise above the supporting surface when the rotor, shifted by the eccentricity e, passes the upper position equals: $\omega = \sqrt{\frac{(m_1+m_2)\times g}{m_1\times e}}$.

Questions and comments:

10.3.1. Eccentricity of rotating elements of mechanical systems, which causes cyclic motions of other elements of the system and the cyclic reactions acting from fixed supports.

10.3.2. Critical elements of motion of rotating elements of mechanical systems, in which the reactions of the supports have limiting values and disconnections are possible of open supports in which the mechanical system is held by weight or pressing forces.

10.3.3. Finding the displacements of the stator, not fixed on a horizontal surface, when the rotor turns with eccentricity, determined considering the constancy of static moment of the rotor and stator total mass, which are not affected by external forces, as a result of which their total acceleration in the plane of the supporting surface is zero, and the velocity of common center of mass is equal to the initial velocity (also zero) and the movement of the common center of mass does not occur.

10.3.4. Determination of reaction in the support when the stator is secured, preventing its horizontal movements, considering the angular velocity of the rotor, it mass and eccentricity.

10.3.5. Opening of joint supporting a loose motor on a horizontal surface, which occurs when, when passing the upper semicircle of the trajectory of the center of mass of a rotor turning with eccentricity, if projection of the centrifugal force exceeds the total weight of the mechanical system (the rotor and stator of motor).

10.4. Indicators (the moments) of inertia of rotational motion of solid bodies, total, proper and transfer moments of inertia, the Huygens–Steiner theorem

The indicators of moments of inertia of bodies are used as the characteristics of the distribution of their masses and the distribution of masses of mechanical systems when considering their rotational motions. In particular, the polar moment of inertia of mass for the mechanical system of rigidly connected elements at points M_k relative to the point 0 is equal: $I_0 = \sum m_k \times r_k^2$, where m_k – the mass of the k–th element of the mechanical system, which is concentrated at the point M_k (Fig. 10.4.1), r_k– the radius–vector to the mass of the point M_k, in which the mass of the k–th element of the system is concentrated from the point 0, relative to which the polar moment of inertia of the mass is sought.

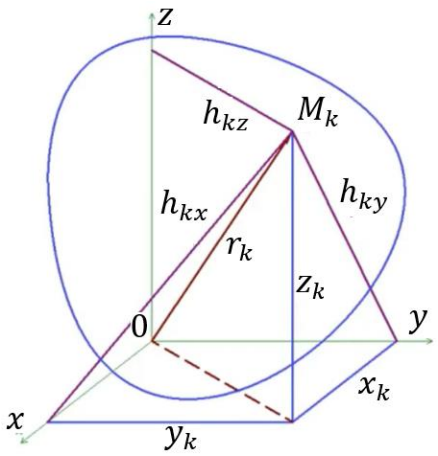


Fig. 10.4.1

For a rigid body, the polar moment of inertia of mass relative to the point 0 is equal: $I_0 = \int r^2 \times \rho \times dV$, where r – radius–vector from the point 0, relative to which the polar moment of inertia is sought, to the point at which the elementary mass is concentrated ($\rho \times dV$ – the mass in an elementary volume dV at mass density ρ of a solid body), the integral is determined by the volume of the body V. The moment of inertia of mass is a scalar value, regardless of whether it is determined relative to point (the polar moment of inertia) or one relative to axis (the axial moment of inertia).

Radius of inertia ρ_I is equal: $\rho_I = \sqrt{I/M}$, where: $M = \sum m_k$ – the total mass of the mechanical system of rigidly connected elements at points M_k or mass of a solid body. Moment of inertia I (polar or axial) one can determine if the radius of inertia and mass of the rigid body (or of the mechanical system of rigidly coupled elements) are known: $I = \rho_I^2 \times M$.

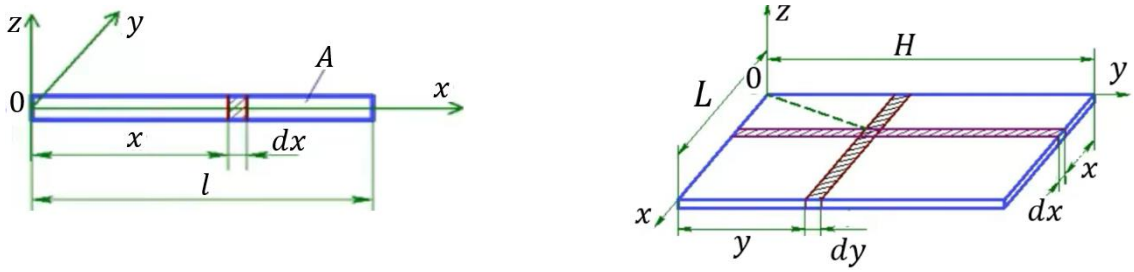
Moment of inertia of mass of the mechanical system of rigidly connected elements relative to the l axis (that is the axial moment of inertia) is equal: $I_l = \sum m_k \times h_k^2$, where h_k– the distance from the axis relative to which the moment of inertia is sought to the point M_k, in which the mass of the k–th element of the system is concentrated, and the moment of inertia of the solid body is equal: $I = \int h^2 \times \rho \times dV$, where h – the distance from the axis l, relative to which the moment of inertia is sought to the point at which the elementary mass is concentrated ($\rho \times dV$), the integral is determined by the volume V of body.

Finding in the Cartesian coordinate system the axial moments of masses inertia (I_x, I_y and I_z) of the mechanical system of rigidly connected elements is as follows: $I_x = \sum m_k \times (y_k^2 + z_k^2)$, $I_y = \sum m_k \times (z_k^2 + x_k^2)$ and $I_z = \sum m_k \times (x_k^2 + y_k^2)$ and axial moments of inertia of the mass of a solid body (by the integration over the volume of the body V): $I_x = \int_0^V (y^2 + z^2) \times \rho \times dV$, $I_y = \int_0^V (z^2 + x^2) \times \rho \times dV$ and $I_z = \int_0^V (x^2 + y^2) \times \rho \times dV$.

Polar moment of inertia mass of the mechanical system of rigidly coupled elements relative to the origin 0 is equal: $I_0 = \sum m_k \times (x_k^2 + y_k^2 + z_k^2)$, polar moment of inertia of a rigid body relative to the origin 0 is determined as: $I_0 = \int_0^V (x^2 + y^2 + z^2) \times \rho \times dV$.

The polar moment of inertia of mass is related to the axial moments of inertia, as: $2 \times I_0 = I_x + I_y + I_z$.

The definition of moments of inertia of the mass of a thin homogeneous rod is illustrated by the diagram (Fig. 10.4.2 a).



(a) (b)

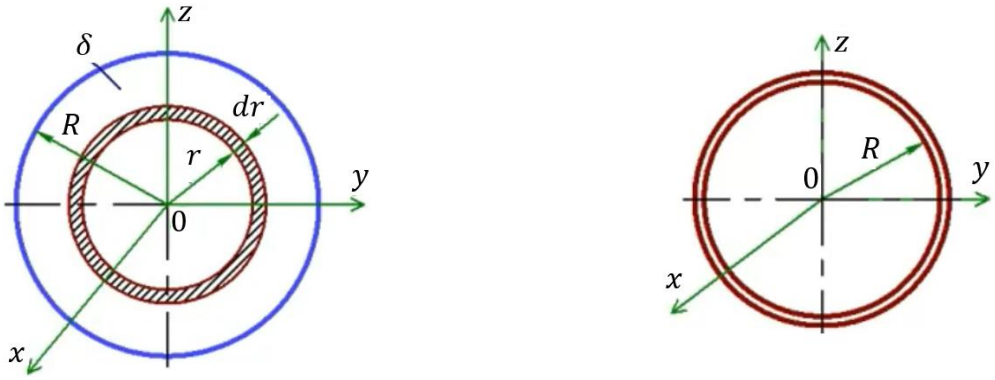
Fig. 10.4.2

The increment in the mass of a thin homogeneous rod (which is the mass of the element of the length of the rod) equals: $dm = \rho \times A \times dx$, where A – the cross–sectional area of homogeneous rod. The rod mass equals: $m = \rho \times A \times l$, where l – the rod length. The moments of inertia of the mass of the rod (I_y and I_z) relative to the axes: y and z, passing perpendicular to the axis of the rod through one of its ends, and the polar moment of inertia I_0 relative to one of the ends of a thin homogeneous rod are equal:

$I_y = I_z = I_0 = \int_0^m x^2 \times dm = \rho \times A \times \int_0^l x^2 \times dx = \rho \times A \times \frac{l^3}{3} = \frac{m \times l^2}{3}$ and $I_x = 0$.

Moments of inertia of the mass of rectangular plate having dimensions: H and L (Fig. 10.4.2 b) one can find relative to the coordinate axes: x, y and z, two of which (x and y) pass along the sides of the plate, by integrating: $I_x = \frac{m \times H^2}{3}$ and $I_y = \frac{m \times L^2}{3}$ and by double integration: $I_z = \int_0^H \int_0^L (x^2 + y^2) \times \rho \times dx \times dy = \frac{m \times (H^2 + L^2)}{3}$.

When finding the moments of inertia of the mass of a solid homogeneous disk (Fig. 10.4.3 a) of uniform thickness δ as an elementary mass, it is relevant to determine the mass of selected ring with a width dr: $dm = \rho \times \delta \times 2 \times \pi \times r \times dr$.



(a) (b)

Fig. 10.4.3

The moment of inertia of mass of homogeneous disk I_x relative to the coordinate axis x and the polar moment of inertia I_0 are equal:

$I_x = I_0 = \int_0^m r^2 \times dm = \rho \times \delta \times 2 \times \pi \times \int_0^R r^3 \times dr = \rho \times \delta \times 2 \times \pi \times \frac{R^4}{4} = \frac{m \times R^2}{2}$,

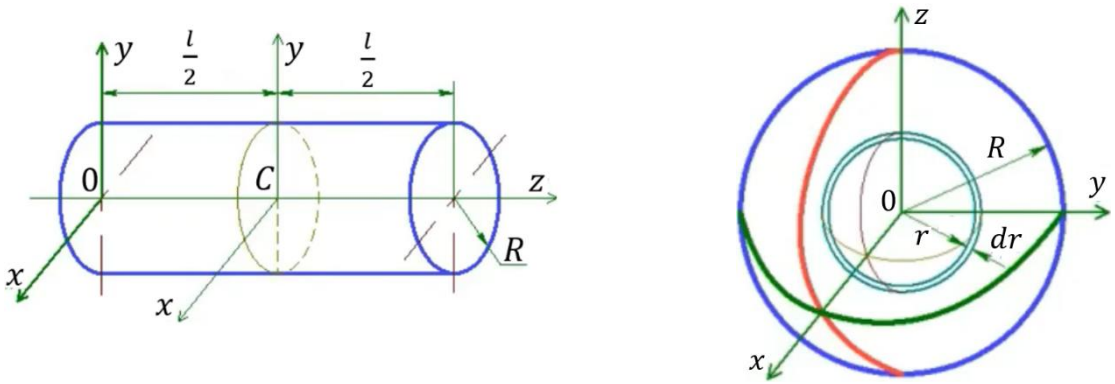
where m – disk mass: $m = \rho \times \pi \times R^2 \times \delta$.

Similarly, one can determine the moments of inertia of the mass of a homogeneous disk (I_y and I_z) relative to the coordinate axes: y and z, which are equal: $I_y = I_z = \frac{m \times R^2}{4}$.

The moments of inertia of the mass of a round thin ring (Fig. 10.4.3 b) relative to the mutually perpendicular coordinate axes: x, y and z, which are the axes of its symmetry, and the polar moment of inertia of mass of the ring are equal: $I_x = I_0 = m \times R^2$ and $I_y = I_z = \frac{m \times R^2}{2}$.

Taking into account the symmetry properties of geometric figures, highlighting the elements of their volumes dV and thus determining the elements of mass dm, one can find by integration the moments of inertia of mass of these figures.

Relative to the mutually perpendicular coordinate axes: x, y and z, which are the axes of symmetry of a circular cylinder (Fig. 10.4.4 a), if pass through its center of gravity C, or through the center 0 of the circular base of the cylinder (with radius R and length l) the moments of inertia of mass of the cylinder and the polar moment of inertia are equal: $I_{Cz} = \frac{m \times R^2}{2}$, $I_{Cx} = I_{Cy} = \frac{m \times l^2}{12}$ (at the passage of coordinate axes through the center of gravity of the cylinder, located at its geometric center) and $I_{0x} = I_{0y} = \frac{m \times l^2}{3}$ – when the coordinate axes: x and y pass in the base of a circular cylinder.

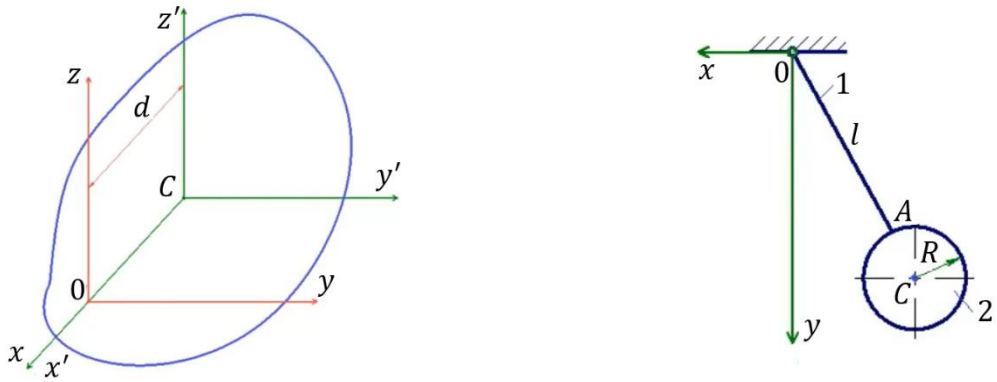


(a) (b)

Fig. 10.4.4

With respect to the mutually perpendicular coordinate axes: x, y and z, which are the axes of symmetry of a solid round ball (Fig. 10.4.4 b), its moments of inertia of mass and polar moment of inertia are equal: $I_x = I_y = I_z = \frac{2}{5} \times m \times R^2$ and $I_0 = \frac{3}{5} \times m \times R^2$, and the mass m of ball is determined by the well–known formula: $m = \rho \times \frac{4}{3} \times \pi \times R^3$.

Theorem of the Huygens–Steiner. Moment of inertia of the mass of the mechanical system of rigidly coupled elements or moment of inertia of a solid body relative to any axis is equal to the sum of the own moment of inertia $I_{Cz'}$ of mechanical system or body relative to a parallel axis passing through the center of mass, and the transferring moment of inertia (product of mass M by square of the distance d^2 between these axes, Fig. 10.4.5 a): $I_{0z} = I_{Cz'} + M \times d^2$.



(a) (b)

Fig. 10.4.5

Exercise The exercise assigns one to determine the moment of inertia of mass I_{0z} relative to an axis passing perpendicular to the plane of the diagram through point 0 of the mechanical system (the pendulum, Fig. 10.4.5 b), consisting of a thin homogeneous rod 1 of mass m and length l and a homogeneous disk 2 of mass m and radius: $R = l/4$.

Moment of inertia of mass I_{0z} of the mechanical system, consisting of two elements rigidly connected to each other, is the sum of two moments of inertia (I_1 and I_2) for both elements of the mechanical system, and for each element of the mechanical system the moment of inertia mass in the general case, is the sum of its own moment of inertia relative to its center of mass and the transferring moment of inertia relative to the comparison axis (which passes through the hinged pendulum in the issue being solved). The sum of the own and transferring moments of mass inertia for the rod is the solution: $I_1 = \frac{m \times l^2}{3}$.

The sum of the own and transferring moments of inertia of the mass of rod of a homogeneous disk 2 of mass m and radius: $R = l/4$ relative to the axis passing through the point 0 equals: $I_2 = \frac{m}{2} \times \left(\frac{l}{4}\right)^2 + m \times \left(\frac{5 \times l}{4}\right)^2 = \frac{51 \times m \times l^2}{32}$.

Moment of inertia mass I_{0z} of the mechanical system is equal:

$I_{0z} = I_1 + I_2 = \frac{m \times l^2}{3} + \frac{51 \times m \times l^2}{32} = \frac{185}{96} \times m \times l^2$.

Questions and comments:

10.4.1. Moments of inertia of the mass of mechanical systems and their elements, which are characteristics of the distribution of masses in them and scalar characteristics of their inertia in rotational motion.

10.4.2. Moments of inertia of the mass relative to the coordinate axes and relative to the pole of rotation (the polar moment of inertia). The radii of inertia relative to axes and the polar radius of inertia.

10.4.3. Axial and polar moments of inertia of a mechanical system consisting of rigidly connected elements.

10.4.4. Axial moments of inertia of mass of the rod relative to perpendicular axes passing through its end perpendicular to the axis of the rod, polar moment of inertia of the rod relative to the origin of coordinates in this system of axes.

10.4.5. Dependence on the dimensions of a rectangular plate of the axial moments of inertia of the mass relative to the coordinate axes passing along the sides of this plate, and relative to the axis passing through one of the vertices of the rectangular shape perpendicular to the plane of the plate.

10.4.6. Moments of inertia of mass of flat solid homogeneous disk relative to perpendicular axes passing through the center of the disk in its plane, as well as the moment of inertia of disk mass relative to the axis passing through the center of the disk perpendicular to its plane, which is equal to the polar moment of inertia of this disk.

10.4.7. Moments of inertia of the mass of ring: the polar and axial moments corresponding to the difference between these moments of inertia of flat solid homogeneous disks (external and internal), the geometric difference of which is the ring.

10.4.8. Moments of inertia of the mass of round solid cylinder relative to the cylinder axis, relative to the axes passing through the center in the plane of the base of the cylinder, and relative to the axes passing through the center of gravity of the cylinder, perpendicular to the cylinder axis.

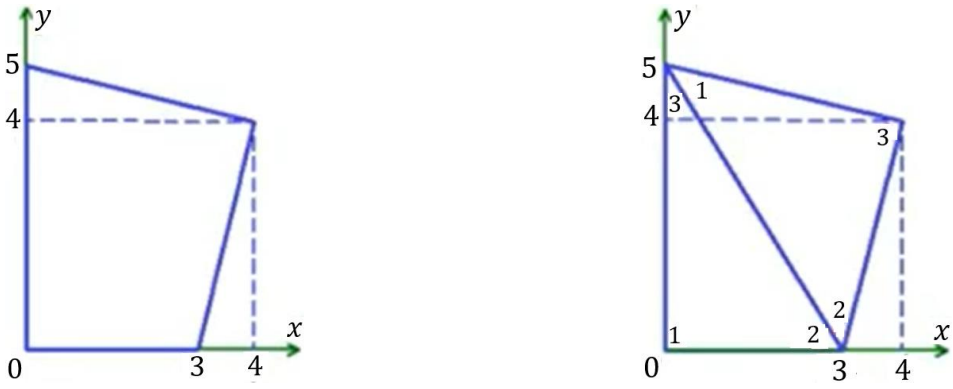
10.4.9. Moments of inertia of the mass of the ball relative to the coordinate axes passing through the center of the ball and the polar moment of inertia of the mass of the ball relative to its center.

10.4.10. Theorem of the Huygens–Steiner about the moment of inertia of the mass of a mechanical system consisting of rigidly connected elements, equal to the sum of the own moments of inertia of each of the elements in the coordinate system associated with the center of gravity of this element and, for axial moments of inertia of the mass, parallel to that axis, relative to which the moment of inertia of the mechanical system is sought, and the transferring moments of inertia, each of which is equal to the product of the mass of this element by the square of the distance from its center of gravity to the axis with respect to which the moment of inertia of the mechanical system is sought (or by the square of the distance to the point with respect to which it is determined polar moment of inertia of the system).

10.5. Determining the position of the center of gravity by coordinates of body shape using computer algebra Mathcad

Exercise 1. The exercise assigns one to determine the area and coordinates of center of the area of the figure (which is the center of gravity of the plane figure shown on the diagram, Fig. 10.5.1 a).

The specified flat figure one can represent as consisting of two triangular elements, vertices, which it is advisable to number (Fig. 10.5.1 b). To perform calculations, it is convenient to use the Mathcad computer algebra system.



(a) (b)

Fig. 10.5.1

The coordinates of the vertices of triangular elements are specified by matrices in the Mathcad (see fragment of page in the Mathcad, Fig. 10.5.2).

$$r := 2 \qquad x1 := \begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \qquad y1 := \begin{bmatrix} 0 & 0 & 5 \\ 5 & 0 & 4 \end{bmatrix}$$

Fig. 10.5.2

Since the given figure for performing calculations is represented in the form of two triangles, then the number of rows in each of the coordinate matrices ($x1$ and $y1$) is two: the number of rows of the matrices is equal to number of triangular elements, and the number of columns is the number of vertices of triangular element.

To determine the projections of the lengths of the sides of both triangles on the coordinate axes (x and y) it is necessary to do the auxiliary matrices (see fragment of the page in the Mathcad, Fig. 10.5.3).

$$x12 := \text{submatrix}(x1, 1, 2, 2, 3) \qquad y12 := \text{submatrix}(y1, 1, 2, 2, 3)$$

$$x2 := \text{augment}\left(x12, x1^{\langle 1 \rangle}\right) \qquad y2 := \text{augment}\left(y12, y1^{\langle 1 \rangle}\right)$$

$$x2 = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} \qquad y2 = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 4 & 5 \end{bmatrix}$$

$$x := x1 - x2 \qquad y := y1 - y2$$

$$x = \begin{bmatrix} -3 & 3 & 0 \\ -3 & -1 & 4 \end{bmatrix} \qquad y = \begin{bmatrix} 0 & -5 & 5 \\ 5 & -4 & -1 \end{bmatrix}$$

Fig. 10.5.3

The parameters of the operation: submatrix(A, ir, jr, ic, jc) are: ir and jr – the extreme row numbers of the original matrix A; ir, jr – the extreme column numbers extracted by the operation: submatrix. The parameters of the operation: augment(A, B): A and B – the matrices A and B to be merged by adding the columns of matrix B to the columns of matrix A.

Determining the lengths of the sides of triangular elements as part of an assigned plane figure in the Mathcad and redefining the symbols of side lengths is shown on a fragment of the page (Fig. 10.5.4).

$$l := \overrightarrow{\sqrt{x^2 + y^2}} \qquad l = \begin{bmatrix} 3 & 5.831 & 5 \\ 5.831 & 4.123 & 4.123 \end{bmatrix} \qquad a := l^{\langle 1 \rangle} \qquad b := l^{\langle 2 \rangle} \qquad c := l^{\langle 3 \rangle}$$

Fig. 10.5.4

When determining the lengths l of sides, in addition to extracting the root of the sum of the squares of the projections of the sides, one needs to apply vectorization so that the operator for determining the lengths of the sides includes the data of projections of the lengths of sides of both triangles on coordinate axes (x and y) as components of the corresponding matrices, and not as integral ones.

The areas of triangular elements in the assigned plane figure one can find using the formula Heron's, which is transformed (fragment of the page in the Mathcad, Fig. 10.5.5).

$$S := \frac{1}{4} \cdot \sqrt{\left(a^2 + b^2 + c^2\right)^2 - 2 \cdot \left(a^4 + b^4 + c^4\right)} \qquad S = \begin{bmatrix} 7.5 \\ 8.5 \end{bmatrix}$$

Fig. 10.5.5

Coordinates of the area centers (x_C and y_C) of each of the triangular parts in the assigned plane figure: $x_C = \frac{x_1+x_2+x_3}{3}$ and $y_C = \frac{y_1+y_2+y_3}{3}$, where: x_1, x_2 and x_3 and y_1, y_2 and y_3 – coordinates of the vertices of each of both triangular parts (see fragment of the page in the Mathcad, Fig. 10.5.6).

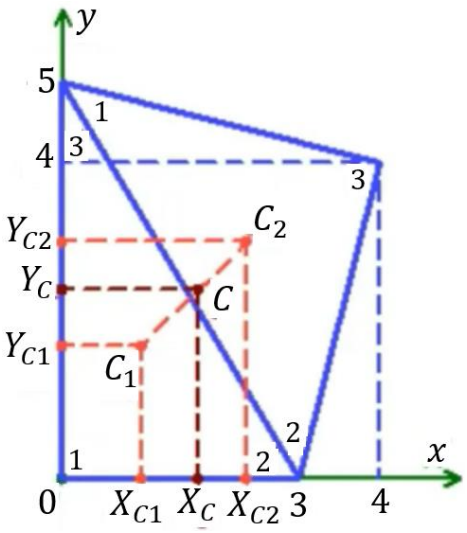
$$i := 1 \,..\, r \qquad j := 1 \,..\, 3$$

$$fx(i) := \frac{1}{3} \cdot \sum_{j=1}^{3} x1_{i,j} \qquad fy(i) := \frac{1}{3} \cdot \sum_{j=1}^{3} y1_{i,j}$$

$$xc := \left\| \begin{array}{l} i \leftarrow 1 \\ \text{for } i \in 1 \,..\, r \\ \quad \left\| \begin{array}{l} a_i \leftarrow fx(i) \\ i \leftarrow i + 1 \end{array} \right. \\ a \end{array} \right. \qquad yc := \left\| \begin{array}{l} i \leftarrow 1 \\ \text{for } i \in 1 \,..\, r \\ \quad \left\| \begin{array}{l} a_i \leftarrow fy(i) \\ i \leftarrow i + 1 \end{array} \right. \\ a \end{array} \right.$$

Fig. 10.5.6

The found coordinates of the area centers (x_{C1}, y_{C1} and x_{C2}, y_{C2}) of each of the r triangular parts in the given plane figure (Fig. 10.5.7 a), as well as the found coordinates of center of the area of given figure (x_C and y_C) are indicated on fragment of the page in the Mathcad (Fig. 10.5.7 b).



(a)

$$xc = \begin{bmatrix} 1 \\ 2.333 \end{bmatrix} \qquad yc = \begin{bmatrix} 1.667 \\ 3 \end{bmatrix}$$

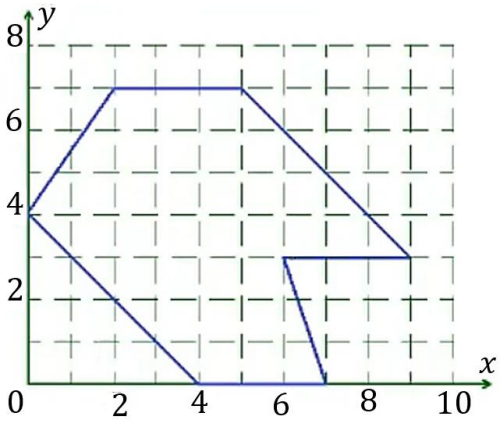
$$Xc := \frac{\sum_{i=1}^{r} xc_i \cdot S_i}{\sum_{i=1}^{r} S_i} \qquad Xc = 1.708$$

$$Yc := \frac{\sum_{i=1}^{r} yc_i \cdot S_i}{\sum_{i=1}^{r} S_i} \qquad Yc = 2.375$$

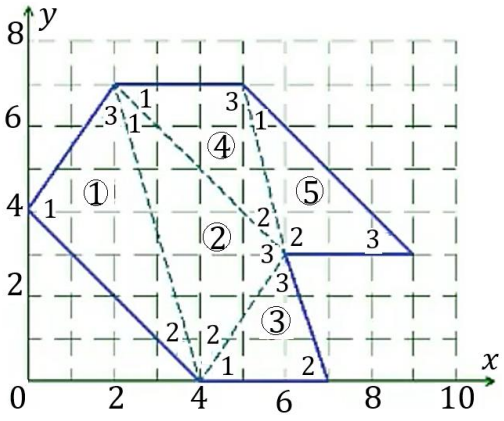
(b)

Fig. 10.5.7

Exercise 2. The exercise assigns one to determine the area and coordinates of the center of area of the figure shown in the diagram (Fig. 10.5.8 a).



(a)



(b)

Fig. 10.5.8

Assigned flat figure one can represent as consisting of five triangular elements, the vertices of which are numbered on the diagram (Fig. 10.5.8 b).

Since the given figure for performing calculations is represented in the form of two triangles, then the number of rows in each of the coordinate matrices ($x1$ and $y1$) is five: the number of rows of the matrices is equal to the number of triangular elements, and the number of columns is the number of vertices of the triangular element.

To determine the projections of the lengths of the sides of five triangles on the coordinate axes (x and y) it is required to do auxiliary matrices (see fragment of the page in the Mathcad, Fig. 10.5.9).

$$r := 5 \qquad x1 := \begin{bmatrix} 0 & 4 & 2 \\ 2 & 4 & 6 \\ 4 & 7 & 6 \\ 2 & 6 & 5 \\ 5 & 6 & 9 \end{bmatrix} \qquad y1 := \begin{bmatrix} 4 & 0 & 7 \\ 7 & 0 & 3 \\ 0 & 0 & 3 \\ 7 & 3 & 7 \\ 7 & 3 & 3 \end{bmatrix}$$

$$x12 := \text{submatrix}(x1, 1, 5, 2, 3) \qquad y12 := \text{submatrix}(y1, 1, 5, 2, 3)$$

$$x2 := \text{augment}\left(x12, x1^{\langle 1 \rangle}\right) \qquad y2 := \text{augment}\left(y12, y1^{\langle 1 \rangle}\right)$$

$$x2 = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 2 \\ 7 & 6 & 4 \\ 6 & 5 & 2 \\ 6 & 9 & 5 \end{bmatrix} \quad y2 = \begin{bmatrix} 0 & 7 & 4 \\ 0 & 3 & 7 \\ 0 & 3 & 0 \\ 3 & 7 & 7 \\ 3 & 3 & 7 \end{bmatrix} \qquad \begin{matrix} x := x1 - x2 \\ y := y1 - y2 \end{matrix} \quad x = \begin{bmatrix} -4 & 2 & 2 \\ -2 & -2 & 4 \\ -3 & 1 & 2 \\ -4 & 1 & 3 \\ -1 & -3 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 4 & -7 & 3 \\ 7 & -3 & -4 \\ 0 & -3 & 3 \\ 4 & -4 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

Fig. 10.5.9

The areas of triangular elements in the assigned plane figure are determined using the Heron's transformed formula. Found areas of triangular elements and coordinates of the area centers (x_{C1} and y_{C1}, x_{C2} and y_{C2}, x_{C3} and y_{C3}, x_{C4} and y_{C4}, x_{C5} and y_{C5}, Fig. 10.5.10), as well as the resulting coordinates of center of the area of assigned figure (x_C and y_C) are given on a fragment of the Mathcad page (Fig. 10.5.11).

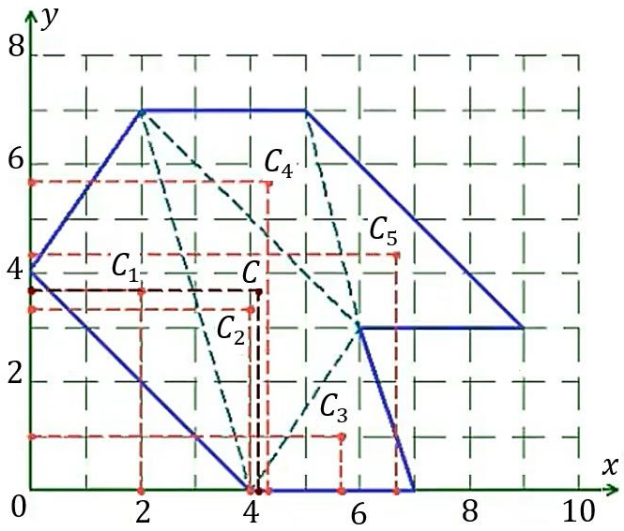


Fig. 10.5.10

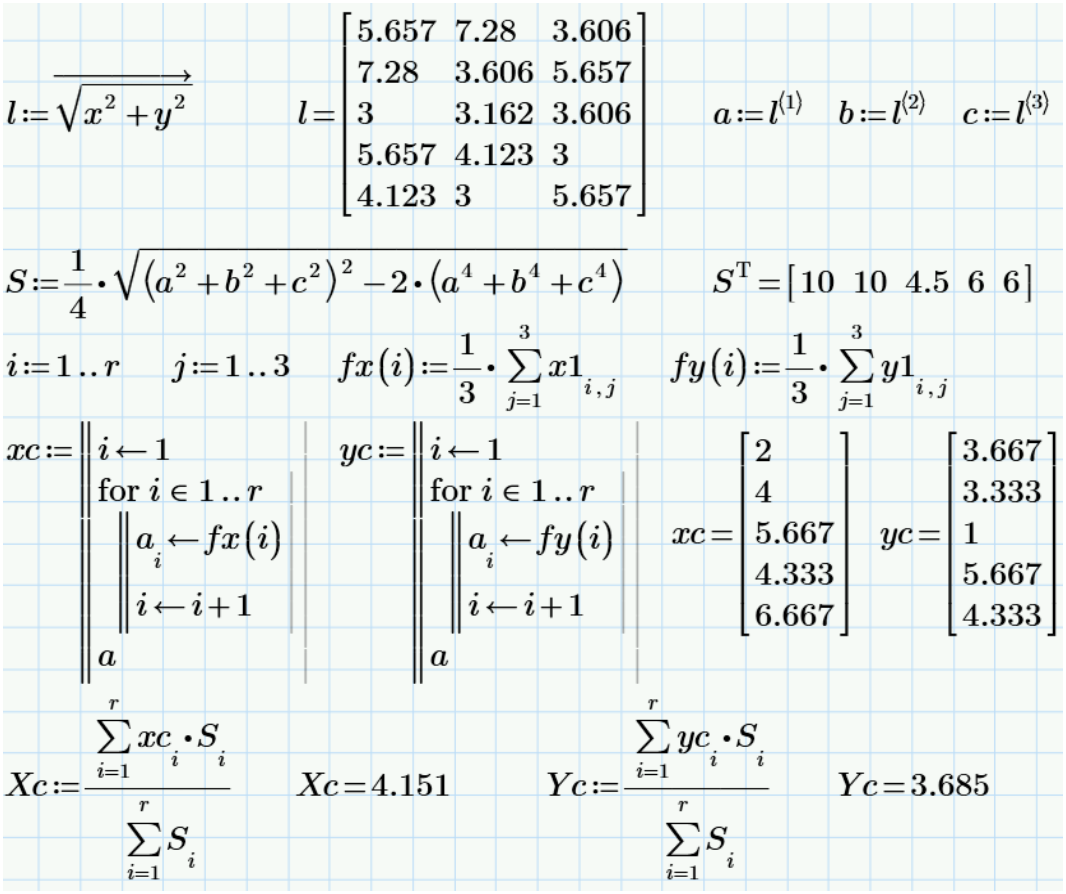


Fig. 10.5.11

Questions and comments:

10.5.1. Specifying the coordinates of the vertices of parts of a plane figure using matrices in the computer algebra system Mathcad. The operations use with matrices in the Mathcad: submatrix and augment to do an auxiliary matrix and to determine projections on the axes of the sides of triangular figures that are parts of the original figure, for which the coordinates of center of the area are determined.

10.5.2. The lengths of the sides and areas of each of the triangular parts of a given flat figure, as well as the coordinates of the centers of the area of these parts, determined in the Mathcad. Using Heron's transferred formula to find the areas of triangles from the lengths of their sides.

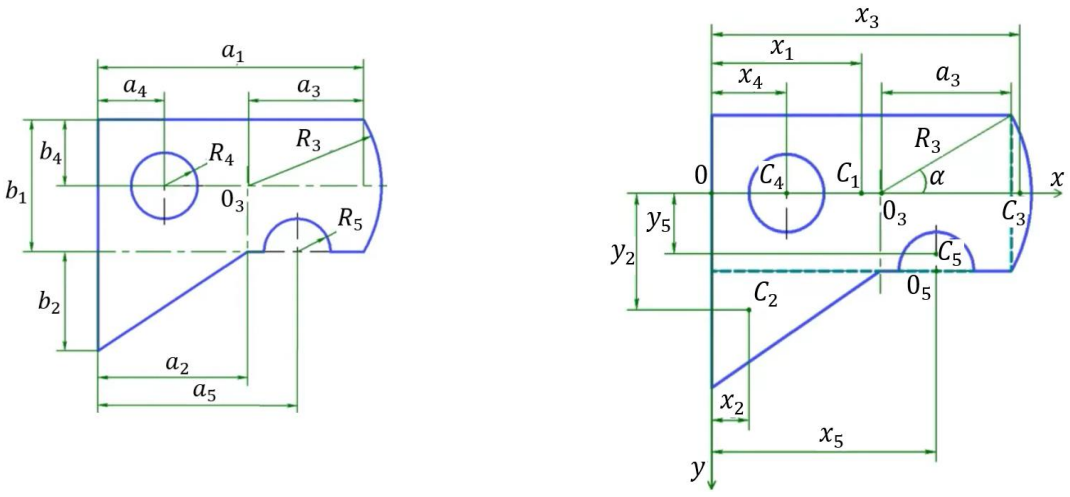
10.5.3. Finding in the Mathcad the coordinates of center of the area of specified figure, considered as consisting of parts, by dividing the total static moments of this figure relative to each of the axes by the total area of the figure.

10.6. The use the addition method to find center of the area of a plane figure

Exercise. The exercise assigns one to determine the coordinates of center of the area of plane figure shown on the scheme (Fig. 10.6.1 a), using the addition method. The assigned flat figure one can represent as consisting of the rectangle with sides a_1 and b_2 and a circular segment of radius R_3, in which a circular hole of radius R_4 and a cutout in the form of a semicircle of radius R_5 are made. The dimensions of a flat figure are shown in the Table 10.6.1, cm.

Table 10.6.1

a_1	b_1	a_2	b_2	a_3	R_3	a_4	b_4	R_4	a_5	R_5
80	40	45	30	34.64	40	20	20	10	60	10



(a) (b)

Fig. 10.6.1

The area S and the coordinates of center of the area (x_C and y_C) of plane figure consisting of five elements one can find using the formulas: $S = \sum_{k=1}^{5} S_k$, $x_C = \frac{\sum_{k=1}^{5} x_k \times S_k}{S}$ and $y_C = \frac{\sum_{k=1}^{5} y_k \times S_k}{S}$, where S_k – the area of k–th element of a plane figure, x_k and y_k – the abscissa and ordinate of center of the area of k–th element of the figure.

The coordinates of center and the area of rectangle 1 (Fig. 10.6.1 a) are determined as: $x_1 = \frac{a_1}{2} = \frac{80}{2} = 40$, cm $y_1 = 0$ and $S_1 = a_1 \times b_1 = 80 \times 40 = 3200$, cm^2.

Coordinates of center and the area of triangle 2 are equal: $x_2 = \frac{a_2}{3} = \frac{45}{3} = 15$, cm, $y_2 = \frac{b_1}{2} + \frac{b_2}{3} = \frac{40}{2} + \frac{30}{3} = 30$, cm and $S_2 = \frac{1}{2} \times a_2 \times b_2 = \frac{1}{2} \times 45 \times 30 = 675$, cm^2.

The coordinates of the center of the area and the area of circular segment 3 are determined as: $a_3 = \sqrt{R_3^2 - \left(\frac{b_1}{2}\right)^2} = \sqrt{40^2 - 20^2} = 34.64$, cm,

$OO_3 = b_1 - a_3 = 80 - 34.64 = 45.36$, cm, $\sin\alpha = \frac{b_1}{2\times R_3} = \frac{40}{2\times 40} = 0.5$, $\alpha = \frac{\pi}{6}$,

$O_3C_3 = \frac{4}{3} \times R_3 \times \frac{\sin^3\alpha}{2\times\alpha - \sin(2\times\alpha)} = \frac{4}{3} \times 40 \times \frac{0.5^3}{2\times\frac{\pi}{6} - \sin\frac{\pi}{3}} = 36.83$, cm,

$x_3 = OO_3 + O_3C_3 = 45.36 + 36.83 = 82.19$, cm and $y_3 = 0$

and $S_3 = \frac{1}{2} \times R_3^2 \times (2 \times \alpha - \sin(2 \times \alpha)) = \frac{1}{2} \times 40^2 \times \left(2 \times \frac{\pi}{6} - \sin\frac{\pi}{3}\right) = 14.48$, cm^2.

The coordinates of the center and the area of the circular cutout 4 are:

$x_4 = 0$, $y_4 = 0$ and $S_4 = -\pi \times R_4^2 = -\pi \times 10^2 = -314.16$, cm^2.

Center coordinates and area of semicircular cutout 5 are equal:

$x_5 = 60$, cm, $y_5 = \frac{b_1}{2} - \frac{4\times R_5}{3\times\pi} = \frac{40}{2} - \frac{4\times 10}{3\times\pi} = 15.76$, cm

and $S_4 = -\frac{\pi\times R_5^2}{2} = -\frac{\pi\times 10^2}{2} = -157.08$, cm^2.

The elements of a plane figure (x_k, y_k and S_k), used in determining its area S and the coordinates of the center of the area (x_C and y_C), are indicated in the Table 10.6.2.

Table 10.6.2

Figure	Rectangle 1	Triangle 2	Round segment 3	Circular neckline 4	Semicircular neckline 5
x_k, cm	40	15	82.19	20	60
y_k, cm	0	30	0	0	15.67
S_k, cm^2	3200	675	14.48	– 314.16	–157.08

Area of flat figure equals:

$S = \sum_{k=1}^{5} S_k = 3200 + 675 + 14.48 - 314.6 - 157.08 = 3418.2$, cm^2.

The coordinates of center of the area (x_C and y_C) of flat figure consisting of five elements shown on the scheme (Fig. 10.6.2) are equal:

$x_C = \frac{\sum_{k=1}^{5} x_k\times S_k}{S} = \frac{40\times 3200+15\times 675+82.19\times 14.48+20\times(-314.16)+60\times(-157.08)}{3418.2} = 36.16$, cm,

$y_C = \frac{\sum_{k=1}^{5} y_k\times S_k}{S} = \frac{0\times 3200+30\times 675+0\times 14.48+0\times(-314.16)+15.67\times(-157.08)}{3418.2} = 5.20$, cm.

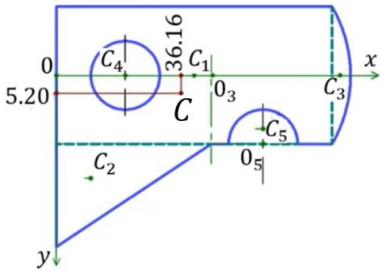


Fig. 10.6.2

Questions and comments:

10.6.1. An approach to find the geometric characteristics of complex figure consisting of regular geometric elements, considering both the material geometric elements and the cuts in them as elements.

10.6.2. The use of the the geometric properties of the material forms and the cuts in them to find the areas, coordinates of their centers, static moments of the areas relative to the coordinate axes, the total area of a flat figure, formed by geometric shapes and cuts, as well as the coordinates of center of the area of this figure.

10.7. The Momentum Change Theorem of a material point

The motion of the k–th material point of a certain mass m_k under the action of external forces $\vec{F}_k$, which determine the trajectory of the point motion, which can be specified by the radius vector $\vec{r}$, and the equation has the form: $m_k \times \frac{d^2\vec{r}}{dt^2} = \vec{F}_k$.

Total of forces acting on a mechanical system includes external forces $\sum \vec{F}_k^e$ (the main vector of external forces) and the internal forces $\vec{F}_k^i$, that is: $\sum \vec{F}_k = \sum \vec{F}_k^e + \sum \vec{F}_k^i$. The methodological obstacles may arise during finding the internal forces for non–free mechanical systems, for elements for which, in addition to external forces, the reactions of connections with other elements of the system are added.

The amount of mechanical motion (the momentum) of material point is a vector magnitude equal to product of mass of the point and its velocity (Fig. 10.7.1): $\vec{q} = m \times \vec{V}$.

The dimension of amount of mechanical motion and of the force momentum in the International System of Units (Système international d'unités, SI): kg×m/s or N×s.

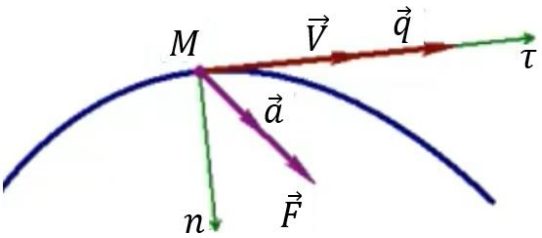


Fig. 10.7.1

Elementary momentum of the force $d\vec{S}$, transmitted to a material point over an elementary time interval dt (for a period of time corresponding to its differential increment) is vector value equal to the product of the force $\vec{F}$ by the time increment: $d\vec{S} = \vec{F} \times dt$.

The momentum transmitted by the force over a finite period of time t_1, is determined by integration and equals: $\vec{S} = \int_0^{t_1} \vec{F} \times dt$.

If the force acting on a material point is constant: $\vec{F} = const$, then impulse of the force during the period of time t_1 one can find by the product: $\vec{S} = \vec{F} \times t_1$.

Projections of the momentum vector of the force on the coordinate axes (x, y and z) are defined as: $S_x = \int_0^{t_1} F_x \times dt$, $S_y = \int_0^{t_1} F_y \times dt$ and $S_z = \int_0^{t_1} F_z \times dt$.

Scalar magnitude of impulse $\vec{S}$ of the force $\vec{F}$ equals: $S = \sqrt{S_x^2 + S_y^2 + S_z^2}$.

Using Newton's second law of mechanics (the inertial force acting on a body is equal to the product of its mass and acceleration), the differential form of the Momentum Change Theorem of a material point one can indicate by lowering the order of differentiation of the equation of the point motion: $m_k \times \frac{d^2\vec{r}}{dt^2} = \vec{F}_k$ by replacing the differentiable function of the coordinate $\vec{r}$ with the velocity function of the material point: $\frac{m_k \times d\vec{V}}{dt} = \vec{F}_k$ or $m_k \times d\vec{V} = \vec{F}_k \times dt$. The time derivative of the momentum of the point ($m_k \times \vec{V}$) is equal to the force acting on the point: $\vec{F}_k$.

Integrating the differential formula of the Momentum Change Theorem of a material point and substituting limits of integration: the velocity equals: $\vec{V} = \vec{V}_0$ at $t = t_0$ and the velocity equals: $\vec{V} = \vec{V}_1$ at $t = t_1$ one can get the integral form of material point Momentum Change Theorem: $m_k \times \int_{\vec{V}_0}^{\vec{V}_1} d\vec{V} = \int_{t_0}^{t_1} \vec{F}_k \times dt$.

The final integral form of material point Momentum Change Theorem is:
$m_k \times \vec{V}_1 - m_k \times \vec{V}_0 = \int_{t_0}^{t_1} \vec{F}_k \times dt$, or $m_k \times (\vec{V}_1 - \vec{V}_0) = \vec{S}_k$.

The change of material point momentum over a certain period of time is equal to the impulse of the external force acting on the point during this period of time.

In projections on the coordinate axes (x, y and z) the equation of the Momentum Change Theorem of a material point in the integral form is:
$m_k \times (V_{1x} - V_{0x}) = S_{kx}$, $m_k \times (V_{1y} - V_{0y}) = S_{ky}$ and $m_k \times (V_{1z} - V_{0z}) = S_{kz}$.

Questions and comments:

10.7.1. Momentum is a vector value equal to the product of material point mass (or the physical body, or mechanical system) and the material point velocity vector.

10.7.2. Non–free mechanical systems, the elements of which are acted not only by external forces, but also internal ones from other elements of the system.

10.7.3. The mechanical impulse of force is a vector value equal to time integral of the action of external force during the period of the force action. The elementary impulse corresponds to the product of force by the differential increment of the time parameter.

10.7.4. Impulse of the constant force vector equals to the product of force by time of it action – the duration of application of force to the mechanical system.

10.7.5. Dimension of indicators of momentum and the mechanical impulse under the action of an external force in the International System of Units SI is: kg×m/s, or N×s.

10.7.6. Differential form of the Momentum Change Theorem of a material point (the change in momentum over a certain time is equal to the mechanical impulse of the external force applied during this time), following from Newton's second law of the mechanics (the inertial force acting on a body is equal to the product of its mass by acceleration), if one considers acceleration is the value of time derivative of velocity and lowers the order of the derivative.

10.7.7. The Momentum Change Theorem in the form of certain integral (when substituting conditions at the boundaries of the time interval) and in the form of a solution to this integral, in which the change in momentum and velocity over a time interval corresponds to the mechanical impulse under the applied external force during this interval.

10.8. Momentum of mechanical system and the Momentum Change Theorem

The measure of motion of rigid body or the mechanical system is the amount of mechanical motion $\bar{Q}$ (the vector value), which is sum of the amounts of mechanical motions of all elements $\vec{q}_k$ of the system (or the integral of increments of the mass of mass of the solid body, on which the external forces act.): $\bar{Q} = \sum_{k=1}^{n} \vec{q}_k = \sum_{k=1}^{n} m_k \times \vec{V}_k$, where n – the number of all elements of mechanical system.

Momentum of the mechanical system (or the solid body) one can get in dependence on it mass: $M = \sum_{k=1}^{n} m_k$ as product by velocity $\vec{V}_C$ of the mass center: $\bar{Q} = M \times \vec{V}_C$.

The theorem on the change in momentum of the mechanical system is known in three forms. The first two forms of the theorem are differential, the third is integral.

Momentum Change Theorem, form 1. The time derivative of the momentum of the mechanical system is equal to the sum of all external forces (the main vector of external forces): $\sum \vec{F}_k^e$, acting on the system (for each of it element $\vec{F}_k^e$): $\frac{d\bar{Q}}{dt} = \sum \vec{F}_k^e$.

The change in the momentum of the k–th element of the mechanical system is equal to the sum of the forces acting on the element of the force system: the external $\vec{F}_k^e$ and internal $\vec{F}_k^i$, which is acting on the part of other $(n-1)$ elements of the system: $\frac{d(m_k \times \vec{V}_k)}{dt} = \vec{F}_k^e + \vec{F}_k^i$, where k =1, …, n.

At the summation of the forces acting on all elements of the system, the change in momentum is determined: $\frac{d\bar{Q}}{dt} = \sum_{k=1}^{n} \frac{d(m_k \times \vec{V}_k)}{dt} = \sum \vec{F}_k^e$, because the sum of internal forces $\vec{F}_k^i$, acting on each element of the mechanical system from other $(n-1)$ elements is equal to zero: $\sum \vec{F}_k^i = 0$.

In projections on the coordinate axes, the Momentum Change Theorem under the influence of external forces on mechanical system has the form: $\frac{dQ_x}{dt} = \sum F_{kx}^e$, $\frac{dQ_y}{dt} = \sum F_{ky}^e$ and $\frac{dQ_z}{dt} = \sum F_{kz}^e$.

Momentum Change Theorem, form 2. The increment in momentum $d\bar{Q}$ of mechanical system over a period of differential time increment is equal to the total mechanical impulse $d\vec{S}_k^e$ under the main vector of external forces: $\sum \vec{F}_k^e$, acting on the system during period of time increment dt. This definition follows from the theorem in the form 1, if both sides of the equation: $\frac{d\bar{Q}}{dt} = \sum \vec{F}_k^e$ is to be multiplied by the period of time increment dt, that is: $d\bar{Q} = \sum \vec{F}_k^e \times dt = \sum d\vec{S}_k^e = d\vec{S}^e$, where $d\vec{S}^e$ – the increment of impulse of external forces $d\vec{S}_k^e$, acting on mechanical system during time increment dt.

In the projections on coordinate axes, the Momentum Change Theorem in the form of an increment in momentum transmitted to mechanical system by external forces has the form: $dQ_x = dS_x^e = \sum_{k=1}^n dS_{kx}^e$, $dQ_y = dS_y^e = \sum_{k=1}^n dS_{ky}^e$ and $dQ_z = dS_z^e = \sum_{k=1}^n dS_{kz}^e$.

Integral form of the Momentum Change Theorem (form 3). The momentum of mechanical system change during any period of time t is equal to the integral of the vector sum of the mechanical impulses of all external forces acting on the system over the same period of time: $\int_0^t d\bar{Q} = \int_0^t d\vec{S}^e$. This follows from the differential form 3 of the Momentum Change Theorem: $d\bar{Q} = d\vec{S}^e$.

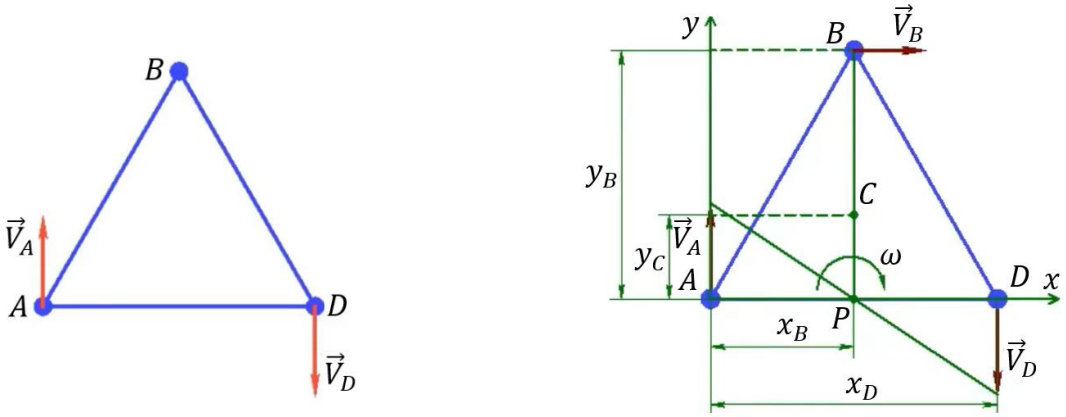
In projections on the coordinate axes the equality of the system momentum change: $(\bar{Q} - \bar{Q}_0)$ for a certain period of time t, as the sum of impulses $\vec{S}^e$ of the external forces, that is: $\bar{Q} - \bar{Q}_0 = \vec{S}^e$ has the form: $Q_x - Q_{0x} = S_x^e$, $Q_y - Q_{0y} = S_y^e$ and $Q_z - Q_{0z} = S_z^e$.

Laws of conservation of momentum of the mechanical system (the consequences of the Momentum Change Theorem):

1. If the main vector of external forces of the system is equal to zero: $\sum \vec{F}_k^e = 0$, then the amount of mechanical motion of the system is of constant magnitude and the direction: $\bar{Q} = const$.

2. If the projection of the main vector of all external forces of system onto one or another coordinate axis is equal to zero, from among the projections of forces: $\sum F_{kx}^e = 0$, or $\sum F_{ky}^e = 0$, or $\sum F_{kz}^e = 0$, then the projection of the momentum of the system on this axis is constant: $Q_x = const$, or $Q_y = const$, or $Q_z = const$.

Exercise 1. The unchangeable mechanical system is assigned, consisting of three material points: A, B and D of the same mass m, located at the vertices of an equilateral rod triangle (Fig. 10.8.1 a).



(a) (b)

Fig. 10.8.1

In the position of the triangle ABD, shown on the diagram, the velocities of the points $\vec{V}_A$ and $\vec{V}_D$ are equal V and are directed perpendicular to the rod AD in opposite directions. The exercise assigns one to determine the main vector of momentum of the system in specified position. The masses of the rods: AB, BD and AD are neglected.

The amount of mechanical motion of specified system of three connected loads located at the vertices of an equilateral triangle is equal to:

$\bar{Q} = M \times \vec{V}_C = m_A \times \vec{V}_A + m_B \times \vec{V}_B + m_D \times \vec{V}_D$, where M – the mass of all elements of the assigned mechanical system, $\vec{V}_C$ – the velocity of center of mass of the mechanical system, m_A, m_B and m_D – the same masses m of material points: A, B and D; $\vec{V}_A$, $\vec{V}_B$ and $\vec{V}_D$ – the velocity of these material points.

The instantaneous velocity center P is located on the segment AD in the middle between the points: A and D. The material point B is located perpendicular to the segment AD, passing through the instantaneous velocity center P (Fig. 10.8.1 b).

Angular velocity of the mechanical system equals: $\omega = \frac{V_A}{AP} = \frac{2 \times V}{a}$, where a – length of a side of a triangle of rods in the mechanical system.

The velocity of the point B, the direction of which is shown on the diagram (Fig. 10.8.1 b), equals in magnitude: $V_B = \omega \times BP = \frac{2 \times V}{a} \times \frac{a \times \sqrt{3}}{2} = V \times \sqrt{3}$.

Determining of position of the center of mass of the mechanical system is as follows: $x_C = \frac{x_A \times m_A + x_B \times m_B + x_D \times m_D}{m_A + m_B + m_D} = \frac{(a/2) \times m + a \times m}{3 \times m} = \frac{a}{2}$

and $y_C = \frac{y_A \times m_A + y_B \times m_B + y_D \times m_D}{m_A + m_B + m_D} = \frac{a \times \sqrt{3}}{2} \times \frac{m}{3 \times m} = \frac{a \times \sqrt{3}}{6}$.

The magnitude of the velocity V_C of the center of mass C of the mechanical system equals: $V_C = \omega \times CP = \frac{2 \times V}{a} \times \frac{1}{3} \times \frac{a \times \sqrt{3}}{2} = V \times \frac{\sqrt{3}}{3}$.

Momentum of the mechanical system is equal to its mass $(3 \times m)$, multiplied by the velocity $\vec{V}_C$ of the center of mass: $\bar{Q} = 3 \times m \times \vec{V}_C$, and the magnitude of the momentum is: $Q = 3 \times m \times V \times \frac{\sqrt{3}}{3} = m \times V \times \sqrt{3}$.

Exercise 2. A load of the mass m is lowered from rest along a rough inclined plane (Fig. 10.8.2) with a friction coefficient f. The exercise assigns one to determine the velocity of the load after the assigned time t after start of motion.

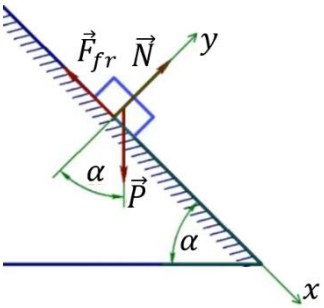


Fig. 10.8.2

The friction force $\vec{F}_{fr}$, proportional to the friction coefficient f and the pressure of the load on the inclined surface: $P \times \cos\alpha$ when the load moves, transmits a braking impulse to the load, reducing the amount of its movement along the inclined plane, and the projection: $P \times \sin\alpha$ of the weight force $\vec{P}$ of the load in the direction of its movement increases the amount of movement of the load. The amount of mechanical motion of the load under the influence of these forces depends on the duration of its movement from the initial state and in projection onto the direction of movement x the momentum equals:

$Q_x - Q_{0x} = S_x^e = P \times \sin\alpha \times t - P \times \cos\alpha \times f \times t,$

$m \times V = m \times g \times \sin\alpha \times t - m \times g \times \cos\alpha \times f \times t$

and the velocity of load in dependence on time t is: $V = (\sin\alpha - \cos\alpha \times f) \times g \times t$.

Exercise 3. On the horizontal platform 1, moving with velocity $\vec{V}_1$, the trolley 2 moves with relative velocity $\vec{V}_{2r}$ (Fig. 10.8.3). At definite time t after an inelastic collision with the stop on the platform, the relative motion of the trolley stops. The exercise assigns one to determine the resulting velocity u of mechanical system, if the mass of the platform is m_1, and of the trolley m_2. Ignore platform motion resistance.

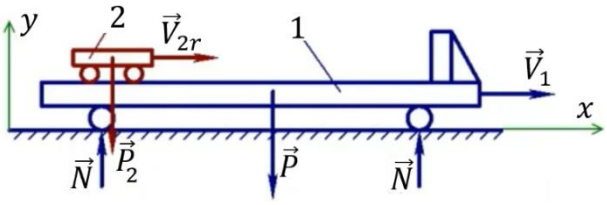


Fig. 10.8.3

In an inelastic collision of trolley with the stop on the platform, their overall momentum does not change, because the external forces wherein do not act on the mechanical system, that's why: $Q_x - Q_{0x} = S_x^e = 0$.

Since after the collision the momentum of the mechanical system is: $Q_x = (m_1 + m_2) \times u$, and before the collision the momentum of the mechanical system, consisting of platform and the trolley, equals: $Q_{0x} = m_1 \times V_1 + m_2 \times (V_1 + V_{2r})$ and $Q_x = Q_{0x}$, then: $(m_1 + m_2) \times u = m_1 \times V_1 + m_2 \times (V_1 + V_{2r})$, therefore, the resulting velocity u of the mechanical system consisting of platform and the trolley after an inelastic collision of the trolley with the stop on platform is equal: $u = \frac{m_1 \times V_1 + m_2 \times (V_1 + V_{2r})}{m_1 + m_2}$.

Questions and comments:

10.8.1. The momentum of a mechanical system (or of a solid body, consisting of parts – its mass elements) is equal to the sum of the momentum of all elements of the system (or of all mass elements of the body).

10.8.2. The vector value of the momentum of mechanical system is equal to the product of the mass of the system (the sum of the mass of all elements of system or the elements of the body mass) by the velocity vector of the center of mass of mechanical system (or the rigid body).

10.8.3. First form (the differential one) of the Momentum Change Theorem of a mechanical system is the equality of time derivative of amount of mechanical motion of the system (that is of all its elements) to sum of all external forces acting on the mechanical system, since the internal forces acting between the elements of the system, when they are added are reset, then they do not lead to a change of the mechanical system momentum.

10.8.4. The second form (the differential one) of the Momentum Change Theorem of mechanical system, following from the first form of this theorem: the increment of momentum of a mechanical system (the product of total mass of all elements of the system by the increment of their common center of mass velocity vector) over the period of differential time increment is equal to the vector of mechanical impulse of all external forces during this period of the time increment.

10.8.5. The third form (the integral one) of the Momentum Change Theorem of a mechanical system: the change of a mechanical system momentum over a period of time is equal to the integral of the vector sum of mechanical impulses of all external forces acting on the system over the same period of time.

10.8.6. The amount of mechanical motion of the system is constant in the magnitude and direction, provided that the main vector of external forces acting on the mechanical system is zero.

10.8.7. The projection of mechanical system momentum onto one or another coordinate axis is of constant value if the projection onto this axis of the main vector of external forces acting on the mechanical system is equal to zero.

10.8.8. Independence on the rotation of the system of amount of its mechanical motion, which depends on mass of all its elements and on the translational motion of the system center of mass.

10.8.9. The amount of motion of a mechanical system under the influence of mechanical impulses of several external forces is equal to the amount of mechanical motion under the influence of the impulse of the main vector of external forces.

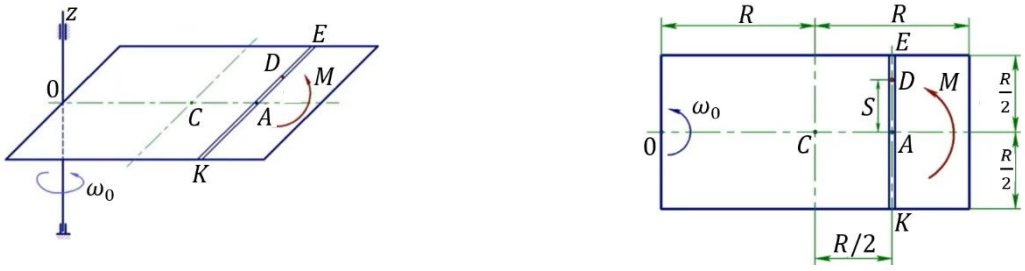
10.8.10. The emergence (or the termination) of a rigid connection between the elements of a mechanical system, which is not affected by external forces in the projection on the direction of its movement, which do not lead to a change of the system center of mass velocity and its momentum.

10.9. The Moment of Momentum Change Theorem of a mechanical system

Exercise. A homogeneous horizontal platform (Fig. 10.9.1) of a rectangular shape with sides: R and $2 \times R$, having a mass m_1, rotated around the vertical axis $0z$ with angular velocity ω_0. Summary of the data assigned of the mechanical system is given below in the Table 10.9.1.

Table 10.9.1

m_1, kg	m_2, kg	R, m	ω_0, s^{-1}	S, m	M, N×m
30	6	1.0	4	$0.6 \times t$	$12 \times t^2$



(a) (b)

Fig. 10.9.1

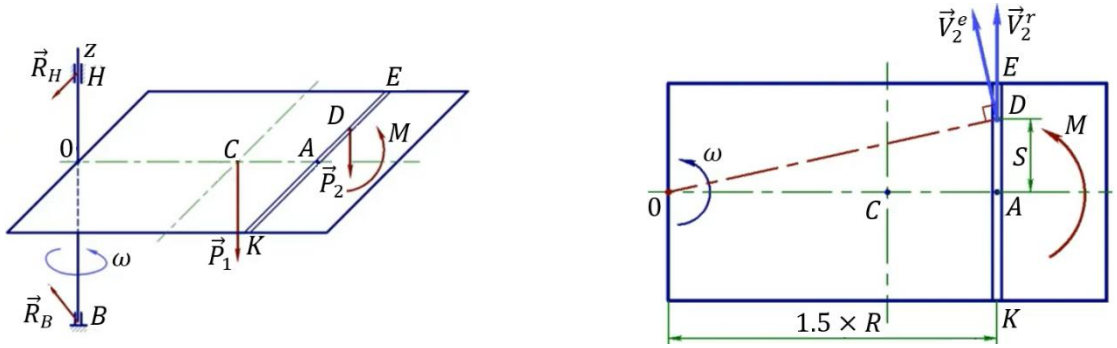
The load D with mass m_2 at the definite time is located at the point A on a straight chute KE. Under the influence of internal forces load D begins to move along the chute in accordance with the pattern: $S = AD = S(t) = 0.6 \times t$. At the same time, a pair of external forces begin to act on the platform, initiating the moment: $M = 12 \times t^2$.

The exercise assigns one to determine the pattern of the angular velocity of the platform change: $\omega = \omega(t)$.

In accordance with the Moment of Momentum Change Theorem the moment of momentum of mechanical system is equal to the action of moment of momentum of external forces on the system, which in the projection of moments onto the z axis has the form: $\frac{dL_z}{dt} = \sum_{k=1}^{n} M_{kz}(F_{kz}^e)$, where L_z – the moment of momentum of body rotation corresponding to the product of it angular velocity and the moment of the body inertia relative to the axis of rotation, n – the number of elements of mechanical system (or the parts of body mass) on which the moments M_k from external forces F_k^e act.

The dimension of the moment of momentum in the International System of Units (Système international d'unités, SI): kg×m^2/s, or N×m×s, or J×s.

To find the moment of momentum in considered issue it is relevant to indicate the external forces on the scheme (Fig. 10.9.2).



(a) (b)

Fig. 10.9.2

When the platform is exposed to moment of the external forces pair: $M = 12 \times t^2$, a time–dependent moment of momentum L_z is provided, which one can find by time integrating of moment of external forces pair, as: $\frac{dL_z}{dt} = M = 12 \times t^2$ and $L_z = 4 \times t^3 + C_1$.

The found moment of momentum L_z of external forces pair is transmitted to the rotating platform L_{1z} and to load moving along the platform L_{2z}, that is: $L_z = L_{1z} + L_{2z}$.

The moment of momentum of the platform, that is it kinetic moment is equal to the product of angular velocity of the platform rotation by moment of it inertia: $L_{1z} = I_z \times \omega$.

Moment of inertia of rectangular platform one can determine as the sum of its own moment of inertia I_{Cz} (relative to the center of gravity of the platform) and the transferring moment of inertia: $m_1 \times (OC)^2$ of the gravity center rotation about the assigned axis z as: $I_z = I_{Cz} + m_1 \times (OC)^2 = \frac{m_1}{12} \times ((2 \times R)^2 + R^2) + m_1 \times R^2 = \frac{17}{12} \times m_1 \times R^2$.

The moment of momentum of the platform L_{1z}, determined depending on the angular velocity of it rotation ω, is equal:

$L_{1z} = I_z \times \omega = \frac{17}{12} \times m_1 \times R^2 \times \omega = \frac{17}{12} \times 30 \times 1^2 \times \omega = 42.5 \times \omega$, kg×m^2/s.

The load D with mass m_2 is interfered in relative motion (that is moving relative to the platform) and in transferring motion – together with the rotating platform. One is to consider as well that the load is a material point and the own moment of inertia of the load is equal to zero.

That's why the moment of momentum of the load D is determined by summation of two terms and equals:

$L_{2z} = m_2 \times V_2^r \times OA + m_2 \times V_2^e \times OD = m_2 \times V_2^r \times OA + m_2 \times OD^2 \times \omega$,

and: $m_2 \times V_2^r \times OA = m_2 \times OA^2 \times \omega_2^r$, where: $\omega_2^r = V_2^r / OA$ – the angular velocity of transferring rotation of a material point relative to the fixed axis of rotation of the plate.

To find the moment of momentum of the load D motion the data is used:

$V_2^r = \dot{S} = 0.6$, m/s, $OA = 1.5 \times R = 1.5$, m

and $OD^2 = OA^2 + AD^2 = 1.5^2 + (0.6 \times t)^2 = 2.25 + 0.36 \times t^2$, m^2.

The moment of momentum of the load D motion is equal:

$L_{2z} = 6 \times 0.6 \times 1.5 + 6 \times (2.25 + 0.36 \times t^2) \times \omega = 5.4 + (13.5 + 2.16 \times t^2) \times \omega$, kg×m^2/s.

Using the Moment of Momentum Change Theorem of the mechanical system, equating the moment of momentum of the system (the platform and load D) and moment of momentum from the impact of a pair of forces on the mechanical system, as:

$42.5 \times \omega + 5.4 + (13.5 + 2.16 \times t^2) \times \omega = 4 \times t^3 + C_1$, one can find the pattern of change of angular velocity ω of the platform rotation, that is:

$56 \times \omega + 5.4 + 2.16 \times t^2 \times \omega = 4 \times t^3 + C_1$.

To find the angular velocity ω of the platform rotation one has to use the initial conditions: $\omega = \omega_0 = 4$, s^{-1} at $t = 0$, $C_1 = 56 \times 4 + 5.4 = 229.4$,

then: $\omega \times (56 + 2.16 \times t^2) = 4 \times t^3 + 224$ and $\omega = \frac{4 \times t^3 + 224}{2.16 \times t^2 + 56}$, radian/s.

Questions and comments:

10.9.1. The equality of the change during certain time of the moment of momentum during rotation of a mechanical system and the moment of momentum of external forces acting during this time (the moment of momentum of the resultant external forces) in accordance with the Moment of Momentum Change Theorem.

10.9.2. Finding the moment of momentum of external forces by time integrating of the moments of external forces (of the moment of resultant external force).

10.9.3. Determination of the moment of inertia of mass during rotation of an element of a mechanical system as the sum of its own moment of inertia relative to the center of gravity of the system element and the transferring moment of inertia as the product of the mass of the element by the square of the distance from the axis of rotation to the center of gravity of this element of the system.

10.9.4. Determination of the moment of momentum of an element of a mechanical system as the product of the moment of inertia of mass relative to the axis of its rotation and the angular velocity of rotation.

10.9.5. Determination of the moment of momentum of an element of mechanical system during its complex motion as the sum of the moment of momentum of its own motion relative to the rotating coordinate system and the moment of momentum of the transferring motion of the rotating coordinate system relative to its axis of rotation.

10.9.6. The proper moment of momentum is equal to the product of the moment of inertia of an element of a mechanical system relative to stationary coordinate system, assuming that the center of gravity of the system element is located in a moving coordinate system – rotating relative to a stationary one with the angular velocity of rotation of a moving coordinate system.

10.10. Kinetic energy of a mechanical system

Kinetic energy is a scalar value dependent only on the inertia (on mass or on the moment of inertia of mass) and on velocity (translational and angular). The kinetic energy is measure of motion of physical bodies, mechanical systems and their elements.

In the first method of the kinetic energy determining of plane motion of a rigid body (Fig. 10.10.1) of a certain mass m and inertia at rotation, characterized by the moment of inertia I_C relative to the axis passing through the center of mass C and perpendicular to the plane of motion of the body, is considered as complex, in which the motion of the center of mass C with velocity $\vec{V}_C$ is translational, and the rotation of the body around the center of mass C with angular velocity ω serves as relative motion, wherein the kinetic energy of the body equals: $T = \frac{m \times V_C^2}{2} + \frac{I_C \times \omega^2}{2}$.

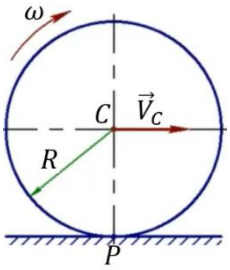


Fig. 10.10.1

In the second method of determining of kinetic energy, the plane motion of a rigid body is considered as rotation around the instantaneous velocity center P with angular velocity ω, and the kinetic energy equals: $T = \frac{I_P \times \omega^2}{2}$, where I_P – moment of inertia of the

body relative to the axis passing through the instantaneous velocity center P, perpendicular to the plane of motion of the body.

Exercise 1. The exercise assigns one to determine the kinetic energy of a flat homogeneous disk of mass m and radius R, which rolls along a horizontal section of the path without slipping, if velocity of center of the disk C equals: $V_C = V$.

The moment of inertia of a flat homogeneous disk relative to the center of mass is equal, as is known: $I_C = \frac{m \times R^2}{2}$. Angular velocity of rotation of the disk when it rolls equals: $\omega = \frac{V_C}{CP} = \frac{V}{R}$. Considering the rolling of disk as a complex motion – the movement of center of mass and rotation of the disk, its kinetic energy can be found as the sum of the transferring energy of translational motion and the energy of relative rotational motion:

$$T = \frac{m \times V_C^2}{2} + \frac{I_C \times \omega^2}{2} = \frac{m \times V^2}{2} + \frac{m \times R^2}{2} \times \frac{V^2}{2 \times R^2} = \frac{3 \times m \times V^2}{4}.$$

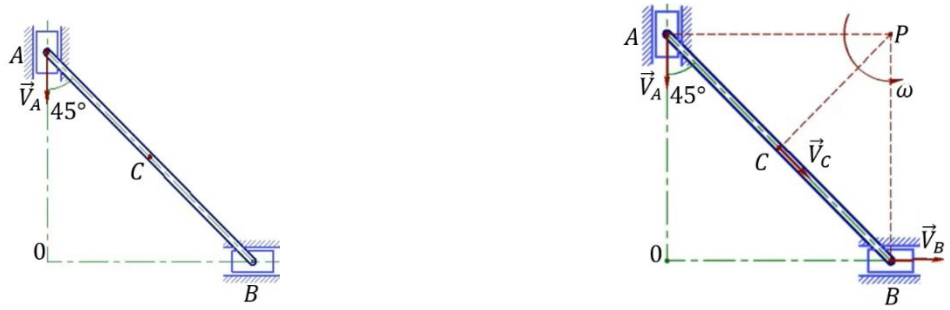
Instantaneous center of velocities is at the point P of the disk support. If the rolling of the disk is considered as rotation with angular velocity ω around the instantaneous center of velocities P (the point of support), then kinetic energy is determined as: $T = \frac{I_P \times \omega^2}{2}$.

The total moment of inertia of the disk I_P when it rotates around an axis that does not pass through the center of gravity is equal to the sum of its own moment of inertia I_C (relative to the axis that passes through the center of mass of the disk C) and the transferring moment of inertia: $m \times (CP)^2$, that is:

$$I_P = I_C + m \times (CP)^2 = \frac{m \times R^2}{2} + m \times R^2 = \frac{3 \times m \times R^2}{2}.$$

Kinetic energy of a rolling disk equals: $T = I_P \times \frac{\omega^2}{2} = \frac{3 \times m \times R^2}{2} \times \frac{\omega^2}{2} = \frac{3 \times m \times V^2}{4}$.

Exercise 2. The exercise assigns one to determine kinetic energy of the ellipsograph ruler (Fig. 10.10.2) with specified mass m, considering it a homogeneous straight rod of length l, if the velocity of the slider A is known: $V_A = V$.



(a) (b)

Fig. 10.10.2

Since instantaneous center of velocities P of ellipsograph ruler lies at the intersection of perpendiculars to the axes of the sliders: A and B then the distance from each of

the sliders to instantaneous center of velocities of the ellipsograph ruler one can find geometrically, as: $AP = BP = l \times \sqrt{2}/2$, and the distance from the middle of the ellipsograph ruler (from its center of mass) to the instantaneous velocity center equals: $CP = l/2$.

The proper moment of inertia of the ellipsograph ruler relative to the center of mass equals: $I_C = \frac{m \times l^2}{12}$.

Projections of the velocities: $\vec{V}_A$ and $\vec{V}_B$ of sliders onto the ellipsograph axis, passing at an angle of $45°$ relative to the axes of each of the sliders: A and B, make it possible to find the velocity magnitude V_C of center of mass C of the ellipsograph ruler, which is equal: $V_C = V_A \times \cos 45° = V/\sqrt{2}$ and its angular velocity is: $\omega = \frac{V_C}{CP} = \frac{V \times \sqrt{2}}{l}$.

Considering the motion of the ellipsograph ruler as complex, that is the displacement of the center of mass of the ruler and its rotation with a certain angular velocity, the kinetic energy of the ruler can be found as the sum of the transferring energy of translational motion and the energy of relative rotational motion:

$$T = \frac{m \times V_C^2}{2} + \frac{I_C \times \omega^2}{2} = \frac{m \times V^2}{2 \times 2} + \frac{m \times l^2}{12} \times \frac{V^2 \times 2}{2 \times l^2} = \frac{m \times V^2}{3}.$$

The instantaneous velocity center P is located at the intersection of perpendiculars to the axes of sliders at the points where the sliders are located: A and B.

The total moment of inertia of mass of the ellipsograph ruler I_P during its rotation around an axis that does not pass through the center of gravity is equal to the sum of its own moment of inertia I_C (relative to an axis that passes through the center of mass of the ruler C) and transferring moment of inertia: $m \times (CP)^2$, that is:

$$I_P = I_C + m \times (CP)^2 = \frac{m \times l^2}{12} + \frac{m \times l^2}{4} = \frac{m \times l^2}{3}.$$

Kinetic energy of the ellipsograph ruler equals: $T = I_P \times \frac{\omega^2}{2} = \frac{m \times l^2 \times V^2 \times 2}{3 \times 2 \times l^2} = \frac{m \times V^2}{3}$.

Questions and comments:

10.10.1. A measure of mechanical motion that depends only on the inertia of physical bodies, mechanical systems and their elements (on their masses or on the moments of inertia of masses) and on velocity (the translational and angular), which is kinetic energy.

10.10.2. Kinetic energy of body (an element of a mechanical system), including the energy of its translational motion, associated with its mass and with the translational speed of the center of mass, and the energy of the rotational motion of the body around the center of mass, characterized by moment of inertia of the body mass and the angular velocity of its rotation.

10.10.3. Determination of the kinetic energy of a body rotating relative to the instantaneous center of velocities, the position of which is determined by the translational

velocity of the center of mass of body and the angular velocity of rotation of the body, and the moment of inertia of the body relative to the instantaneous center of velocities, which equals to its own moment of inertia relative to the center of mass in total with the transferable moment of inertia of the body when its center of mass rotates relative to the instantaneous center of velocities.

10.11. The use of the Kinetic Energy Change Theorem to determine the elements of motion of a mechanical system

In addition to kinetic energy, which is the measure of the motion of mechanical systems and their elements and the characteristic of their inertia and velocity, the law of energy conservation also covers the potential energy, which, firstly, is part of the total mechanical energy, along with kinetic energy, and secondly, the potential energy acts not as a function of movement and velocity, but as a function of the coordinates of the system elements under influence of the conservative forces, i.e. those whose work when acting on a body equals zero when the body moves along a closed path or during rotation.

It is possible to assume that the potential energy remains unchanged in many cases in the calculations of moving mechanical systems. The same assumption is methodologically included in the Kinetic Energy Change Theorem (under the assumption of constancy of potential energy), according to which the difference in the values of the kinetic energy of a mechanical system (or energy received by the system) over a certain period ($T - T_0$) corresponds to the sum of the work of forces acting on the elements of the system: $T - T_0 = \sum A_k^e + \sum A_k^i$.

The total of external forces work: $\sum A_k^e$ under certain conditions one can correlate with change of the potential energy of mechanical system, for example, when a load moves under the influence of gravity, and this is what sets the mechanical system in motion.

Because the work of internal forces ($\sum A_k^i$, the transfer of energy between the elements of a mechanical system) does not change the energy of motion of the entire system, i.e. the work of internal forces does not cause the change of kinetic energy of the system, which kinetic energy change thus is zero: $\sum A_k^i = 0$, then the Kinetic Energy Change Theorem of mechanical system can be defined as the equality of change in the kinetic energy of mechanical system and the total work of external forces: $\sum A_k^e$, acting on the elements of this system: $T - T_0 = \sum A_k^e$.

If the mechanical system was motionless at the initial moment and its kinetic energy was initially equal to zero: $T_0 = 0$, then subsequently the kinetic energy of the system is equal to total work of the external forces: $T = \sum A_k^e$.

Exercise. The mechanical system (Fig. 10.11.1) consists of loads 1 and 2 having masses: m_1 =0 and m_2 =6 kg, cylindrical homogeneous roller 3 with the mass: m_3 =2

kg and stepped pulleys 4 and 5 (the mass: m_4 =3 kg and m_5 =0) with step radii: r_4 =0.2 m, R_4 =0.3 m, r_5 =0.2 m and R_5 =0.4 m.

The coefficient of friction of loads 1 and 2 on supporting surfaces is equal: f =0.1. The weights, roller and pulleys are connected to each other by means of non–stretchable flexible connections thrown over the pulleys.

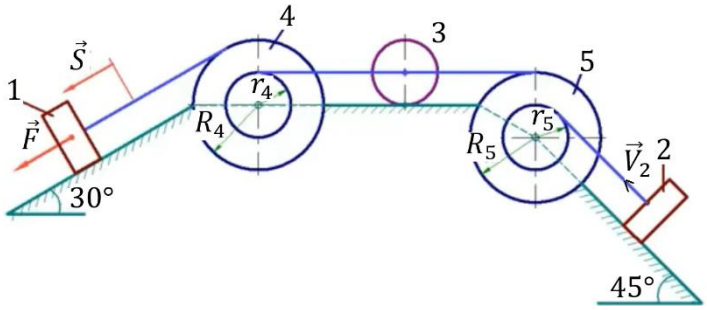


Fig. 10.11.1

The system begins to move from a stationary state under the influence of force: $F(S) = 50 \times (3 + 4 \times S)$, N, which is a function of the movement of the load 1.

Pulley 5 is subject to a constant moment of resistance: M_5 =0.3 N×m. Consider the mass of the pulley to be evenly distributed over the rim. The exercise assigns one to determine the velocity $\vec{V}_2$ of the load 2 at the moment of time when load 1 moves a distance: $S = S_1 = 1.2$ m.

Kinetic energy of assigned mechanical system equals: $T = T_1 + T_2 + T_3 + T_4 + T_5$.

To solve the issue and to determine the value of speed V_2 of the load 2 one has to obtain the indicators of translational and angular velocities of elements of the assigned mechanical system (Fig. 10.11.2) by velocity V_2 sought of the load 2: $\omega_5 = \frac{V_2}{r_5}$,

$V_3 = R_5 \times \omega_5 = \frac{R_5}{r_5} \times V_2$, $\omega_3 = \frac{V_3}{r_3} = \frac{R_5}{r_3 \times r_5} \times V_2$, $\omega_4 = \frac{V_3}{r_4} = \frac{R_5}{r_4 \times r_5} \times V_2$

and $V_1 = R_4 \times \omega_4 = \frac{R_4 \times R_5}{r_4 \times r_5} \times V_2$.

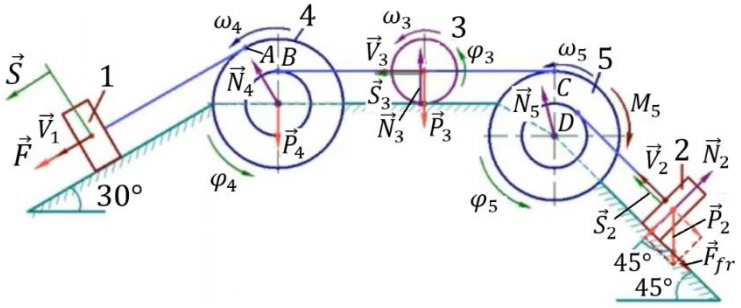


Fig. 10.11.2

The dependences of translational and angular velocities of the elements of assigned mechanical system on the desired speed V_2 of load 2 are summarized in Table 10.11.1.

Table 10.11.1

k	1	2	3	4	5
V_k	$\frac{R_4 \times R_5}{r_4 \times r_5} \times V_2$	V_2	$\frac{R_5}{r_5} \times V_2$	–	–
ω_k	–	–	$\frac{R_5}{r_3 \times r_5} \times V_2$	$\frac{R_5}{r_4 \times r_5} \times V_2$	$\frac{V_2}{r_5}$

Moments of inertia of roller 3 and pulley 4 are equal: $I_3 = \frac{m_3 \times r_3^2}{2}$ and $I_4 = m_4 \times R_4^2$.

Dependences of the kinetic energy of five assigned elements of the mechanical system on sought speed V_2 of the load 2. The kinetic energy of load 1 is zero: T_1 =0, because mass of load 1 is zero (m_1 =0).

Kinetic energy of mass 2: m_2 =6 kg equals: $T_2 = \frac{m_2 \times V_2^2}{2} = \frac{6 \times V_2^2}{2} = 3 \times V_2^2$.

The kinetic energy of a homogeneous roller 3 with mass: m_3 =2 kg includes the energy of translational motion of the center of mass of the roller and the energy of rotation of the roller relative to the center of mass and equals:

$$T_3 = \frac{m_3 \times V_3^2}{2} + \frac{I_3 \times \omega_3^2}{2} = \frac{m_3}{2} \times \left(\frac{R_5}{r_5} \times V_2\right)^2 + \frac{1}{2} \times \frac{m_3 \times r_3^2}{2} \times \left(\frac{R_5}{r_3 \times r_5} \times V_2\right)^2 =$$

$$= \frac{2}{2} \times \left(\frac{0.4}{0.2} \times V_2\right)^2 + \frac{1}{2} \times \frac{2}{2} \times \left(\frac{0.4}{0.2} \times V_2\right)^2 = 6 \times V_2^2.$$

Kinetic energy of step pulley 4 with mass: m_4 =3 kg equals:

$$T_4 = \frac{I_4 \times \omega_4^2}{2} = \frac{1}{2} \times m_4 \times R_4^2 \times \left(\frac{R_5}{r_4 \times r_5} \times V_2\right)^2 = \frac{1}{2} \times 3 \times \left(\frac{0.3 \times 0.4}{0.2 \times 0.2}\right)^2 \times V_2^2 = 13.5 \times V_2^2.$$

The kinetic energy of the stepped pulley 5 is zero: T_5 =0, because the mass of step pulley 5 is zero (m_5 =0). The total magnitude of the kinetic energy of five assigned elements of mechanical system depending on the velocity of load 2 is determined by summing and equals: $T = \sum T_k = 22.5 \times V_2^2$.

The values of the kinetic energy of five elements of the mechanical system depending on the velocity V_2 sought of load 2 are given in the summary, Table 10.11.2.

Table 10.11.2

T_1	T_2	T_3	T_4	T_5	T
0	$3 \times V_2^2$	$6 \times V_2^2$	$13.5 \times V_2^2$	0	$22.5 \times V_2^2$

The kinetic energy of the mechanical system changes under the influence of work of an external force: $F(S) = 50 \times (3 + 4 \times S)$, N, which is a function of the movement of load 1, of the constant moment of resistance: M_5 =0.3 N×m, acting on the pulley 5, of the frictional forces and of weights of the loads moving along inclined surfaces.

Considering the applied force $F(S)$, as well as the weight force P_2, the assigned moment M_5, as well as the friction force F_{fr2} the work of external forces is equal to: $\sum A_k^e = A(F) + A(P_2) + A\left(F_{fr2}\right) + A(M_5)$.

Other external forces do not affect the mechanical system.

Since the external force $F(S)$ is dependence on motion of the load 1 (that is on S_1), then to find the work of external forces and moment: $A(F)$, $A(P_2)$, $A\left(F_{fr2}\right)$ and $A(M_5)$ one has to determine the translational movements on which external forces work: F, P_2 and F_{fr2} and the moment M_5 at the time when load 1 moves a distance:

$S = S_1 = 1.2$ m, that is: $\varphi_4 = \frac{S_1}{R_4} = \frac{1.2}{0.3} = 4$, radian,

$S_3 = r_4 \times \varphi_4 = \frac{r_4}{R_4} \times S_1 = \frac{0.2}{0.3} \times 1.2 = 0.8$, m,

$\varphi_5 = \frac{S_3}{R_5} = \frac{r_4}{R_4 \times R_5} \times S_1 = \frac{0.2}{0.3 \times 0.4} \times 1.2 = 2$, radian

and $S_2 = r_5 \times \varphi_5 = \frac{r_4 \times r_5}{R_4 \times R_5} \times S_1 = \frac{0.2 \times 0.2}{0.3 \times 0.4} \times 1.2 = 0.4$, m.

Summary. The Table 10.11.3 contains the dependencies of displacements on which the external forces: F, P_2 and F_{fr2} and the moment M_5 on the assigned displacement: $S = S_1 = 1.2$ m.

Table 10.11.3

k	1	2	3	4	5
S_k, m	S_1	$\frac{r_4 \times r_5}{R_4 \times R_5} \times S_1$	$\frac{r_4}{R_4} \times S_1$	–	–
	1.2	0.4	0.8	–	–
φ_k, radian	–	–	$\frac{r_4}{R_4 \times r_3} \times S_1$	$\frac{S_1}{R_4}$	$\frac{r_4 \times S_1}{R_4 \times R_5}$
	–	–	$0.8/r_3$	4.0	2

The work of external forces and the moment: $A(F)$, $A(P_2)$, $A\left(F_{fr2}\right)$ and $A(M_5)$ are determined as: $A(F) = \int_0^{S_1} F \times dS = 50 \times \int_0^{S_1} (3 + 4 \times S) \times dS =$

$= 50 \times (3 \times S_1 + 2 \times S_1^2) = 50 \times (3 \times 1.2 + 2 \times 1.2^2) = 324$, N×m (that is J),

$A(P_2) = -P_2 \times \sin 45° \times S_2 = -m_2 \times g \times \sin 45° \times S_2 =$

$= -6 \times 9.81 \times 0.707 \times 0.4 = -16.65$, N×m

and $A(F_{fr2}) = -f \times N_2 \times S_2 = -f \times m_2 \times g \times \cos 45° \times S_2 =$

$= -0.1 \times 6 \times 9.81 \times 0.707 \times 0.4 = -1.66$, N×m,

$A(M_5) = -M_5 \times \varphi_5 = -0.4 \times 2.0 = -0.8$, N×m.

The results of calculating the work of external forces and the moment: $A(F)$, $A(P_2)$, $A(F_{fr2})$ and $A(M_5)$ are given in the summary Table 10.11.4.

Table 10.11.4

$A(F)$	$A(P_2)$	$A(F_{fr2})$	$A(M_5)$
$50 \times (3 \times S_1 + +2 \times S_1^2)$	$-f \times m_2 \times g \times \times \cos 45° \times S_2$	$-f \times N_2 \times S_2$	$-M_5 \times \varphi_5$
324, N×m	−16.65, N×m	−1.66, N×m	−0.8, N×m

The total work of external forces and the moment: $A(F)$, $A(P_2)$, $A(F_{fr2})$ and $A(M_5)$ equals: $\sum A_k^e = 324 - 16.65 - 1.66 - 0.8 = 304.89$, N×m (that is J).

The velocity V_2 of load 2 one can find by equating the kinetic energy of the moving one of the mechanical system: $T = 22.5 \times V_2^2$ and the work of external forces and the moment: $A(F)$, $A(P_2)$, $A(F_{fr2})$ and $A(M_5)$. The velocity V_2 of load 2 at the moment of time equals: $V_2 = \sqrt{\frac{304.89}{22.5}} = 3.68$, m/s when load 1 moves at distance: $S = S_1 = 1.2$ m.

Questions and comments:

10.11.1. Indicators included in the law of energy conservation of a system which is not upon external forces action: the kinetic energy, which characterizes the motion of mechanical system and its elements and depends on their translational and angular velocities and on the characteristics of inertia (the masses and moments of inertia of masses), and the potential energy, characterizing the coordinates of elements of mechanical systems location and the conservative forces acting on these elements, the work of which when moving along a closed path or during rotation is equal to zero.

10.11.2. The kinetic energy of mechanical system equals to sum of kinetic energy values of all moving elements of the system. The energy of each moving element of a mechanical system is either equal to the total of the kinetic energy of its translational and rotational motions, taking into account the inertia of the body in translational and rotational motions, or equal to the energy of rotational motion of an element of mechanical system relative to its instantaneous center of velocities, taking into account the inertia of the body when rotating relative to the center of velocities.

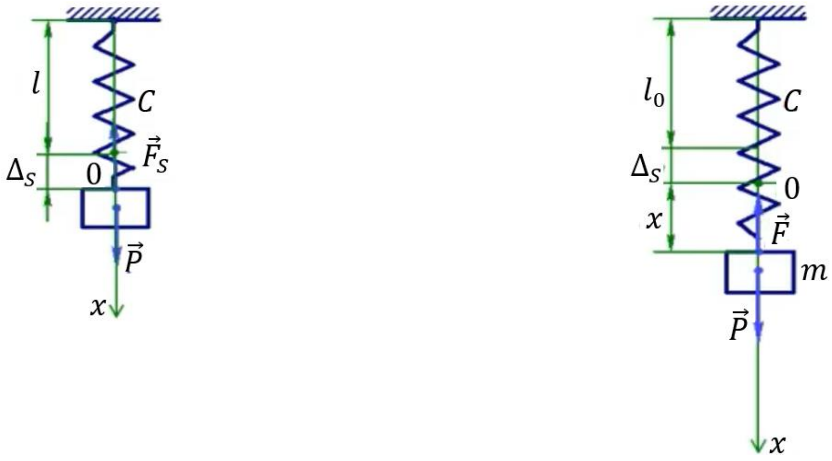
10.11.3. The Kinetic Energy Change Theorem, considering zero influence of the work of internal forces, acting between the elements of a mechanical system, on the change in the kinetic energy of total system, showing the equality of the kinetic energy change and the work of external forces acting on the elements of mechanical system.

10.11.4. Determination of motion indicators (the translational or angular velocities) of mechanical system interconnected elements by the motion indicator of one of these elements, for example, by the velocity of a moving load doing work that causes the movement of the elements of the mechanical system and changes their kinetic energies.

Chapter 11. Vibrations of mechanical systems under the influence of external forces, weight, inertia and elasticity

11.1. Basic characteristics of free vibrations of a material point under the influence of weight and elastic forces

Equation of oscillations of a material point (Fig. 11.1.1, the concentrated load on a spring) has the form: $m \times \ddot{x} = P - F$, where $m \times \ddot{x}$ – the force of inertia acting on material point (the load) when moving with acceleration $\ddot{x}$, P – the load weight: $P = m \times g$, $g = 9.81$ m/s^2, F – the force of elasticity, acting on the load on the spring side: $F = c \times (x + \Delta_s)$, c – the spring constant coefficient, Δ_s – the static spring deformation.

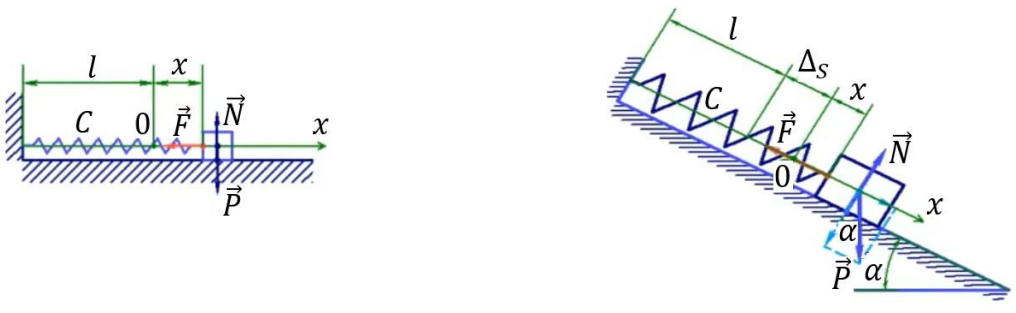


(a) (b)

Fig. 11.1.1

The equation of free vibrations of material point one can perform in the form: $\ddot{x} + k^2 \times x = 0$, where: $k = \sqrt{c/m} = \sqrt{g/\Delta_s}$ – the circular (cyclic) frequency of free vibrations of material point on a spring (the dimension: s^{-1} or radian/s).

Options for the location of load (the material point) on a spring when it moves along horizontal or inclined surfaces are shown on the schemes (Fig. 11.1.2 a and b).



(a) (b)

Fig. 11.1.2

When load on a spring moves along a horizontal surface, the influence of weight force is zero: $P = 0$, and the equation has the form: $m \times \ddot{x} = -F$ or $\ddot{x} + \frac{c}{m} \times x = 0$.

When load moves along an inclined surface (Fig. 11.1.2 b) the equation has the following form: $m \times \ddot{x} = P_x - F$ (at the absence of friction), $P_x = m \times g \times \sin\alpha$ and $F = c \times (x + \Delta_S)$.

Solution of linear homogeneous equation of free vibrations of a material point (the load) on spring without friction is in the form: $x = C_1 \times \cos(k \times t) + C_2 \times \sin(k \times t)$, and the velocity of the material point when moving without friction when defined in general equals: $\dot{x} = -C_1 \times k \times \sin(k \times t) + C_2 \times k \times \cos(k \times t)$.

To find the integration constants (C_1 and C_2) one can use the initial conditions, in particular: $x = x_0$ and $\dot{x} = V_0$ at $t = 0$, then the integration constants are equal: $C_1 = x_0$ and $C_2 = V_0/k$, and the law of motion of material point without friction on a spring has the form: $x = x_0 \times \cos(k \times t) + \frac{V_0}{k} \times \sin(k \times t)$.

In the case when vibrations of a material point (the load on a spring) without friction have an initial phase φ_0 and an initial amplitude A and the integration constants considering the initial phase, one can define in the form: $C_1 = A \times \sin\varphi_0$ and $C_2 = A \times \cos\varphi_0$, then the law of motion of material point without friction on a spring is:
$x = A \times \sin(k \times t + \varphi_0)$.

Elements of free vibrations without friction are shown on the graph (Fig. 11.1.3).

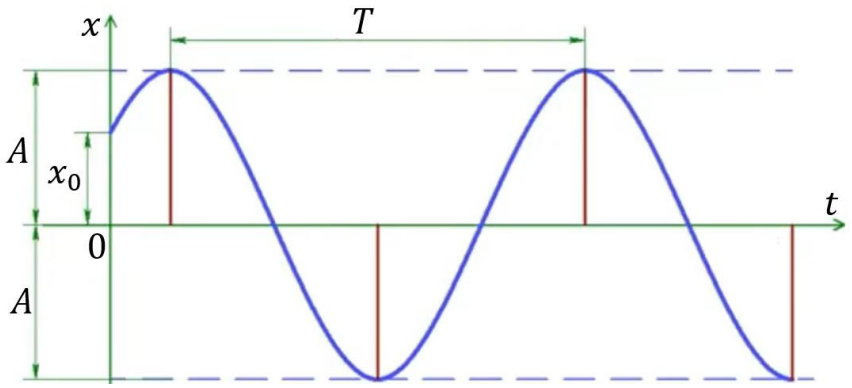
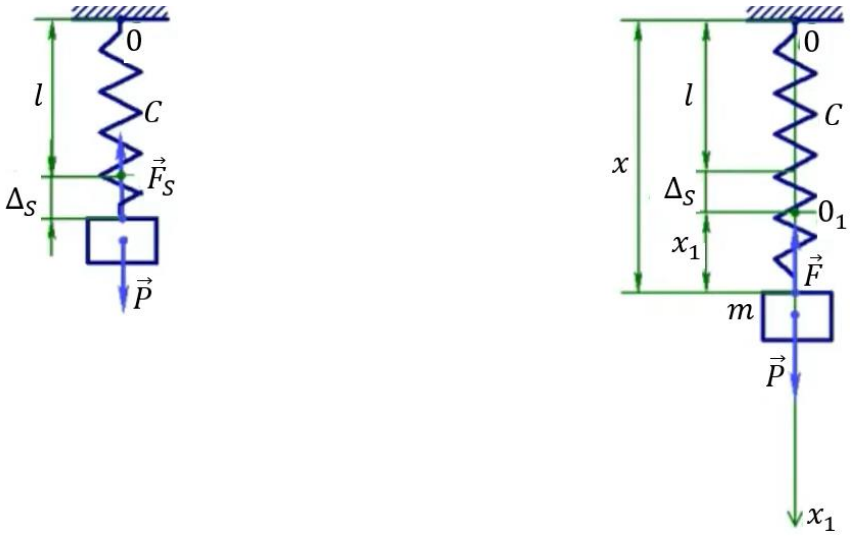


Fig. 11.1.3

The main elements of free vibrations in mechanics are the following: the amplitude of free vibrations: $A = \sqrt{x_0^2 + \frac{V_0^2}{k^2}}$, the initial phase φ_0 oscillations: $\tan\varphi_0 = \frac{x_0 \times k}{V_0}$; circular (the cyclic) frequency of free oscillations equals: $k = \sqrt{c/m} = \sqrt{g/\Delta_S}$; the period of free oscillations is: $T = 2 \times \pi/k$.

Exercise. The exercise assigns one to find the law of vertical free oscillations of a point load suspended on a weightless spring (Fig. 11.1.4 a) relative to the point 0 of the spring suspension, if the load begins to move from a static position of balance with initial

velocity: $V_0 = 1$ m/s. Spring length without deformation is: $l \approx 30.2$ cm, and it static deformation under the action of load equals: Δ_s=9.81 cm (9.81×10^{-2} m).



(a) (b)

Fig. 11.1.4

The solution is based on integration of the equation of free vibrations of a material point relative to the position of balance 0_1: $m \times \ddot{x} = P - F$, where: $m \times \ddot{x}$ – the force of inertia acting on material point (the load) when moving with acceleration $\ddot{x}$; $P = m \times g$ – the load weight, $g = 9.81$ m/s^2, $F = c \times (x + \Delta_s)$ – the force of elasticity acting on the load from side of the spring, c – the spring constant coefficient, Δ_s – the spring static deformation.

By substituting mathematical equations, one can convert the original equation to the form: $m \times \ddot{x} = m \times g - c \times (x + \Delta_s)$, since: $m \times g - c \times \Delta_s = 0$,
then one can transform the equation: $\ddot{x} + k^2 \times x = 0$, where:
$k = \sqrt{c/m} = \sqrt{g/\Delta_s} = \sqrt{9.81/(9.81 \times 10^{-2})} = 10$ – the cyclic frequency of free vibrations of material point on a spring, s^{-1} (or the radian/s).

General form of resulting formula for finding the load displacement:
$x = C_1 \times \cos(k \times t) + C_2 \times \sin(k \times t)$ and formula for finding the load velocity:
$\dot{x} = -C_1 \times k \times \sin(k \times t) + C_2 \times k \times \cos(k \times t)$.

Considering the initial data: $x = x_0 = 0$ and $\dot{x} = V_0$ at $t = 0$, the integration constants will be equal: $C_1 = x_0 = 0$ and $C_2 = \frac{V_0}{k} = \frac{1}{10} = 0.1$, and the law of motion of a material point on spring has the form: $x = 0.1 \times \sin(10 \times t)$, m.

Since without the load oscillations, the length of a statically stretched spring relative to suspension 0 equals the sum of length of the spring without load and deformation, that is: $l \approx 30.2$ cm the magnitude of static deformation of spring: Δ_s=9.81 cm, that is:

$l + \Delta_s = 30.2 + 9.8 = 40$, cm, then the resulting law of motion of the material point on the spring finally has the form: $x = 0.1 \times \sin(10 \times t) + 0.4$, m.

The solution can be done in another way (Fig. 11.1.4 b), determining the displacement from the spring suspension point: $x = x_1 + \Delta_s + l$, while the elastic force is: $F = c \times (x - l)$.

The resulting differential equation: $m \times \ddot{x} = m \times g\ - c \times (x - l)$, after transformation has the form: $\ddot{x} + \frac{c}{m} \times x = \frac{c}{m} \times (\Delta_s + l)$.

The solution to this linear inhomogeneous equation one can find in the form of the sum of two solutions: $x = x_1 + x_2$.

The first solution x_1 of the equation characterizes the oscillations of a point load suspended on a weightless spring equals: $x_1 = C_1 \times \cos(k \times t) + C_2 \times \sin(k \times t)$, and the second solution x_2 – the static extension of spring from the weight of load is: $x_2 = A$ and $A = \ l + \Delta_s = 30.2 + 9.81 \approx 40$, cm.

The definition of derivative of the function x_1 of oscillations of a point load suspended on a spring is considered above by finding the velocity $\dot{x}$ depending on the displacement x. After set of displacement and velocity values for the initial conditions: $x_1 = x_{10} = 0$ and $\dot{x}_{10} = V_0$ at $t = 0$, one can find the integration constants: $C_1 = x_{10} = 0$ and $C_2 = \frac{V_0}{k} = \frac{1}{10} = 0.1$.

After set the integration constants, the resulting law of motion of the material point on the spring finally has the form: $x = 0.1 \times \sin(10 \times t) + 0.4$, m.

Questions and comments:

11.1.1. The action on a material point (the concentrated load), or on a body, or on an element of a mechanical system of inertial forces proportional to accelerations (the second derivative of displacements with respect to the time parameter) and forces proportional to displacements (the elasticity forces of a spring, the forces of excess buoyancy when immersed in a liquid, etc.), leading to solutions in the form of harmonic oscillations.

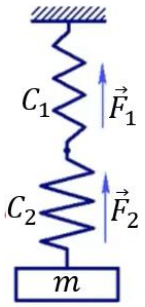
11.1.2. Equation of free oscillations of a material point (the concentrated load) without environmental resistance and solution of this equation.

11.1.3. Circular (the cyclic) frequency of free vibrations of material point, the dependence of cyclic frequency on the characteristics of rigidity and the mass.

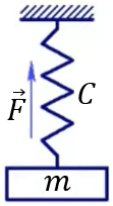
11.1.4. Integration constants of the equation of free oscillations, the determination of initial phase of oscillations and the initial amplitude.

11.2. The influence of the mass and stiffness of elements of mechanical systems on the characteristics of their vibrations, the influence of parallel and series connection of springs on the characteristics of vibrations

The stiffness of the springs when they are connected in series (Fig. 11.2.1 a and b) is determined taking into account the stiffness of the individual springs (c_1 and c_2) and taking into account the equality of force in the springs, when they are connected in series: $F = F_1 = F_2$, as: $F_1 = c_1 \times \Delta_1$, $F_2 = c_2 \times \Delta_2$, $F = c \times \Delta$ and $\Delta_1 + \Delta_2 = \Delta$, that is in this case (when springs are connected in series) the springs stiffnesses (c_1 and c_2) and the rigidity of the integral connection c are determined as: $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$ and $c = \frac{c_1 \times c_2}{c_1 + c_2}$.



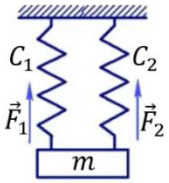
(a)



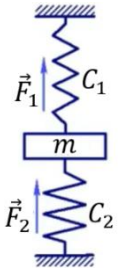
(b)

Fig. 11.2.1

The stiffness when connecting springs in parallel (Fig. 11.2.2 a) is determined considering the stiffnesses of individual springs (c_1 and c_2) and taking into account the equality of deformations of the springs when connecting them in parallel: $\Delta_1 = \Delta_2 = \Delta$, as: $F_1 + F_2 = F$, that is the stiffness at a parallel connection of springs is equal to the sum of the stiffnesses of the connected springs: $c = c_1 + c_2$.



(a)



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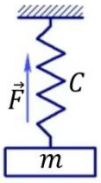
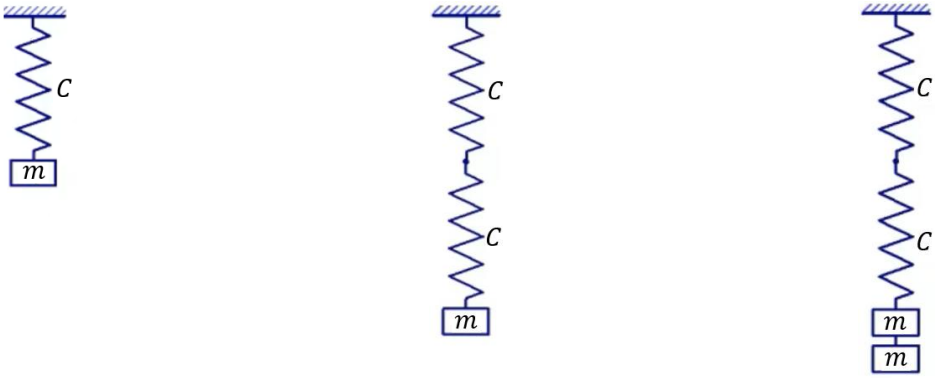


(c)

Fig. 11.2.2

Exercise 1. The material point of assigned mass m, suspended on spring with specified stiffness c, performs free undamped vertical oscillations (Fig. 11.2.3 a), starting its motion from the position of the balance with velocity V_0. The exercise assigns one to

determine how the amplitude A of vibrations of the material point will change, if, under equal conditions, to introduce a second spring of the same stiffness c, connected in series with the first spring (Fig. 11.2.3 b). What will happen to the period of free oscillations if two loads each of specified mass m are suspended on two springs connected in series (Fig. 11.2.3 c) ?



(a) (b) (c)

Fig. 11.2.3

Determination of the ratio of amplitudes of oscillations of a material point (A_2/A_1) if under equal conditions to introduce a second spring of the same stiffness c, connected in series with the first. The stiffness c_2 of an equivalent system at a series connection of two springs of equal stiffness equals: $\frac{1}{c_2} = \frac{1}{c} + \frac{1}{c}$ and $c_2 = \frac{c}{2}$.

Circular (the cyclic) frequency of free vibrations of a material point on a spring equals: $k_1 = \sqrt{c/m}$ and $k_2 = \sqrt{c_2/m} = \sqrt{c/(2 \times m)}$.

When determining the amplitudes (A_1 and A_2) one has to use the initial conditions and to determine the integration constants (C_1 and C_2), which are the coefficients of the displacement formula for vibrations of the material point:

$x = C_1 \times \cos(k \times t) + C_2 \times \sin(k \times t)$, or $x = x_0 \times \cos(k \times t) + \frac{V_0}{k} \times \sin(k \times t)$.

The following initial conditions are used: $x = x_{01} = x_{02} = x_0 = 0$ and $\dot{x} = V_{01} = V_{02} = V_0$ at $t = 0$.

Amplitude A_1 of vibrations of material point when it is suspended on one spring equals: $A_1 = \sqrt{x_{01}^2 + \frac{V_{01}^2}{k_1^2}} = \sqrt{0^2 + \frac{V_0^2}{(c/m)^2}} = V_0 \times \sqrt{\frac{m}{c}}$

and $A_2 = \sqrt{x_{02}^2 + \frac{V_{02}^2}{k_2^2}} = \sqrt{0^2 + \frac{V_0^2}{(c/(2\times m))^2}} = V_0 \times \sqrt{\frac{m}{c}} \times \sqrt{2}$.

The ratio of the amplitudes of oscillations of a material point, if, under equal conditions, to introduce a second spring of the same stiffness c, connected in series with the first one, equals: $A_2/A_1 = \sqrt{2}$.

Cyclic (the circular) frequency of free oscillations will also decrease by $\sqrt{2}$ times if a second load of specified mass m is suspended at two springs connected in series (Fig. 11.2.3 c). When suspending one load of mass m the cyclic frequency of free vibrations, which is equals: $k_3 = \sqrt{c_2/(2 \times m)} = \sqrt{c/(2 \times 2 \times m)} = \frac{k_2}{\sqrt{2}} = \frac{k_1}{\sqrt{2} \times \sqrt{2}} = \frac{k_1}{2}$, the period of free oscillations, wherein (that is at the suspension of two masses m to two series–connected stiffness springs c), increases by $\sqrt{2}$ times compared to the option of suspending one mass m to two series–connected springs and twice as compared to the period free oscillations at suspension of one mass m to one spring.

Exercise 2. The exercise assigns one to determine the ratio of amplitudes (A_3/A_1) of vibrations of material point (the load) at the following differing conditions (Fig. 11.2.4 a): in option 1 the spring stiffness is equal: $c_1 = c$, and the mass of the load is: $m_1 = m$, and in option 3 the spring stiffness is equal: $c_3 = \frac{c}{2}$, and the mass of the load: $m_3 = 2 \times m$.

(a) (b)

Fig. 11.2.4

The cyclic frequency of free oscillations in option 1 indicated above is determined as: $k_1 = \sqrt{c/m}$, and in the option 3 the cyclic frequency is equal:
$k_3 = \sqrt{c_3/m_3} = \sqrt{c/(2 \times 2 \times m)} = \frac{1}{2} \times \sqrt{c/m} = \frac{1}{2} \times k_1$.

In option 1 of the load with specified mass m suspension on one spring of the stiffness equal to: $c_1 = c$ the amplitude A_1 of it vibrations defined above is: $A_1 = V_0 \times \sqrt{\frac{m}{c}}$.

The amplitude A_3 of vibrations of material point of mass: $m_3 = 2 \times m$ in the option 3 at the material point suspension on one spring of stiffness: $c_3 = \frac{c}{2}$ equals:

$A_3 = \sqrt{x_{03}^2 + \frac{V_{03}^2}{k_3^2}} = \sqrt{0^2 + \frac{2^2 \times m \times V_0^2}{c^2}} = 2 \times V_0 \times \sqrt{\frac{m}{c}}$ and $A_3/A_1 = 2$.

Exercise 3. The load of specified mass: $m = 1$ kg is suspended on three springs with the stiffness coefficients: $c_1 = 2$ N/cm and $c_2 = c_3 = 7$ N/cm (Fig. 11.2.4 b). The

exercise assigns one to find the circular frequency k of free vertical harmonic oscillations of the load and the frequency v.

When three springs are connected in parallel, their total stiffness is equal:
$c = c_1 + c_2 + c_3 = 2 + 7 + 7 = 16$, N/cm, that is: $c = 16 \times 10^2$ N/m.

The cyclic frequency equals: $k = \sqrt{c/m} = \sqrt{16 \times 10^2/1} = 40$, s^{-1} or radian/s.

The frequency of harmonic oscillations (the number of oscillation periods passing per unit time) dependent on the cyclic frequency is:
$v = k/(2 \times \pi) = 40/(2 \times \pi) = 6.37$, Hz (periods of harmonic oscillations per second).

Questions and comments:

11.2.1. Mechanical system rigidity by serial and the parallel connection of springs.

11.2.2. The influence of additional mass (the mass doubling) attached to a concentrated load on the cyclic frequency and on the period of oscillations, as well as on the amplitude of material point oscillations.

11.2.3. The influence of an attached spring of same stiffness on the cyclic frequency, period and the amplitude of oscillations.

11.2.4. The $(2 \times \pi)$–ratio of harmonic oscillations frequency and cyclic (the circular) frequency.

11.3. Oscillations of a mechanical system with one degree of freedom (the single–mass system)

One considers vibrations of the mechanical system with one degree of freedom (Fig. 11.3.1), which is a statically determined elastic weightless rod at which a load (the material point) is attached. In the absence of vibrations, the mechanical system has static deflection. The load at its motion is affected by: external force $F(t)$, including the weight of the load, the force of inertia F_I, when the load moves with acceleration, and the force of elasticity R under bending of the rod. The forces act independently, and the load displacement y occurs due to these forces: $y = y_F + y_{FI} + y_R$.

At the substitution of forces defined by unit deflection: δ_{11}, corresponding to force: $P = 1$, the independent displacements: y_F, y_{FI} and y_R one can find as:
$y_F = \delta_{11} \times F(t)$, $F_l = -m \times \ddot{y}$, $y_{Fl} = \delta_{11} \times F_l = -\delta_{11} \times m \times \ddot{y}$ and $y_R = \delta_{11} \times R$,
and displacement of the load wherein is determined by the equation:
$y = \delta_{11} \times F(t) - \delta_{11} \times m \times \ddot{y} - \delta_{11} \times R$.

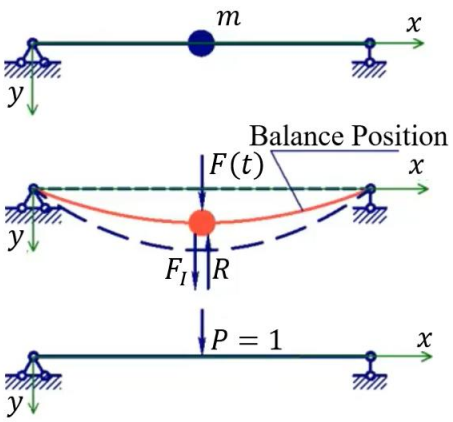


Fig. 11.3.1

The oscillation equation of the mechanical system with one degree of freedom in general form is: $\ddot{y} + \omega^2 \times y + \frac{1}{m} \times R = \frac{1}{m} \times F(t)$, where: $\omega^2 = \frac{1}{\delta_{11} \times m}$.

In the absence of resistance forces, the equation of free vibrations has the form: $\ddot{y} + \omega^2 \times y = 0$. Taking into account the integration constants (C_1 and C_2), this equation has the following solutions: $y = C_1 \times \cos(\omega \times t) + C_2 \times \sin(\omega \times t)$, $\dot{y} = V = -C_1 \times \omega \times \sin(\omega \times t) + C_2 \times \omega \times \cos(\omega \times t)$, where V – the load velocity, and $\ddot{y} = a = -C_1 \times \omega^2 \times \cos(\omega \times t) - C_2 \times \omega^2 \times \sin(\omega \times t)$, a – the load acceleration.

When finding the integration constants (C_1 and C_2) the initial conditions are used: $y = y_0$ and $\dot{y} = V_0$ at $t = 0$.

The integration constants one can find as: $C_1 = y_0$ and $C_2 = \frac{V_0}{\omega}$.

At the set of integration constants, the solution to the oscillation equation of mechanical system with one degree of freedom without friction has the form (Fig. 11.3.2): $y = y_0 \times \cos(\omega \times t) + \frac{V_0}{\omega} \times \sin(\omega \times t)$, or $y = A \times \sin(\omega \times t + \varphi)$,

where: $A = \sqrt{y_0^2 + \frac{V_0^2}{\omega^2}}$ – the vibration amplitude of the mechanical system,

$\varphi = \arctan\left(\frac{y_0 \times \omega}{V_0}\right)$ – the initial phase of oscillation.

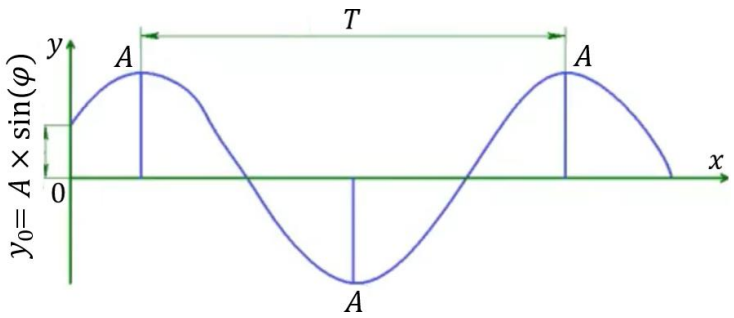
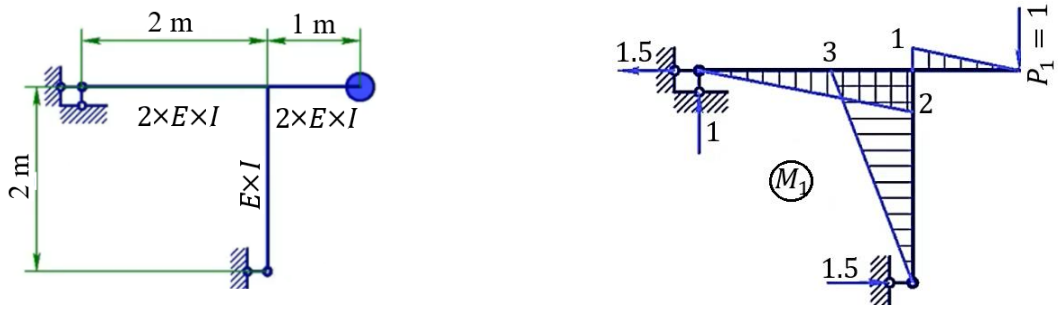


Fig. 11.3.2

Free vibrations of the mechanical system are characterized by following elements. The period of free oscillations is: $T = 2 \times \pi/\omega$. Circular (the cyclic) frequency equals: $\omega = 2 \times \pi/T$. Harmonic vibration frequency is: $f = 1/T$.

For considered mechanical system the formula can be used to determine the angular frequency: $\omega = \sqrt{\frac{1}{\delta_{11} \times m}} = \sqrt{\frac{g}{\delta_{11} \times P}} = \sqrt{\frac{g}{y_S}}$, where y_S – the deflection of the mechanical system under consideration in the position of it static balance.

Exercise. The exercise assigns one to determine the frequency and period of natural oscillations of the single–mass mechanical system (Fig. 11.3.3), if the weight of material point is: $P = 1$ kN, and bending stiffness of the vertical rod equals: $E \times I = 10^4$ kN×m^2.



(a) (b)

Fig. 11.3.3

Natural vibration frequency of assigned single–mass mechanical system equals: $\omega = \sqrt{\frac{g}{\delta_{11} \times P}}$, where $g = 9.81$ m/s^2 – the gravitational acceleration.

To find displacement (the deflection) δ_{11} under unit force: $P_1 = 1$ one can use the Mohr's integral: $\delta_{11} = \sum \int \frac{M_1 \times M_1}{E \times I} \times dx$, and one needs to determine the diagram of moments under unit force (Fig 11.3.3 b).

Displacement under unit force δ_{11}, determined using the Mohr's integral equals:

$$\delta_{11} = \frac{1}{2 \times E \times I} \times \left(\frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \times 2 + \frac{1}{2} \times 1 \times 1 \times \frac{2}{3} \times 1\right) + \frac{1}{E \times I} \times \frac{1}{2} \times 2 \times 3 \times \frac{2}{3} \times 3 =$$

$= 7.5 \times 10^{-4}$, m/kN.

Circular frequency of the single–mass mechanical system free oscillations is:

$$\omega = \sqrt{\frac{g}{\delta_{11} \times P}} = \sqrt{\frac{9.81}{7{,}5 \times 10^{-4} \times 1}} = 114.4, \text{ s}^{-1} \text{ or radian/s}$$

The period of free oscillations is: $T = 2 \times \frac{\pi}{\omega} = 2 \times \frac{\pi}{114.4} = 0.055$, s

Harmonic vibration frequency is equal: $f = \frac{1}{T} = \frac{1}{0.055} = 18.2$, Hz

Questions and comments:

11.3.1. Oscillating elastic mechanical system characterized by rigidity and the mass. Determination of the cyclic frequency, amplitude and the initial phase of oscillations of an elastic mechanical system.

11.3.2. Finding the stiffness index in the direction of motion of concentrated mass and determining the stiffness index as the inverse of the displacement index of a unit force, calculated by the use of the Mohr's integral.

11.4. Determination of the natural characteristics of free vibrations of a multi–mass mechanical system

The equations of free vibrations of a multi–mass (n–mass) mechanical system (Fig. 11.4.1) relate elastic forces arising during n motions of the system, and inertial forces: $F_i = -m_i \times \ddot{y}_i$, $i = 1, \ldots, n$, which can be performed in the form of the canonical equations of the force method, the terms of which are elastic forces proportional to displacements and the inertial forces proportional to accelerations:

$y_1 = -m_1 \times \ddot{y}_1 \times \delta_{11} - m_2 \times \ddot{y}_2 \times \delta_{12} - \cdots - m_n \times \ddot{y}_n \times \delta_{1n}$,

$y_2 = -m_1 \times \ddot{y}_1 \times \delta_{21} - m_2 \times \ddot{y}_2 \times \delta_{22} - \cdots - m_n \times \ddot{y}_n \times \delta_{2n}$,

……,

$y_n = -m_1 \times \ddot{y}_1 \times \delta_{n1} - m_2 \times \ddot{y}_2 \times \delta_{n2} - \cdots - m_n \times \ddot{y}_n \times \delta_{nn}$.

The solution of the equations of free vibrations of n–mass mechanical system one can represent as:

$y_1 = A_1 \times \sin(\omega \times t + \varphi)$,

$y_2 = A_2 \times \sin(\omega \times t + \varphi)$,

……,

$y_n = A_n \times \sin(\omega \times t + \varphi)$.

For acceleration function set: $\ddot{y}_1$, $\ddot{y}_2$ … $\ddot{y}_n$ solution of the equations of free oscillations of the n–mass mechanical system has the form:

$\ddot{y}_1 = -A_1 \times \omega^2 \times \sin(\omega \times t + \varphi)$,

$\ddot{y}_2 = -A_2 \times \omega^2 \times \sin(\omega \times t + \varphi)$,

……,

$\ddot{y}_n = -A_n \times \omega^2 \times \sin(\omega \times t + \varphi)$.

After the set of the derivatives into the original equations they take the form:

$A_1 = A_1 \times \omega^2 \times \delta_{11} + A_2 \times \omega^2 \times \delta_{12} + \cdots + A_n \times \omega^2 \times \delta_{1n}$,

$A_2 = A_1 \times \omega^2 \times \delta_{21} + A_2 \times \omega^2 \times \delta_{22} + \cdots + A_n \times \omega^2 \times \delta_{2n}$,

…….. ,

$A_n = A_1 \times \omega^2 \times \delta_{n1} + A_2 \times \omega^2 \times \delta_{n2} + \cdots + A_n \times \omega^2 \times \delta_{nn}$.

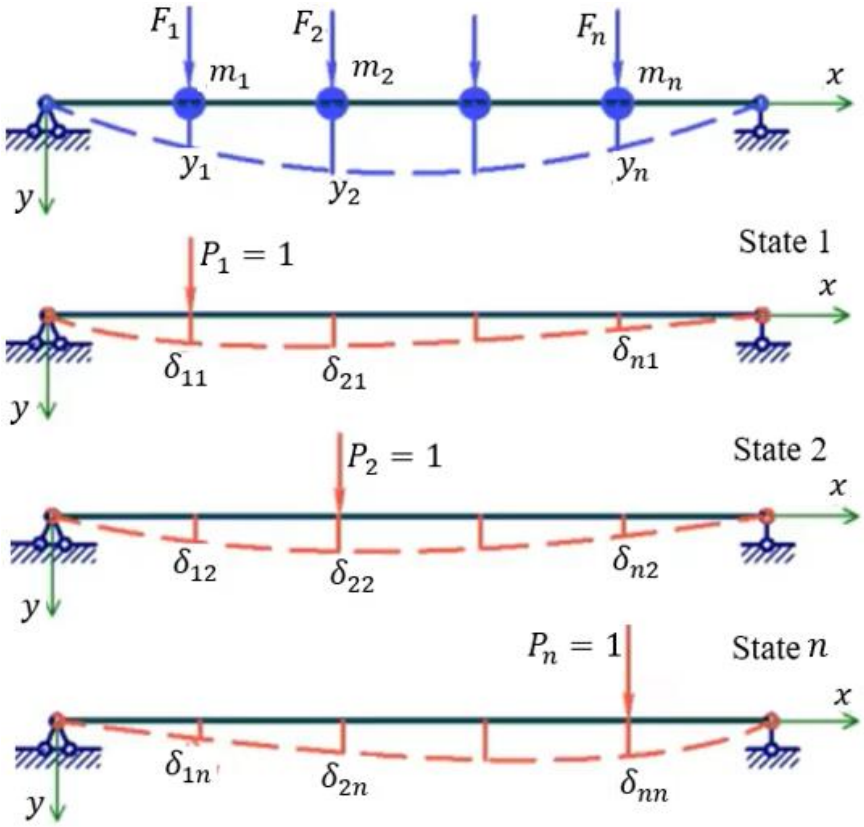


Fig. 11.4.1

In the order of further transformations, one can substitute the indicator of natural frequency ω of free oscillations of the n– mass mechanical system with indicators of the eigen number (the eigenvalue): $\lambda = \frac{1}{\omega^2}$, the system of equations takes the form:

$$(m_1 \times \delta_{11} - \lambda) \times A_1 + m_2 \times \delta_{12} \times A_2 + \cdots + m_n \times \delta_{1n} \times A_n = 0,$$
$$m_1 \times \delta_{21} \times A_1 + (m_2 \times \delta_{22} - \lambda) \times A_2 + \cdots + m_n \times \delta_{2n} \times A_n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$m_1 \times \delta_{n1} \times A_1 + m_2 \times \delta_{2n} \times A_2 + \cdots + (m_n \times \delta_{nn} - \lambda) \times A_n = 0.$$

A trivial solution to the resulting system of equations means that all amplitudes are equal to zero: $A_1 = A_2 = A_n = 0$.

The amplitude matrix $[A]$ can be defined as the product of the compliance matrix $[\delta]$ and the diagonal mass matrix $[m]$:

$$[\delta] = \begin{bmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_{n1} & \delta_{n2} & \ldots & \delta_{nn} \end{bmatrix} \text{ and } [m] = \begin{bmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & m_n \end{bmatrix}.$$

The eigen numbers and natural frequencies of free vibrations n of the n–mass mechanical system. If at least one amplitude is not equal to zero, then is equal to zero the determinant of the matrix of coefficients of system of canonical equations (the characteristic frequency equation):

$$\begin{vmatrix} (m_1 \times \delta_{11} - \lambda) & m_2 \times \delta_{12} & \dots & m_n \times \delta_{1n} \\ m_1 \times \delta_{21} & (m_2 \times \delta_{22} - \lambda) & \dots & m_n \times \delta_{2n} \\ \dots & \dots & \dots & \dots \\ m_1 \times \delta_{n1} & m_2 \times \delta_{2n} & \dots & (m_n \times \delta_{nn} - \lambda) \end{vmatrix} = 0.$$

By revealing the determinant one can obtain a characteristic equation – the polynomial of n –degree: $\lambda^n + b_1 \times \lambda^{n-1} + b_2 \times \lambda^{n-2} + \cdots + b_{n-1} \times \lambda + b_n = 0$.

The roots of this characteristic equation: λ_i $(i = 1, \dots, n)$ serve as eigenvalues of the system of equations: $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

Spectrum of natural frequencies: $\omega_i = \sqrt{\frac{1}{\lambda_i}}$ of free oscillations of the n–mass mechanical system: $\omega_1 < \omega_2 < \cdots < \omega_n$.

The minimal frequency ω_1 is the fundamental tone frequency.

Movements of concentrated masses (the material points, loads) of the mechanical system (Fig. 11.4.1) show the deformed axis of the rod at which the masses are fixed, which characterizes the shape of the vibrations.

The displacement of each mass m_j equals: $y_j = \sum_{i=1}^{n} A_{ij} \times \sin(\omega_i \times t + \varphi_i)$.

Properties of the modes of main vibrations of the multi–mass mechanical system are as follows. Property 1. For each i–th vibration mode for the j–th and k–th masses (m_j and m_k) the ratio of displacements and amplitudes is a constant: $\frac{y_{ji}}{y_{ki}} = \frac{A_{ji}}{A_{ki}} = const$.

Property 2. Main modes of vibration are orthogonal.

The sum of the products of masses m_j by the amplitudes of two main modes of the vibration A_{ji} and A_{jk} for different frequencies ($\omega_i \neq \omega_k$) of natural vibrations of a mechanical system is zero: $\sum_{j=1}^{n} m_j \times A_{ji} \times A_{jk} = 0$, that is, the work of the forces of one main mode of vibration on the motiona of other main mode of vibration is zero, which indicates a superposition of the main modes of vibration.

The matrix form of the resulting system of homogeneous linear equations of free vibrations of the n–mass mechanical system has the form: $([A] - \lambda \times [E]) \times \{V\} = \{0\}$, where $[E]$ – the unit diagonal matrix, $\{V\} = \{A_1, A_2, \dots A_n\}^T$ – the eigenvector (vector of the amplitudes) of the matrix $[A]$, that is, such a non–zero vector $\{V\}$, which, when multiplied by the square matrix $[A]$ turns into itself with numerical coefficient (the factor) λ, which is the eigenvalue of the matrix $[A]$.

Questions and comments:

11.4.1. Equations of free vibrations of the multi–mass mechanical system in the canonical form of the force method, the terms of elastic forces proportional to displacements, and the inertial forces proportional to accelerations of masses.

11.4.2. The form of the equations solving of free oscillations of multi–mass mechanical system, finding the form of accelerations by double differentiation of the form of displacements, substituting into the canonical equations of the force method.

11.4.3. The matrix of amplitudes of free vibrations of multi–mass mechanical system equal to the product of the compliance matrix (the displacements of mechanical system in all directions from unit forces in each direction) by the diagonal mass matrix, along the diagonal of which all masses of the mechanical system are indicated, and the remaining elements of the matrix are zeroes.

11.4.4. Characteristic equation of free oscillations of multi–mass mechanical system, natural frequencies of free oscillations and eigenvalues, the frequency of the fundamental tone – the smallest of the set of solutions to the characteristic equation natural frequency, the spectrum of natural frequencies of free oscillations of the multi–mass mechanical system.

11.4.4. Proportionality of the amplitudes of different masses motions of the multi–mass mechanical system and the amplitudes of these motions for each form of free vibrations of the multi–mass system.

11.4.6. Orthogonality of the vibration main modes of multi–mass mechanical system, i.e. the equality to zero of work done by the forces of one main mode of vibration on the displacements of the other main mode of vibration, which indicates the superposition of vibration main modes.

11.4.7. Matrix form of the system of homogeneous linear equations of free oscillations of the multi–mass mechanical system.

11.5. Spectrum of natural frequencies and main modes of free vibrations of a frame with two concentrated masses, use of the Mathcad computer algebra system to find the spectrum and main modes

Exercise. The exercise assigns one to determine the spectrum of natural frequencies and the main modes of free vibrations of the assigned two–mass mechanical system (the frames with loads, Fig. 11.5.1). The concentrated masses are specified: $m_1 = m$ and $m_2 = 2 \times m$, as well as the bending rigidities are assigned of the vertical sections of the frame ($E \times I$) and the horizontal sections ($2 \times E \times I$).

Mechanical system (the frame) has two degrees of freedom, shown on the scheme by the displacements of the loads (y_1 and y_2) during elastic deformation of the frame, and the characteristic equations of free vibrations of the loads have the form:

$(m_1 \times \delta_{11} - \lambda) \times A_1 + m_2 \times \delta_{12} \times A_2 = 0,$

$m_1 \times \delta_{21} \times A_1 + (m_2 \times \delta_{22} - \lambda) \times A_2 = 0.$

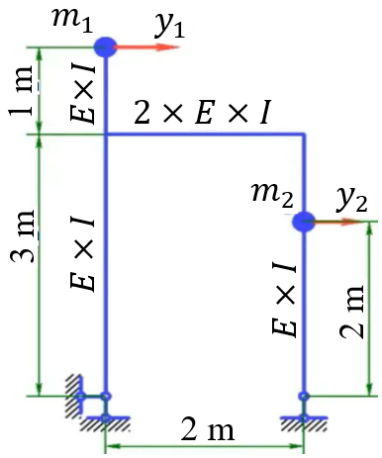
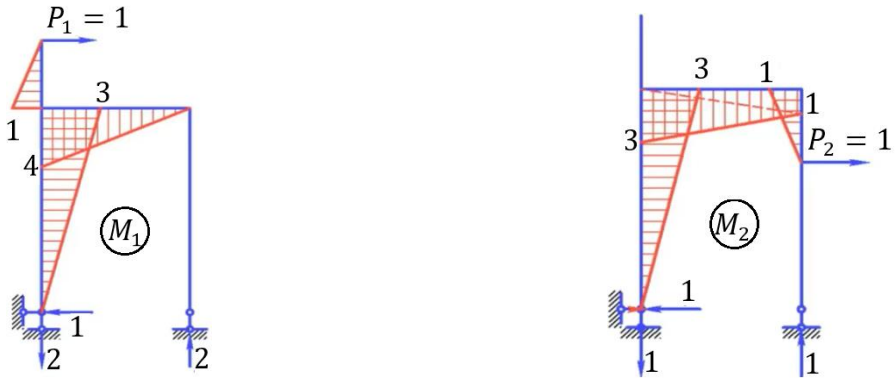


Fig. 11.5.1

To find the deflections under action of unit forces included in the characteristic equations: $P_1 = 1$ and $P_2 = 1$ the determined diagrams are used of the bending moments (Fig. 11.5.2 a and b). These diagrams are multiplied by the sections and the displacement in each of the two directions under the action of each of two unit forces are determined using the Mohr's integral: $\delta_{ij} = \sum \int \frac{M_i \times M_j}{E \times I} \times dx$.



(a) (b)

Fig. 11.5.2

The displacements δ_{ij} under frame bending by unit forces, which are elements of its compliance matrix, when the data introduce are determined as:

$$\delta_{11} = \frac{1}{E \times I} \times \frac{1}{2} \times 3 \times 3 \times \frac{2}{3} \times 3 + \frac{1}{E \times I} \times \frac{1}{2} \times 1 \times 1 \times \frac{2}{3} \times 1 + \frac{1}{2 \times E \times I} \times \frac{1}{2} \times 2 \times 4 \times \frac{2}{3} \times 4 =$$

$$= \frac{44}{3 \times E \times I},$$

$$\delta_{12} = \delta_{21} = \frac{1}{E \times I} \times \frac{1}{2} \times 3 \times 3 \times \frac{2}{3} \times 3 + \frac{1}{E \times I} \times \frac{1}{2} \times 2 \times 4 \times \left(\frac{2}{3} \times 3 + \frac{1}{3} \times 1\right) = \frac{23}{3 \times E \times I},$$

and $\delta_{22} = \frac{1}{E \times I} \times \frac{1}{2} \times 3 \times 3 \times \frac{2}{3} \times 3 +$

$$+ \frac{1}{2 \times E \times I} \times \left(\frac{1}{2} \times 2 \times 3 \times \frac{2}{3} \times 3 + \frac{1}{2} \times 2 \times 1 \times \frac{2}{3} \times 1\right) + \frac{1}{E \times I} \times \frac{1}{2} \times 1 \times 1 \times \frac{2}{3} \times 1 = \frac{38}{3 \times E \times I}.$$

By drawing up the characteristic equations and determine the eigenvalues, it is relevant to combine the frame bending rigidity data and masses (m_1 and m_2) with sought eigenvalues λ in the form of a generalized parameter $\bar{\lambda}$, which equals:
$\bar{\lambda} = \lambda \times \frac{3\times E\times I}{m} = \frac{3\times E\times I}{\omega^2\times m}$.

As a function of the generalized parameter $\bar{\lambda}$ the system of characteristic equations for an elastic frame with two concentrated masses has the form:
$(44 - \bar{\lambda}) \times A_1 + 46 \times A_2 = 0,$
$23 \times A_1 + (76 - \bar{\lambda}) \times A_2 = 0.$

The expanding of a system of characteristic equations determinant:

$$Det = \begin{vmatrix} (44 - \bar{\lambda}) & 46 \\ 23 & (76 - \bar{\lambda}) \end{vmatrix} = 0$$

gives the characteristic equation as a function of the generalized parameter $\bar{\lambda}$:
$\bar{\lambda}^2 - 120 \times \bar{\lambda} + 2286 = 0.$

The solutions of the characteristic equation is the values of the generalized parameter $\bar{\lambda}$, which equals to: $\bar{\lambda}_{1,2} = 60 \pm \sqrt{3600 - 2286} = 60 \pm 36.25$, $\bar{\lambda}_1 = 96.25$ and $\bar{\lambda}_2 = 23.75$, as well as the values of the spectrum of natural frequencies of free vibrations for the assigned two–mass mechanical system (the frame with masses):

$\omega_1 = \sqrt{\frac{3\times E\times I}{\bar{\lambda}_1\times m}} = \sqrt{\frac{3\times E\times I}{96.25\times m}} = 0.1765 \times \sqrt{\frac{E\times I}{m}}$

and $\omega_2 = \sqrt{\frac{3\times E\times I}{\bar{\lambda}_2\times m}} = \sqrt{\frac{3\times E\times I}{23.75\times m}} = 0.3524 \times \sqrt{\frac{E\times I}{m}}$, where ω_1 – the frequency of the fundamental tone – the smallest of the frequencies.

Determination of the spectra of natural frequencies of free vibrations for assigned two–mass mechanical system one can do using the property of orthogonality of the main vibration modes: for different frequencies ($\omega_i \neq \omega_k$) the sum of the products of masses m_j by the amplitudes of the main modes of natural oscillations A_{ji} and A_{jk}:
$\sum_{j=1}^{n} m_j \times A_{ji} \times A_{jk} = 0$, that is for the characteristic equation of a mechanical system:
$23 \times A_{11} + (76 - \bar{\lambda}_1) \times A_{21} = 0,$
at $A_{11} = 1$, $23 \times 1 + (76 - 96.25) \times A_{21} = 0$ and $A_{21} = 1.1358$,
at $A_{12} = 1$, $23 \times 1 + (76 - 23.75) \times A_{22} = 0$ and $A_{22} = -0.4402$.

For assigned two–mass mechanical system the eigenvectors (vector of the amplitudes) of two main modes of vibration are equal: $\{V\} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.1358 \end{Bmatrix}$
and $\{V\} = \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.4402 \end{Bmatrix}$.

The main modes of vibration are shown on the schemes (Fig. 11.5.3 a and b).

(a) (b)

Fig. 11.5.3

The verification of the main vibration modes orthogonality (for a mechanical system with two degrees of freedom): $\sum_{j=1}^{2} m_j \times A_{j1} \times A_{j2} = m_1 \times A_{11} \times A_{12} +$ $+m_2 \times A_{21} \times A_{22} = m \times 1 \times 1 + 2 \times m \times 1.1358 \times (-0.4402) \approx 0.$

Using the Mathcad computer algebra system to determine the spectrum of natural frequencies and the main modes of free vibrations of the two–masses mechanical system (fragment of the page in the Mathcad is shown on the scheme, Fig. 11.5.4).

$$\delta := \begin{bmatrix} \frac{44}{3} & \frac{23}{3} \\ \frac{23}{3} & \frac{38}{3} \end{bmatrix} \qquad m := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad M := \mathrm{diag}(m) \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad A := \delta \cdot M$$

$$A = \begin{bmatrix} 14.667 & 15.333 \\ 7.667 & 25.333 \end{bmatrix} \qquad E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda := \mathrm{genvals}(A, E) \qquad \lambda = \begin{bmatrix} 32.083 \\ 7.917 \end{bmatrix}$$

$$V := \mathrm{genvecs}(A, E) \qquad V = \begin{bmatrix} 0.88 & 1 \\ 1 & -0.44 \end{bmatrix} \qquad \frac{1}{0.88} = 1.136$$

Fig. 11.5.4

Questions and comments:

11.5.1. Form of characteristic equations of free vibrations of an elastic frame with concentrated loads.

11.5.2. Determining of diagrams of bending moments under the action of unit forces acting on an elastic frame in the directions of possible motions of concentrated loads and the use of the Mohr's integral to find these motions (the elements of compliance matrix of system of the force method canonical equations).

11.5.3. A system of characteristic equations for determining the eigenvalues of free vibrations of an elastic frame with two concentrated loads, grouping the eigenvalues with indicators of masses of the loads and the rigidities of elastic frame in accordance with the elements of the compliance matrix (the displacements under action of unit forces).

11.5.4. Finding the determinant of the characteristic equations system given the roots of the final characteristic equation and determining the natural frequencies of free oscillations of two–masses elastic frame.

11.5.5. Finding the amplitudes of natural vibrations to determine the spectrum (the distribution of amplitudes over frequencies) using the property of orthogonality of the main modes of vibration (i.e., the equality to zero of the work of forces of one main mode of vibration on the displacements of the other main mode).

11.5.6. Determination in the Mathcad using the «genvals» operation of the vector of eigenvalues of square matrix of amplitudes, serving as numerical coefficients (the eigenvalues of the amplitude matrix), by which the eigenvector of amplitudes, when multiplied by the amplitude matrix, turns into itself in the proportions of these coefficients.

Chapter 12. The linear calculations of ship rolling and motion in water

The oscillations and motion of a body in a liquid medium, as well as linear calculations of the ship roll, are considered from the point of view of using mechanics methods to determine the elements of oscillation and the motion of mechanical system within the framework of the used premises of incompressible fluid mechanics (the hydromechanics), which themselves are considered as methodological constraints. At the same time, methods for solving problems in mechanics are revealed, and the definition of the initial data and the methodological limitations used are considered specified in the relevant subjects, in particular, in fluid mechanics.

12.1. Classification of forces and moments acting on a ship during it motion and roll, the coordinate systems and the systems of equations of motion, the elements of ship roll

To formulate the constraints and the assumptions in issues of finding the elements of a ship's rolling and its controlled motion, the variants of rectangular Cartesian coordinate systems are usually used [14, 19, 22 and 28]:

1. System of coordinate planes: the main plane is a horizontal one passing along the bottom of the ship, the diametral plane is the vertical longitudinal plane of symmetry of the ship and the midship–section plane is the transverse vertical one, passing in the middle of ship waterline length. Associated with the ship the right Cartesian coordinate system $0xyz$ is used, whose axes lie at the intersection of planes, in which the geometric characteristics of the shape of ship hull are usually defined (Fig. 12.1).

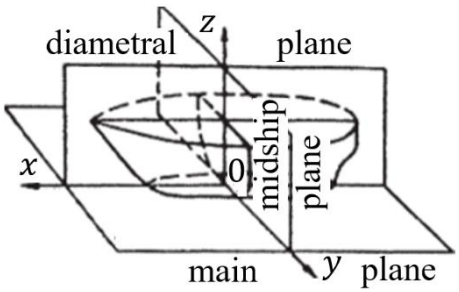


Fig. 12.1

2. Moving coordinate system $Gxyz$ associated with ship. Its center coincides with the center of gravity of the ship G, the abscissa axis Gx is directed to the bow, the ordinate axis Gy – to port side – for the right coordinate system, in which the elements of the ship's roll are usually determined, or to the starboard side (with the left coordinate system, which

is common when forming equations in problems of ship controllability) and the applicate axis Gz is directed upward.

In this coordinate system it is convenient to determine the forces and moments acting on the ship, as well as the translational and angular velocities and accelerations. The system $Gxyz$, like considered others, is inertial, and the laws of dynamics apply in it.

The $Gxyz$ system for ship controllability problems study is usually left–handed, i.e. the «left screw» rule is applied: if one rotates the Gx axis relative to the Gz axis in the direction of the Gy axis and align the axis of the left screw with the Gz axis, then the direction of the screw motion coincides with the direction of the axis Gz.

The others coordinate systems in the issues of controllability of ship are also left–handed. In contrast, the coordinate systems used in the issues of ship rolling are, as a rule, the right–handed, and in them a similar rule of the «right screw», is valid, and the Gy axis is directed to the port side, which is due to mathematical expediency. This difference is manifested in the fact that the signs of some forces and moments, as well as linear and the angular accelerations, velocities and displacements in the equations of ship rolling and its controllability are opposite.

The «right screw» means that when the abscissa axis is rotated relative to the applicate axis in the direction of the ordinate axis, the motion of the «right screw» coincides with the direction of the applicate axis.

3. A fixed coordinate system $0\xi\eta\zeta$, the origin of which coincides with the center of mass G at the initial moment of time, and the directions of the axes at this moment may coincide with the directions of the axes of the coordinate system associated with the ship (although they may differ).

Typically, the translational and angular velocities and accelerations are determined in a coupled coordinate system, which, as noted, is inertial, and the displacements are calculated by integrating these velocities in a fixed coordinate system, and this transition is of kinematic nature.

4. Semi–coupled coordinate system $G\xi_1\eta_1\zeta_1$, the axes of which are parallel to the axes of the fixed system, and the origin is associated with the moving center of mass G.

Considering that a ship in water has six degrees of freedom – three translational degrees of freedom and three rotational, its instantaneous position in a stationary system can be characterized by six elements – three linear motions: ξ, η and ζ, identical with the axes of the fixed coordinate system, and three angular motions: the roll angle θ (the rotation of ship relative to abscissa longitudinal axis Gx), the trim angle φ (the rotation relative to ordinate axis Gy) and the yaw angle χ (the rotation around the axis of applicate Gz).

Despite the different geometric meanings, the listed elements of the ship motion are considered to be components of single vector: $\{\xi, \eta, \varsigma, \theta, \varphi, \chi\}$, since the mathematical operations that can be performed on these coordinates are uniform.

The directions of the vectors: θ, φ and χ, which characterize the angular displacements relative to the axes in the right coordinate system, are sought using the «right screw» rule, the same applies to time derivatives of angular displacements – the angular velocities and accelerations, accordingly, in the left coordinate system to find them directions the «left screw» rule applies. In the right coordinate system, the positive direction of the ship's roll is to the starboard side, the trim is to the bow, the positive course change is counterclockwise, when viewed from above – against the axis of the applicate.

In addition to the considered coordinates, the movement of ship on the surface of water is characterized by following elements:

– the drift angle β_D between the velocity vector and abscissa axis Gx, the curvature of trajectory, the linear velocity of a certain point on the wetted surface of hull of the ship;

– the course CR, i.e. the angle between associated longitudinal axis Gx, which usually lies in the center plane of the ship, and the abscissa axis of fixed coordinate system 0ξ, which in the maritime navigation is directed north;

– the velocity angle φ_v, that is the angle between the direction of ship velocity vector and the abscissa axis of the fixed coordinate system 0ξ.

There is a relationship between the course CR of ship on water surface, the velocity angle φ_v and the drift angle β_D: $\varphi_v = CR + \beta_D$.

Despite the similarity of some terms in the issues about ship rolling and its controllability, there are also differences. When determining the ship rolling elements, the classification of forces and moments contains the following categories:

1. The inertial forces and moments caused by the massiveness of the ship and water and proportional to accelerations.

2. Damping forces and moments of resistance during translational motion and the rotation, caused by the viscosity and massiveness of water when energy is absorbed by waves generated by moving ship.

3. The restoring forces and moments – depending on translational and angular movements and returning ship to the equilibrium position.

4. Disturbing forces and moments – deviating ship from the equilibrium position, for example, the forces and moments of wave nature.

On regular waves, the disturbing forces include the main part and the diffraction part, so–called A.N. Krylov's one [22]. The main part is determined by assumption that the ship is not an obstacle to waves propagation, figuratively speaking, that the ship is «transparent» to waves. The A.N. Krylov's part is caused by the diffraction of waves going around the ship.

In the issues on ship controllability the following classification of acting forces and moments into categories is common:

1. Driving forces and moments (the control efforts) generated by ship propulsors and controls.

2. The external forces and moments, which sources are in the aquatic environment and do not have a causal dependence on ship motion.

3. Reactive forces and moments from water environment arising by the ship motion.

Reactive forces and moments acting on a moving ship from the side of water environment can, in turn, be subdivided into the following categories:

– the inertial, proportional to accelerations translational and angular, i.e. their projections, the masses and moments of inertia, and the viscous ones proportional to velocities (translational and angular) and their projections, or the squares of velocities projections;

– or the positional forces and moments, acting during translational motion, and the damping ones arising during rotation.

When determining the elements of controlled ship motion, the equilibrium equations do not consist of restoring forces and moments, in contrast to the equations, used to calculate the elements of ship roll, and they consist of the driving forces and moments, that are the control efforts.

The set of proportionality coefficients between the inertial forces and moments, on one hand, and the translational and angular accelerations of ship in water, on other hand, includes: the ship displacement Δ, the moments of inertia of ship relative to the coordinate axes: I_X, I_Y and I_Z, the added masses and moments of inertia: μ_{ij} at $i = 1, \dots 6$ and $j = 1, \dots 6$.

The indices correspond to translational motions (the velocities, accelerations and forces) in the direction of the axes: abscissa, ordinate and applicate are: $i = 1, 2, 3$ or $j = 1, 2, 3$ and to angular movements (the angular velocities, angular accelerations and moments of force) under rotation relative to the axes: abscissa, ordinate and applicate, accordingly: $i = 4, 5, 6$ or $j = 4, 5, 6$.

The first index denotes the projections of the components of the inertial force on the axes: abscissa Gx (at $i = 1$), ordinate Gy (at $i = 2$) and the applicate Gy (at $i = 3$), or the projections of the moment of inertial forces during rotation with acceleration relative to the axes: Gx (at $i = 4$), Gy (at $i = 5$) and Gz (at $i = 6$), included in the equations for each of the directions of ship roll: for the longitudinal–horizontal direction (periodic reciprocating motions of ship in the direction of Gx axis), for transverse–horizontal direction (the motion along Gy axis) and the vertical direction (along Gz axis), for roll (the angular motion of ship relative to Gx axis), for pitching (turns of ship around Gy axis), and yaw (turns relative to Gz axis).

The methods for the added masses and moments of inertia finding in fluid mechanics are subject of experimental research and the numerical calculations [14]. To find them in fluid mechanics two methods are usually used:

– the added masses of ship in water and its moments of inertia are sought theoretically for triaxial ellipsoids, which semi–axes correspond to the ship particulars (half the design length, half the width and the draught);

– they are determined numerically using the hypothesis of plane sections, that is the assumption that the ship, during it transverse translational and rotational motions is flow around flat cross–sections (i.e., around theoretical frames).

While graphs of the dependences of the coefficients of added masses on the shape elements of typical contours are used, thus, the added mass is sought for each cross–section of hull separately, and then the numerical integration is performed along the ship length.

Resistance during the rocking of the ship and during its movement arises because the energy of the ship and the attached mass of water is absorbed mainly by waves diverging from the rocking ship, as well as by the viscous movement of water. To find the damping coefficients, this energy would need to be estimated. But such a problem does not have an analytical solution and goes beyond the scope of linear wave theory.

The emerging damping forces (the fluid resistance) are nonlinear with respect to velocity of the body, but to find analytical solutions in the issues of determining the elements of pitching and controlled motion of ship in water the simplifying hypothesis of the linearization of damping forces can be used, that is the assumption of a linear dependence of forces and moments of resistance on the corresponding velocities of motion of the ship in water (the translational and angular).

In the issue of finding the phase elements of motion (the displacement, velocity and acceleration) of ship in water, the use of damping coefficients is required, that is the indicators of proportionality between the emerging forces or the moments of resistance and the velocities of ship in water (the translational and angular ones).

The set of damping coefficients, as well as the coefficients of added masses when a body moves in a fluid, corresponds to the number of degrees of freedom of a moving body (in the general case, six degrees – three translational and three rotational): λ_{ij} at $i = 1, \dots 6$ and $j = 1, \dots 6$.

Previously discussed formal analogies between translational and angular displacements, which make it possible to combine them as components into the common vector, as well as similar definitions of the components of the vectors of translational and angular velocities and accelerations, also apply to form of the components of force vectors and moments of forces. The directions of these moments are usually determined by the «right screw» rule, and vector operations on the projections of forces formally coincide with the operations on projections of moments of forces.

Therefore, the moments and forces are often combined as components of a vector – the resultant force. Six general components of this vector are collectively denoted the forces, despite the fact that three of them are the moments.

Ultimately, these formal similarities in each specified category: the translational and angular displacements, velocities, accelerations, forces and moments of forces lead to generalization of the inertial characteristics of ship and the aquatic environment: the masses and moments of inertia, which are proportionality coefficients between forces and moments of inertia (the components of the forces and moments resultants), on one hand, and the accelerations, accordingly the translational and angular, on the other hand.

For this reason, the added masses of water and the added moments of inertia of water are usually considered as components of matrix of the added masses, although some of these components are added moments of inertia. Specified added indexes indicate the relationship between inertial forces and moments and the translational and angular accelerations.

Formally, the same ordering rule applies to resistance coefficients (the damping ones), which are elements of the matrix of the damping coefficients for translational and rotational (the angular) ship motion in water, as well as combined motion, when, for example, the rotational motion causes the damping forces to translational motion (or translational motion causes rotational resistance).

The same attributes the added masses, that are the the coefficients of proportionality of inertial forces in each direction of rotational or the translational motion and the accelerations (translational or the angular) in each corresponding direction. This corresponds with formal similarities when performing the vector operations on inertial forces and moments.

System of equations of ship motion during roll is drawing up by summing the projections of forces (and moments): the inertial R_{ij}, the damping F_{ij} and the restoring V_i.

The sum of projections of these forces and moments is equal to the projection of the disturbing force N_i (or the moment): $\sum_{j=1}^{6} R_{ij} + \sum_{j=1}^{6} F_{ij} + V_i = N_i$, $i = 1, \dots 6$, where: $R_{ij} = R_{ij}((M^* + M), \ddot{X})$ – the mathematical notation of the matrix of inertial forces; $F_{ij} = F_{ij}(\Lambda, \dot{X})$ – the damping force matrix; $V_i = V_i(\Delta, h, h_\varphi, x_W, y_W, S_W, X)$ – the matrix of restorative forces; $X = \{\xi, \eta, \varsigma, \theta, \varphi, \chi\}$ – the vector of displacement coordinates (linear and the angular): along the abscissa axis ξ, ordinate axis η and applicate axis ς, roll angle θ (rotation relative to the abscissa axis), trim φ (the rotation relative to the ordinate axis, which when calculating pitching elements is usually directed towards port side of ship, and the coordinate system is right) and yaw χ (rotation relative to the applicate axis); h and h_φ – the magnitudes of initial transverse and longitudinal metacentric heights of ship (the vertical distance from the center of gravity of the ship to the metacenter – the instantaneous center relative to which the ship turns when heeling or trimming); S_W – the

waterline area; x_W and y_W – the abscissa and ordinate of the center of the waterline area; $\dot{X}$ and $\ddot{X}$ – the vectors of time derivatives of coordinates; M^*, M and Λ – the matrices: own masses and moments of inertia of ship, the added masses and damping (the resistance) coefficients, in the general case (for a body with six degrees of freedom moving in a fluid).

The matrices: M^*, M and Λ have the forms:

$$M^* = \begin{bmatrix} \Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I_X & 0 & 0 \\ 0 & 0 & 0 & 0 & I_Y & 0 \\ 0 & 0 & 0 & 0 & 0 & I_Z \end{bmatrix}, \; M = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} & \mu_{45} & \mu_{46} \\ \mu_{51} & \mu_{52} & \mu_{53} & \mu_{54} & \mu_{55} & \mu_{56} \\ \mu_{61} & \mu_{62} & \mu_{63} & \mu_{64} & \mu_{65} & \mu_{66} \end{bmatrix}$$

and $\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} & \lambda_{16} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & \lambda_{25} & \lambda_{26} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} & \lambda_{35} & \lambda_{36} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} & \lambda_{45} & \lambda_{46} \\ \lambda_{51} & \lambda_{52} & \lambda_{53} & \lambda_{54} & \lambda_{55} & \lambda_{56} \\ \lambda_{61} & \lambda_{62} & \lambda_{63} & \lambda_{64} & \lambda_{65} & \lambda_{66} \end{bmatrix}$,

where Δ – the mass of a body immersed in water (the ship displacement); I_X, I_Y and I_Z – the moments of inertia when the body rotates with acceleration relative to the axes: abscissa, ordinate and applicate; μ_{ij} – the coefficient of added mass, that is element of the matrix of added masses, showing the influence on a body in an aquatic environment of forces or moments of inertial nature in direction i, when the body moves with translational or angular acceleration in direction j and is a coefficient of proportionality between acceleration in direction j and that arising as a result inertia of water force (or moment) acting from the water on immersed body in the direction i; λ_{ij} – the damping coefficient when a body moves in an aquatic environment, which is the proportionality factor between the velocity in direction j and the resistance force (or moment) arising from the water acting on the body in direction i.

The restoring and disturbing forces in vector notation have the form:

$V = \{V_i, i = 1, \dots 6\} = \{0,0, V_3(X), V_4(X), V_5(X), 0\}$

and $N = \{N_i, i = 1, \dots 6\} = \{N_X, N_Y, N_Z, M_X, M_Y, M_Z\}$.

It would be mentioned that the restoring forces and moments depending on the motions X of ship: $V_3(X)$, $V_4(X)$ and $V_5(X)$ on the water, i.e. preventing motion and returning the ship to the equilibrium position, occur in the vertical direction $V_3(X)$ parallel to the applicate axis, as forces of excess buoyancy or excess weight of the ship when it heaves vertically relative to the level of the equilibrium waterline, or as moments, returning the ship to the equilibrium position when it rotates relative to the x–axis $V_4(X)$, that is when ship roll, or when turning relative to the ordinate axis $V_5(X)$ – the restoring moment during pitching.

The matrix of own inertial characteristics of ship M^* has a diagonal form. The matrices of added masses M and damping coefficients Λ are symmetrical about the main diagonal: $\mu_{ij} = \mu_{ji}$ and $\lambda_{ij} = \lambda_{ji}$.

The matrix notation of the system of equations in the issues of ship rolling has the form: $(M^* + M) \times \ddot{X} + \Lambda \times \dot{X} + V = N$.

In its expanded form, the system of six equations of ship motion in water has more complex form, although careful analysis of the formulas indicates their certain uniformity:

$$(\Delta + \mu_{11}) \times \ddot{\xi} + \mu_{12} \times \ddot{\eta} + \mu_{13} \times \ddot{\varsigma} + \mu_{14} \times \ddot{\theta} + \mu_{15} \times \ddot{\varphi} + \mu_{16} \times \ddot{\chi} +$$
$$+\lambda_{11} \times \dot{\xi} + \lambda_{12} \times \dot{\eta} + \lambda_{13} \times \dot{\varsigma} + \lambda_{14} \times \dot{\theta} + \lambda_{15} \times \dot{\varphi} + \lambda_{16} \times \dot{\chi} + 0 = N_x,$$

$$\mu_{21} \times \ddot{\xi} + (\Delta + \mu_{22}) \times \ddot{\eta} + \mu_{23} \times \ddot{\varsigma} + \mu_{24} \times \ddot{\theta} + \mu_{25} \times \ddot{\varphi} + \mu_{26} \times \ddot{\chi} +$$
$$+\lambda_{21} \times \dot{\xi} + \lambda_{22} \times \dot{\eta} + \lambda_{23} \times \dot{\varsigma} + \lambda_{24} \times \dot{\theta} + \lambda_{25} \times \dot{\varphi} + \lambda_{26} \times \dot{\chi} + 0 = N_y,$$

$$\mu_{31} \times \ddot{\xi} + \mu_{32} \times \ddot{\eta} + (\Delta + \mu_{33}) \times \ddot{\varsigma} + \mu_{34} \times \ddot{\theta} + \mu_{35} \times \ddot{\varphi} + \mu_{36} \times \ddot{\chi} +$$
$$+\lambda_{31} \times \dot{\xi} + \lambda_{32} \times \dot{\eta} + \lambda_{33} \times \dot{\varsigma} + \lambda_{34} \times \dot{\theta} + \lambda_{35} \times \dot{\varphi} + \lambda_{36} \times \dot{\chi} +$$
$$+\rho \times g \times S_W \times \varsigma + \Delta \times g \times h \times \theta / y_W + \Delta \times g \times h_\varphi \times \varphi / x_W = N_z,$$

$$\mu_{41} \times \ddot{\xi} + \mu_{42} \times \ddot{\eta} + \mu_{43} \times \ddot{\varsigma} + (I_x + \mu_{44}) \times \ddot{\theta} + \mu_{45} \times \ddot{\varphi} + \mu_{46} \times \ddot{\chi} +$$
$$+\lambda_{41} \times \dot{\xi} + \lambda_{42} \times \dot{\eta} + \lambda_{43} \times \dot{\varsigma} + \lambda_{44} \times \dot{\theta} + \lambda_{45} \times \dot{\varphi} + \lambda_{46} \times \dot{\chi} +$$
$$+\Delta \times g \times h \times \theta - \Delta \times g \times h_\varphi \times \varphi \times y_W / x_W + \rho \times g \times S_W \times y_W = M_x,$$

$$\mu_{51} \times \ddot{\xi} + \mu_{52} \times \ddot{\eta} + \mu_{53} \times \ddot{\varsigma} + \mu_{54} \times \ddot{\theta} + (I_y + \mu_{55}) \times \ddot{\varphi} + \mu_{56} \times \ddot{\chi} +$$
$$+\lambda_{51} \times \dot{\xi} + \lambda_{52} \times \dot{\eta} + \lambda_{53} \times \dot{\varsigma} + \lambda_{54} \times \dot{\theta} + \lambda_{55} \times \dot{\varphi} + \lambda_{56} \times \dot{\chi} +$$
$$+\Delta \times g \times h_\varphi \times \varphi - \Delta \times g \times h \times \theta \times x_W / y_W + \rho \times g \times S_W \times x_W = M_y,$$

$$\mu_{61} \times \ddot{\xi} + \mu_{62} \times \ddot{\eta} + \mu_{63} \times \ddot{\varsigma} + \mu_{64} \times \ddot{\theta} + \mu_{65} \times \ddot{\varphi} + (I_z + \mu_{66}) \times \ddot{\chi} +$$
$$+\lambda_{61} \times \dot{\xi} + \lambda_{62} \times \dot{\eta} + \lambda_{63} \times \dot{\varsigma} + \lambda_{64} \times \dot{\theta} + \lambda_{65} \times \dot{\varphi} + \lambda_{66} \times \dot{\chi} + 0 = M_z,$$

where h and h_φ – the initial transverse metacentric height and longitudinal metacentric height; $g = 9.81$ m/s^2 – the acceleration of gravity.

In the indicated system of equations, the disturbing forces and moments are specified in projections on the coordinate axes: $N = \{N_x, N_y, N_z, M_x, M_y, M_z\} = \{N_i\}$.

The dot symbols above the symbols of linear and angular coordinates correspond to single or double time differentiation. Considered formulas are universal in terms of the dimensions used, which, when substitute values, have to be mutually consistent.

When determining the elements of ship rocking, the assumptions about the ship symmetry relative to diametral and the midship planes make it possible to simplify the

equations corresponding to main kinds of rocking (the heaving, roll and pitching: $i = 3, 4$ and 5).

These assumptions also allow one to adopt the principle of superposition, that is the independence among themselves of the main types of ship roll, and the matrices of the coefficients of added masses and damping coefficients take on a diagonal form: $\mu_{ij} = 0$ and $\lambda_{ij} = 0$, if $i \neq j$.

In this case there is no need for double indexation of added masses and damping coefficients, and the equations in the issue of ship rocking can be solved independently:

$(\Delta + \mu_3) \times \ddot{\varsigma} + \lambda_3 \times \dot{\varsigma} + \rho \times g \times S_W \times \varsigma = N_z,$

$(I_x + \mu_4) \times \ddot{\theta} + \lambda_4 \times \dot{\theta} + \Delta \times g \times h \times \theta = M_x,$

$\left(I_y + \mu_5\right) \times \ddot{\varphi} + \lambda_5 \times \dot{\varphi} + \Delta \times g \times h_\varphi \times \varphi = M_y.$

As part of solutions of the issues of finding the elements of ship rocking in regular waves and when determining the spectral characteristics of rocking in irregular waves, the results of solving of the isolated homogeneous differential equations of roll in calm water are used (with zero disturbing influences: N_z, M_x and M_y, when the right side of equations is zero): $(\Delta + \mu_3) \times \ddot{\varsigma} + \lambda_3 \times \dot{\varsigma} + \rho \times g \times S_W \times \varsigma = 0,$

$(I_x + \mu_4) \times \ddot{\theta} + \lambda_4 \times \dot{\theta} + \Delta \times g \times h \times \theta = 0,$

$\left(I_y + \mu_5\right) \times \ddot{\varphi} + \lambda_5 \times \dot{\varphi} + \Delta \times g \times h_\varphi \times \varphi = 0.$

The cause of ship rocking in calm water can be a short–term impact that deviates the ship from the balanced position. Ship rocking occurs after the original impact has ceased or changed. Methodologically, the elements of ship rocking in calm water are used in solving the equations of rocking in rough seas. In addition to assessing the sustainability of a ship according to certain conventional standards of maritime classification societies, the elements of ship rocking in regular and irregular waves are associated with assessing the seaworthiness of the sustainability of ship based on study of existing physical laws, taking into account that the possible capsizing of an intact ship has a dynamic nature of roll under extreme conditions.

Isolated linear homogeneous equations of ship rolling on calm water have the form:

$\ddot{\varsigma} + 2 \times \nu_3 \times \dot{\varsigma} + \omega_3^2 \times \varsigma = 0, \nu_3 = \frac{\lambda_3}{2 \times (\Delta + \mu_3)}, \omega_3 = \sqrt{\frac{\rho \times g \times S_W}{(\Delta + \mu_3)}},$

$\ddot{\theta} + 2 \times \nu_4 \times \dot{\theta} + \omega_4^2 \times \theta = 0, \nu_4 = \frac{\lambda_4}{2 \times (I_x + \mu_4)}, \omega_4 = \sqrt{\frac{\Delta \times g \times h}{(I_x + \mu_4)}},$

$\ddot{\varphi} + 2 \times \nu_5 \times \dot{\varphi} + \omega_5^2 \times \varphi = 0, \nu_5 = \frac{\lambda_5}{2 \times (I_y + \mu_5)}, \omega_5 = \sqrt{\frac{\Delta \times g \times h_\varphi}{(I_y + \mu_5)}},$

where ν_3, ν_4 and ν_5 – the decrements (multipliers in the exponent) of the attenuation of the heaving, roll and pitching of a ship in calm water due to the viscous resistance of water and the dissipation of the energy of the rocking of the ship by the waves it generates; ω_3,

ω_4 and ω_5 – the indicators of angular (the cyclic) frequency of natural oscillations of a ship in water for the main types of its rocking.

The general solutions of isolated linear homogeneous equations of ship rocking on calm water have the form: $\varsigma = \varsigma_0 \times e^{-\nu_3 \times t} \times \cos(\omega_3 \times t + \beta_3)$, $\theta = \theta_0 \times e^{-\nu_4 \times t} \times \cos(\omega_4 \times t + \beta_4)$ and $\varphi = \varphi_0 \times e^{-\nu_5 \times t} \times \cos(\omega_5 \times t + \beta_5)$, where ς_0, θ_0 and φ_0 – the initial amplitudes of heaving, roll and the pitching on calm water; β_3, β_4 and β_5 – the initial phases of the heaving, roll and pitching.

Questions and comments:

12.1.1. The coordinate systems for finding the elements of pitching and controlled motion of ship: the moving coordinate system (connected with moving ship), in which it is convenient to find the forces acting on the ship, the system origin and the axes directions are corresponded with ship motion in water, the fixed coordinate system (the geographic), in which it is possible to determine the motion of ship (at the initial moment the moving coordinate system may coincide with the fixed one), the semi–connected coordinate system, which origin is associated with moving ship, and the axes are parallel to ones in the fixed coordinate system.

12.1.2. Directions of axes of the Cartesian right–hand coordinate system, determined by rule of the «right screw»: when the abscissa axis is rotated relative to the applicate axis in the direction of the ordinate one, the rotating «right screw», which axis coincides with the applicate one, moves in the direction of the applicate axis, otherwise, the coordinate system is left, and the «left screw» fixed with x–axis rotating toward y–one moves in the direction of the applicate axis.

12.1.3. The right coordinate system used when studying ship rocking: the abscissa x–axis is directed to the bow of the ship, the ordinate y–axis is to the left side, and the applicate z–axis is upward, and the left coordinate system is used when finding elements of ship controlled motion, differing in the direction of the ordinate axis – to the starboard side of the ship for the left coordinate system, the abscissa axis, which is directed to the bow, and the applicate axis is upward.

12.1.4. Positive directions of angular movements of a ship in the right coordinate system: the roll – to starboard side, trim – to the bow and positive course change – counterclockwise, when viewed from above towards the applicate axis.

12.1.5. The elements of ship motion on the surface of water besides of the coordinates of trajectory and the angular movements are: the drift angle, which is between the velocity vector and the abscissa axis, the course angle is between the longitudinal axis, associated with the ship, and the abscissa axis of fixed coordinate system, which in maritime navigation is directed to the north.

12.1.6. The categories of forces when determining the elements of ship rocking: the inertial forces and moments caused by the massiveness of ship and water and proportional to accelerations, the damping forces and moments of resistance of water, the restoring forces and moments that counteract the deviation of ship from the initial equilibrium and return the ship into equilibrium after the cessation of disturbing forces and moments.

12.1.7. The categories of forces acting on ship when determining the elements of its controlled motion: the driving forces and moments (the control efforts), generated by ship propulsors and controls, the external forces and moments, which sources are in the aquatic environment and do not have a causal dependence on ship motion, as well as the reactive forces and moments from water environment arising by ship motion.

12.1.8. Reactive forces and moments from the water environment, arising as a result of ship motion, are distributed into inertial (that is proportional to acceleration projections) and viscous (proportional to the velocity projections or squares of velocity projections), or into positional forces and moments acting during translational motion, and the damping ones – during the angular rotation.

12.1.9. Differences between the categories of forces acting on ship, the projections of which are included in the equations of ship rocking and in the equations of motion of a ship on water surface.

12.1.10. Matrix of coefficients of added masses of a ship in water, which elements are proportionality coefficients between inertial forces and moments and the accelerations in each direction, which number corresponds to the number of degrees of freedom (three translational and three angular).

12.1.11. Methods of added masses and moments of inertia finding in hydromechanics: the substitute of ship with it geometric model of a triaxial ellipsoid, which semi–axes correspond to ship particulars (half of each ship size), numerical determining of added masses and moments of inertia under the assumption that ship, in the transverse motion, is flow around water along the flat cross–sections.

12.1.12. The hypothesis of damping forces linearization, that is the assumption about linear dependence of projections of forces and the moments of resistance on the corresponding projections of translational and angular velocities.

12.1.13. Matrix the damping coefficients, that are the indexes of proportionality between the emerging forces or moments of resistance and the velocities of ship in water (the translational and the angular ones).

12.1.14. The system of equations of ship motion during it rocking, the matrix form of the equations system, the diagonal form of the matrix of ship inertial characteristics, the symmetry of matrices of added masses and damping coefficients relative to the main diagonal.

12.1.15. Uniformity of the equations of ship motion in water with the expanded form of the system of these equations.

12.1.16. Isolated equations of the main kinds of rocking (heaving, roll and the pitching), the separate application of which (superposition of the main kinds of rocking) is justified by assumptions of the ship symmetry relative to it diametric plane and midship section. Diagonal form of matrices of added masses and damping coefficients for isolated equations of main kinds of ship rocking in their superposition.

12.1.17. The general solutions of isolated linear homogeneous equations for main kinds of ship rocking on calm water.

12.2. Determination of own and associated inertia characteristics using empirical formulas for the main kinds of ship rocking

To use methods for solving equations when finding the elements of a ship's pitching motion and its controlled motion on the surface of the water, it is usually necessary to determine preliminary data or calculate certain elements of motion in an initial approximation. In particular, one has to know ship's own characteristics during translational and rotational motion and the associated inertia characteristics, taking into account water inertia impact on the ship. The inertia of water is considered by the use of methods of the experimental research or the numerical methods, for which the hypothesis is usually used of flow around a ship along it flat sections contours [14].

When solving the equations describing the motion of ship in water, in particular, the equations of the main kinds of ship rocking (heaving, roll and the pitching) for initial approximations it makes sense to use empirical formulas to find the inertia characteristics of ship and the associated inertia characteristics of water. For these purposes the known empirical formulas can be used [22]. For example, to calculate the own moment of inertia of ship I_x relative to the x–axis, used in the equations of roll relative to the longitudinal axis depending on ship displacement Δ and on it particulars: the waterline breadth B, the applicate of the center of gravity of ship Z_G from the main plane, passing along the bottom, on side height H (the ship depth), on ship overall completeness coefficient δ, that is the ratio of ship volumetric displacement (its immersed volume W) to the product of ship length along the waterline L by the breadth B and draught T, on waterline completeness coefficient C_α, i.e. the ratio of waterline area to area of rectangle, which length and width are equal to ship length L along the waterline and breadth B, and on other ship particulars.

In particular, the following empirical formulas can be used to calculate ship own moment of inertia I_x [22]: $I_x \approx 0.125 \times \Delta \times B^2$. In this case, the radius of inertia of ship relative to the abscissa axis i_x does not depend on the displacement Δ, is proportional to breadth B and equals: $i_x = \sqrt{I_x/\Delta} \approx 0.354 \times B$; $I_x \approx \frac{\Delta}{12} \times (B^2 + 4 \times Z_G^2)$ – the Dwyre's

formula, in which the applicate of the center of gravity relative to the main plane for merchant ships of common types can be adopted in the interval: $Z_G/H = 0.56 \div 0.69$, and the radius of inertia of ship relative to x–axis is equal to: $i_x \approx 0.289 \times \sqrt{B^2 + 4 \times Z_G^2}$; $I_x \approx \Delta \times \left(\frac{C_\alpha^2 \times B^2}{11.4 \times \delta} + \frac{H^2}{12}\right)$ – the Yu.A. Shimansky's formula, within the assumptions, which, the radius of inertia of the ship relative to the x–axis is equal to: $i_x \approx \sqrt{\frac{C_\alpha^2 \times B^2}{11.4 \times \delta} + \frac{H^2}{12}}$.

The dimension of the indicators of the own moment of inertia of ship I_x relative to the abscissa axis is as follows: ton×m^2, if the dimension of the used displacement indicator Δ is tons, and the sizes of ship (L, B, H and T) have dimensions in meters.

Since the empirical approach to find the inertia characteristic of such a complex mechanical system as a ship, which depends on the distribution of masses, allows certain ambiguity in the found indicator of the own moment of inertia of ship I_x relative to the x–axis, it makes sense to estimate characteristic intervals of the resulting values of the radius of inertia i_x, which are determined depending on ship breadth B (Fig. 12.2.1 a, b and c), in order to see the degree of mutual correspondence of resulting values of the radius of inertia of ship i_x, determined using different empirical formulas, which also characterizes the correspondence of values of the moment of inertia I_x relative to the axis abscissa x.

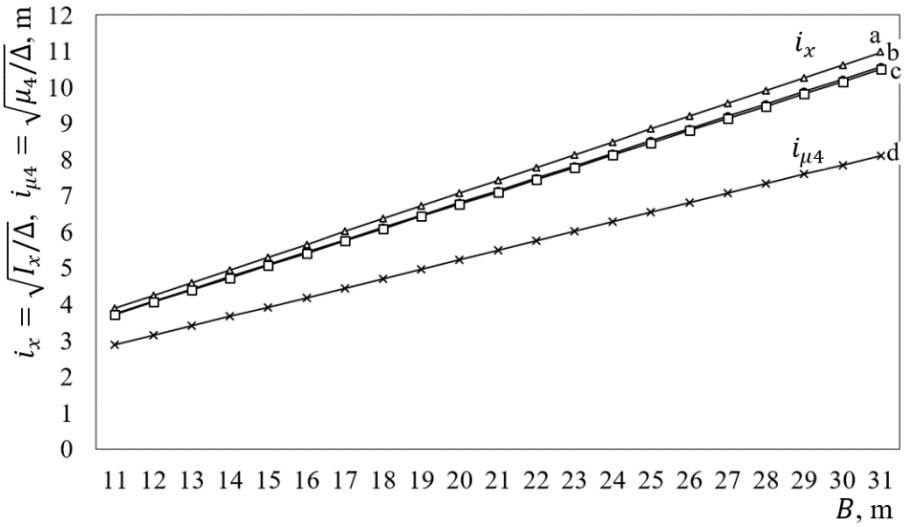


Fig. 12.2.1

Besides, to determine the indicated graphs, it is extra assumed that the ratio of the ship's side height to draft H/T is equal to two, the overall completeness coefficient is: δ =0.74, and the waterline completeness coefficient is: C_α =0.89.

One would indicate that within the interval of ships characteristic sizes (in the breadth range from 11.0 to 31.0 m, Fig. 12.2.1) the radius of ship's own moment of inertia relative to the abscissa axis is: $i_x = \sqrt{I_x/\Delta}$, assessed within the framework of the premises of empirical formulas: the formula of Yu.V. Remez [22] (Fig. 12.2.1 a), the Dwyre's

formula (Fig. 12.2.1 b) and Yu.A. Shimansky's formula (Fig. 12.2.1 c) has fairly close values, which differ from each other by no more than 5 %. This serves as an argument for practical use of the simplest of these formulas, given by Yu.V. Remez:
$I_x \approx 0.125 \times \Delta \times B^2$ and $i_x \approx 0.354 \times B$.

The associated moment of water inertia μ_4 in the issue of ship rolling at initial approximation can be determined based on its correlation with ship own moment of inertia I_x, as: $\mu_4 \approx q_4 \times I_X$. Methodological difficulties would be expected when finding the proportionality coefficient q_4.

If there are observations of the roll period of ship T_4 it is possible to determine total moment of inertia relative to the x–axis: $(I_x + \mu_4)$, by use of formula, corresponded with solution of linear homogeneous equation of roll in calm water:

$$T_4 = 2 \times \pi/\omega_4 = 2 \times \pi \times \sqrt{\frac{(I_x+\mu_4)}{\Delta \times g \times h}}.$$

In the initial or the preliminary calculations, the proper roll period of ship in calm water can be found by use of the empirical «captain's» formula [22, P. 134], as: $T_4 = k_\theta \times B/\sqrt{h}$, s, where k_θ – the coefficient having values: for sea passenger ships – in the interval from 0.71 to 0.75, for linear universal dry cargo ships 0.81, for small universal dry cargo ships empty or in ballast, as well as for ships transporting liquid cargo (petroleum products, liquefied natural gas, etc.) – about 0.88, for universal dry cargo ships with cargo and with a supply of liquid cargo in tanks in the presence of a free surface, the coefficient k_θ is from 0.73 to 0.78 (for the mass of liquid cargo from 5 to 20% in relation to ship displacement), for the ships of all types for ocean industrial fishery – about 0.80, for the coastal fishery ships with the fish tanks containing live catch in the water – about 0.60, for small–tonnage coastal fishery boats – about 0.95, etc.

To transform the «captain's» formula, one can reduce the indexes in the numerator and denominator: $\pi \sim \sqrt{g}$, considering that with such a reduction the coefficient k_θ in the empirical «captain's» formula has the dimension: s/m^{½}.

Having also determined the own moment of inertia of the ship itself I_x relative to the x–axis and associated moment (the added) of inertia of water μ_4 by the displacement of ship Δ and the corresponding total radius of inertia i_{x4}, as: $I_x + \mu_4 = \Delta \times {i_{x4}}^2$, when substitute formulas, one can find: $2 \times \pi \times \sqrt{\frac{(I_x+\mu_4)}{\Delta \times g \times h}} \approx 2 \times \sqrt{\frac{\Delta \times {i_{x4}}^2}{\Delta \times h}} = 2 \times \frac{i_{x4}}{\sqrt{h}} = k_\theta \times \frac{B}{\sqrt{h}}$ and determine the total radius of inertia i_{x4} within the assumptions of the formulas used, as: $i_{x4} = \frac{k_\theta}{2} \times B$.

If one uses coefficient of the empirical «captain's» formula for ships carrying liquid cargo: $k_\theta = 0.88$, then total radius of inertia i_{x4} is equal to: $i_{x4} = \frac{0.88}{2} \times B \approx 0.44 \times B$,

while the own moment of inertia of ship I_x relative to abscissa axis and the associated moment of inertia of water μ_4 add up, as: $I_x + \mu_4 = 0.194 \times \Delta \times B^2$.

And the associated moment of inertia of water μ_4 relative to the abscissa axis deducting the ship own moment of inertia: $I_x \approx 0.125 \times \Delta \times B^2$ from the total moment of inertia is: $\mu_4 = \Delta \times B^2 \times (0.194 - 0.125) = \Delta \times B^2 \times 0.069$, accordingly, the radius of the associated moment of inertia of water, within the assumptions of empirical formulas used for ship carrying liquid cargo, is equal to (Fig. 12.2.1 d): $i_{\mu 4} = \sqrt{\mu_4/\Delta} = 0.262 \times B$.

Besides, one can find the coefficient of proportionality of the associated and own moments of inertia of the ship during the roll, which is: $q_4 = \frac{\mu_4}{I_x} \approx \frac{0.069 \times \Delta \times B^2}{0.125 \times \Delta \times B^2} = 0.55$.

The own moment of inertia of ship relative to the ordinate axis Gy in the associated coordinate system for the issues of ship pitching one can be find using the empirical formulas of G.E. Pavlenko, as: $I_y \approx 0.065 \times C_\alpha \times \Delta \times L^2$, or $I_y \approx 0.0625 \times \Delta \times L^2$.

The radius of inertia relative to transverse axis Gy is equal accordingly to:
$i_y = \sqrt{I_y/\Delta} \approx 0.255 \times L \times \sqrt{C_\alpha}$, or $i_y \approx 0.25 \times L$, (Fig. 12.2.2 a).

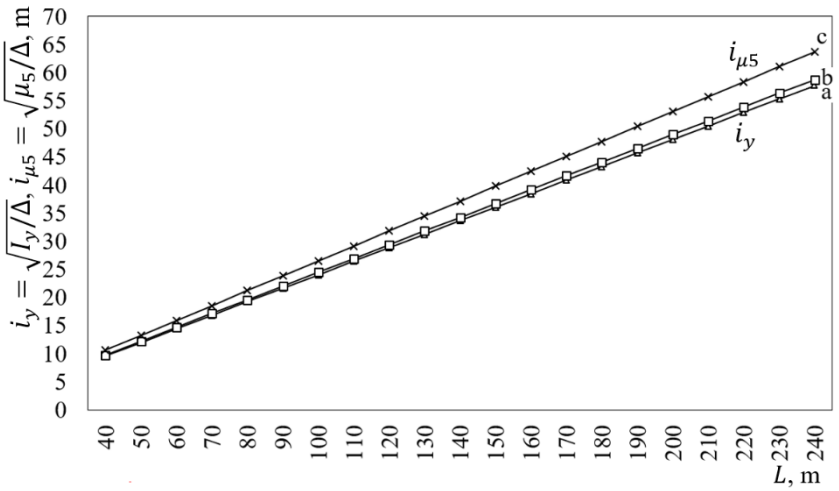


Fig. 12.2.2

There are known the empirical formulas for finding the own period of pitching, depending on ship draught [22, P. 132]: $T_5 = 2.5 \times \sqrt{T}$,
or $T_5 = 1.789 \times \sqrt{(1 + 0.42 \times C_\psi \times (B/T)) \times T}$.

Considering the previously derived formula for determining the natural frequency and the period of ship pitching: $T_5 = 2 \times \pi/\omega_5 = 2 \times \pi \times \sqrt{\frac{(I_y + \mu_5)}{\Delta \times g \times h_\varphi}}$, making a reduction of the values in the numerator and denominator: $\pi \sim \sqrt{g}$ (in the case where the acceleration of gravity: $g = 9.81$ m/s^2 is determined in the International System of Units SI), the substitute of the empirical formula of Pavlenko G.E. for ship own moment of inertia relative

to the ordinate axis Gy: $I_y \approx 0.065 \times C_\alpha \times \Delta \times L^2$, one can find the associated moment of inertia of water during pitching as: $\mu_5 = 1.563 \times T \times \Delta \times h_\varphi - 0.065 \times C_\alpha \times \Delta \times L^2$.

In general approximation, the longitudinal metacentric height h_φ (the distance from the center of ship gravity to the longitudinal metacenter – the center of curvature of the circle along which the center of ship underwater volume moves during the ship trim), which is included in the variables of above indicated formula, can be considered approximately equal to the longitudinal metacentric radius R_φ (the distance from the center of the immersed volume to the longitudinal metacenter), which is determined by dividing the moment of inertia I_f of the waterline area relative to the transverse axis passing through the center of this area, that is relatively to midship section within framework of the assumption of symmetry of ship immersed shape relative to the midship section plane, by the value of immersed volume of the hull, as: $h_\varphi \approx R_\varphi = \frac{I_f \times 1.025}{\Delta}$, where 1.025 ton/m^3 – the usual mass density of sea water.

The moment of inertia of ship waterline area one can find using the approximate empirical formula: $I_f \approx \frac{C_\alpha \times L^3 \times B}{16}$.

Since the displacement of ship is equal to: $\Delta = 1.025 \times \delta \times L \times B \times T$, then the longitudinal metacentric height can be determined in the approximation of the specified empirical formula as: $h_\varphi \approx \frac{I_f \times 1.025}{\Delta} = \frac{C_\alpha \times L^3 \times B \times 1.025}{16 \times 1.025 \times \delta \times L \times B \times T} = \frac{C_\alpha \times L^2}{16 \times \delta \times T}$.

When set the indicator of the longitudinal metacentric height h_φ the empirical formula for associated water inertia radius $i_{\mu 5}$ for finding of the attached moment of inertia relative to the ordinate axis Gy (Fig. 12.2.2 b) has the form:

$i_{\mu 5} = \sqrt{\mu_5/\Delta} \approx L \times \sqrt{C_\alpha} \times \sqrt{0.098/\delta - 0.065}$,

or: $i_{\mu 5} \approx L \times \sqrt{C_\alpha} \times \sqrt{0.050 \times (1 + 0.42 \times C_\psi \times (B/T))/\delta - 0.065}$, (Fig. 12.2.2 c).

With the empirical approach, it is relevant to assume that during vertical motion of ship (that is during translational periodic motions of ship along the applicate axis Gz relative to the equilibrium position of ship at its draught by waterline) the associated mass of water μ_3 is proportional to the displacement: $\mu_3 \approx q_3 \times \Delta$, and the proportionality coefficient q_3 in the initial approximation can be determined depending on the particulars of ship shape (C_α and B/T), as: $q_3 \approx 0.40 \times C_\alpha \times B/T$ (Fig. 12.2.3 a).

The own period of ship heaving in water depending on it draught T one can find approximately by the empirical formula as: $T_3 = 2.5 \times \sqrt{T}$, s. In this case, the coefficient of associated mass of water during ship heave equals (Fig. 12.2.3 b):

$q_3 = \frac{\mu_3}{\Delta} \approx 1.563 \times \frac{C_\alpha}{\delta} - 1$.

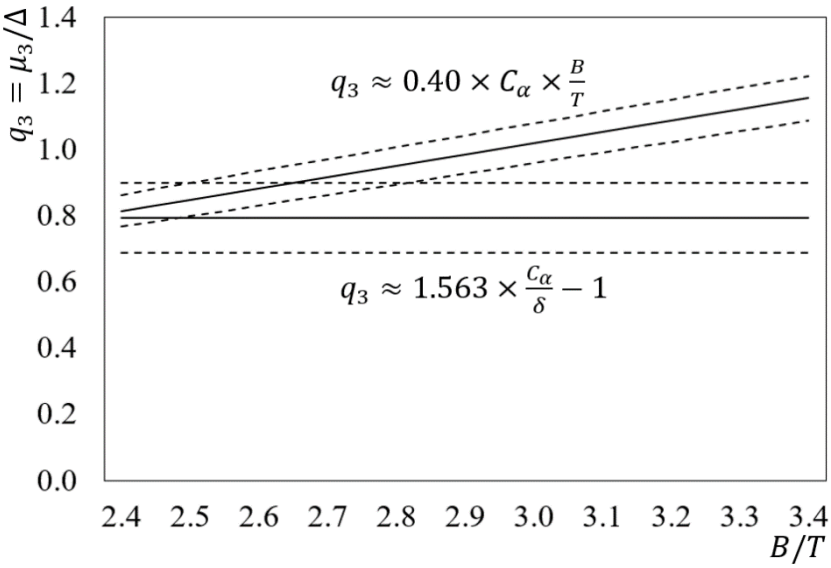
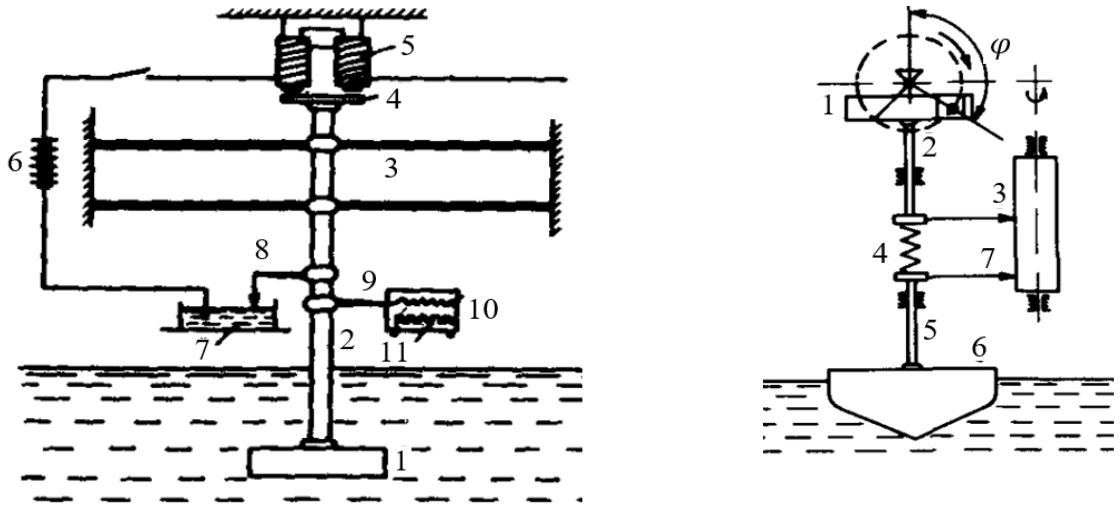


Fig. 12.2.3

A comparison of the graphs (Fig. 12.2.3 a and b) obtained using various empirical formulas indicates, that when using these formulas the definite scatter in the values of the inertia characteristics of ship in water occurs, especially considering since these inertia characteristics depend not only on ship particulars (the dimensions) and on size characteristics (the displacement, etc.), but also on frequency of the corresponding rocking of ship.

Those, finding the added masses and moments of inertia is a nonlinear problem, and the results (the inertia characteristics) influence the data, which includes the frequency characteristics of rocking, which in turn affect the inertia characteristics of ship in water.

Therefore, in the practical work, it is relevant to use numerical methods for more accurate finding of the associated masses of water and moments of inertia [14], or the experimental research methods, for example, the use of the installations like ones shown on the schemes (Fig. 12.2.4 a and b, Fig. 12.2.5).



(a) (b)

Fig. 12.2.4

The diagram (Fig. 12.2.4 a) shows a device for imparting oscillatory motions to model 1 immersed in water, which, using a rod 2, is attached to springs 3 fixed in the foundation of this installation. The rod 2 in the upper part ends with metal plate 4, which is attracted by an electric magnet 5, supplied with electrical power from the battery 6. A contact breaker (7 and 8) is included in the circuit, which turns off the current when the plate 4 is attracted by electromagnet 5 and rod 2 in it uppermost position. Under the action of elastic force of the compressed springs 3, the rod 2 with model 1 goes down, the electrical circuit closes and the process repeats. Using the recorder (9 and 10), the vibrations of the model, as well as time stamps, are recorded on tape 11, and subsequently the recording of the vibrations of the model in water is used to determine the associated masses of water.

The installation for determining the attached masses of the hulls located on the surface of the water is shown in the diagram (Fig. 12.2.4 b).

The crank mechanism 1 does a reciprocating motion and imparts to rod 2 the vertical movements of the form: $y = r \times \cos(\sigma \times t)$, r – crank length, σ – crank cyclic velocity. These motions are recorded on tape using recording device 3. A spring 4 is attached to the lower end of rod 2, connected by its other end to rod 5, on which the model 6 is attached. Motion of the rod 5 and model 6 is recorded by the second recording device 7.

The associated masses of water (the added masses) one can determine by measuring the accelerations during vibration of floating models on the surface of water (Fig. 12.2.5). To measure the accelerations the model 2 is placed in container 1 on the surface of the water, and is fixed to the frame by 5 strings 4 in those places where the presence of oscillations node points is expected to be when model vibrates.

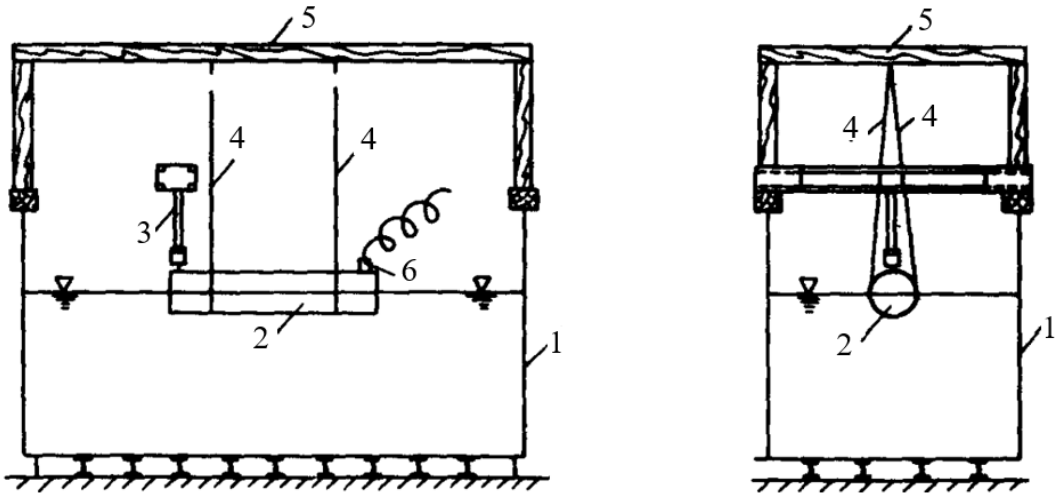


Fig. 12.2.5

The mechanism 3 is installed on the model, exciting vibrations, and a device for measuring the accelerations when the model vibrates.

Questions and comments:

12.2.1. The experimental and numerical methods for determining the characteristics of inertia during oscillations (translational and the angular) and when ship moves with acceleration in water.

12.2.2. Associated with the nonlinearity and dependence of the inertia characteristics on the oscillations in water frequency characteristics, which are mutually dependent on the inertia characteristics, the difficulties which arise when finding the added inertia characteristics during ship oscillations in water.

12.2.3. Ship sizes and the particulars of it shape influencing own characteristics of inertia and the attached characteristics of ship inertia in water: the displacement and the ship own moments of inertia, the length, breadth, draught and side height (the depth), the ratios of ship dimensions, the shape coefficients of ship submerged hull: the overall completeness, the waterline completeness, the midship section completeness and the longitudinal sharpness.

12.2.4. Clarification of found values of te own moments of inertia, characterizing the distribution of masses of ship hull by applying various empirical formulas and the comparing of ship inertia obtained indicators.

12.2.5. Definition of the added moments of inertia of ship in water and the radii of the attached moments of inertia using the property of additivity of inertia indicators – the equality of the inertia indicator of a ship in water to the sum of its own inertia indicator and the associated indicator of inertia in water (the added masses and moments of inertia).

12.2.6. Determination of the added moments of inertia and masses by observations of the period and frequency of the main kinds of ship rocking or by the values of the rocking period found using the empirical formulas.

12.2.7. The use of the ship dimensions and the particulars of it hull shape: the waterline length, draught, waterline completeness and the overall completeness coefficients to find the longitudinal metacentric radius (the distance from the center of the underwater volume to the metacenter, that is the center of curvature of the circle along which the center of the underwater volume moves during the trim of ship), which serves as an approximation of indicator of the longitudinal metacentric height of ship (the distance from center of gravity to the metacenter).

12.2.8. Proportionality of the associated mass of water during ship heave to the displacement and proportionality of added moments of inertia of water during roll and pitch to the natural moments of inertia of ship relative to x–axis and the ordinate axis.

12.2.9. The principle of experimental installations operation for generating the oscillations of models in water and these oscillations recording for subsequent determination of associated masses and moments of inertia of water: when generating oscillations of

model in water using a magnet and an electric switch, when generating oscillations of a model on the surface of water using a mechanism with a rotating crank and a progressively moving connecting rod, when using a vibration generator and fixing the model at points that are nodes of the model's vibration modes, which eliminates the influence of the fastening factor on the vibrations.

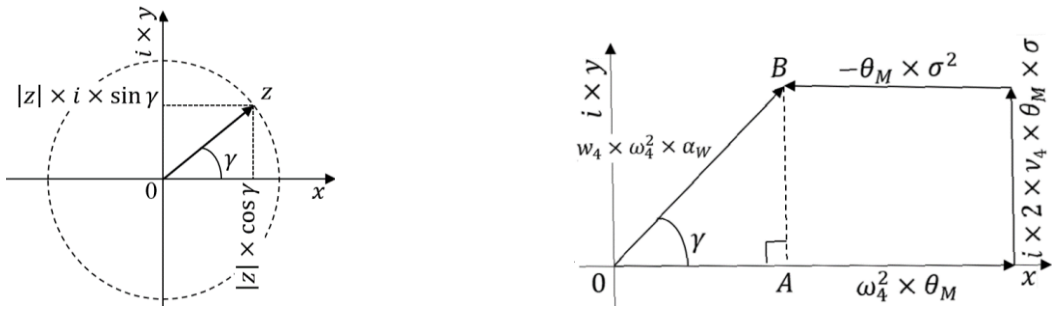
12.3. Finding the roll elements of ship on regular waves using the complex amplitudes method

In many cases the forces and moments acting on a ship in water have nonlinear dependence on the translational and angular motions of ship, as well as on time derivatives of motions – on the velocities and accelerations. Since the nonlinear differential equations usually do not have analytical solutions, the hypothesis of linearization of nonlinear dependencies can be used in the practical issues.

The complex numbers can be used, which show geometric (the vector) analogies of differential equations solutions on the complex coordinate plane when solving the linear differential equations, for example, the isolated equations of ship rocking, when the real part of the complex number is indicated along x–axis, and the imaginary part – along the ordinate axis.

It would be mentioned that the original linear differential equations, for example, the equations covering the elements of ship rocking (the cyclic motions of ship during the rocking, the velocity and acceleration), are usually formulated in real form (the item 12.1) in projections of forces or the moments, and their solutions too are defined in real coordinates, although for the process of these equations solving it is convenient to use the complex form of the equations terms.

The complex numbers general information is as follows. Complex number z has the form (Fig. 12.2.4 a): $z = x + i \times y$, where x – is the real part of complex number; $i \times y$ – the imaginary part of a complex number; $i = \sqrt{-1}$ – the imaginary unit. The modulus of complex number is equal to: $|z| = \sqrt{x^2 + y^2}$.



(a) (b)

Fig. 12.3.1

Complex number also has the exponential form: $z = |z| \times e^{i \times \gamma}$, where: $\gamma = artg\left(\frac{y}{x}\right) = arcos\left(\frac{x}{z}\right)$ – the angle of rotation of the complex number vector z relative to the abscissa axis, and trigonometric form: $z = |z| \times (\cos\gamma + i \times \sin\gamma)$, which are connected by the formula of the L. Euler: $e^{i \times \gamma} = \cos\gamma + i \times \sin\gamma$, which is of great practical significance in solving of the differential equations.

The use of the complex amplitudes method can be demonstrated by the example of solving the isolated linear inhomogeneous equation of ship rolling (the transverse angular oscillations relative to the abscissa axis Gx in the coordinate system associated with ship) on regular waves: $\ddot{\theta} + 2 \times \nu_4 \times \dot{\theta} + \omega_4^2 \times \theta = w_4 \times \omega_4^2 \times \alpha_W \times e^{i \times \sigma \times t}$, where α_W – the angle of maximum of wave slope; $\nu_4 = \frac{\lambda_4}{2 \times (I_x + \mu_4)}$ – ship rolling attenuation decrement; λ_4 – ship rolling damping coefficient due to resistance during dissipation of ship rolling energy in water space; $\omega_4^2 = \frac{\Delta \times g \times h}{(I_x + \mu_4)}$ – the square of natural frequency of ship roll; w_4 – the reduction coefficient depending on proportionality of ship breadth and the wave length; $e^{i \times \sigma \times t}$ – complex exponential form of notation of harmonic oscillations function with definite frequency σ.

The given linear inhomogeneous differential equation for the roll of ship on a regular wave of a sinusoidal profile, running transversely onto the ship from the port side (in accordance with a right–hand coordinate system), has also trigonometric form:
$\ddot{\theta} + 2 \times \nu_4 \times \dot{\theta} + \omega_4^2 \times \theta = w_4 \times \omega_4^2 \times \alpha_W \times \sin(\sigma \times t)$, in which on the left side, the first term is of an inertial nature (from the projection of the inertial moment arising during roll with acceleration $\ddot{\theta}$ onto the abscissa axis Gx), the second term corresponds to the linearized (proportional to the angular velocity $\dot{\theta}$ to the first power) moment of resistance of water to roll, the third term corresponds to the projection onto the x–axis Gx of the restoring moment, proportional to the roll angle θ, and the function on the right side corresponds to the changing angle of the wave slope: $\alpha = \alpha_W \times \sin(\sigma \times t)$ and the resulting periodic heeling moment under the oncoming regular waves of a sinusoidal profile.

According to the well–known Cauchy problem in mathematics, the solution of a linear inhomogeneous equation can be determined as the sum of the general solution of the corresponding homogeneous equation: $\ddot{\theta} + 2 \times \nu_4 \times \dot{\theta} + \omega_4^2 \times \theta = 0$, which describes the roll of ship in calm water with natural frequency ω_4, damping with time, and a particular solution of the inhomogeneous equation – with the frequency of oncoming waves σ and with corresponding roll amplitude due to maximum roll angle θ_M at the initial conditions.

Considering that damped oscillations, which are a solution of the linear homogeneous equation of ship rolling in calm water, do not affect the elements of ship rolling if it is of sufficient duration, to describe the steady processes of ship roll, one can limit himself

to a particular solution of the inhomogeneous equation describing forced oscillations (with the frequency of oncoming waves σ).

In complex form, a particular solution of the inhomogeneous equation (the dependence on the time parameter of ship rolling angle in regular seas) taking into account sought roll amplitude θ_M corresponds to the right side of the original equation and has the form: $\theta = \theta_M \times e^{i\times\sigma\times t}$, accordingly, the time derivatives (the angular velocity and angular acceleration during roll) are equal: $\dot{\theta} = \theta_M \times i \times \sigma \times e^{i\times\sigma\times t}$ and $\ddot{\theta} = -\theta_M \times \sigma^2 \times e^{i\times\sigma\times t}$, where: $i = \sqrt{-1}$ – the imaginary unit.

After set the particular solution form of the inhomogeneous equation and time derivatives of this solution into the original equation and after transforming (the common factor reducing: $e^{i\times\sigma\times t}$) one can get the following equation:
$-\theta_M \times \sigma^2 \times e^{i\times\sigma\times t} + 2 \times \nu_4 \times \theta_M \times i \times \sigma \times e^{i\times\sigma\times t} + \omega_4^2 \times \theta_M \times e^{i\times\sigma\times t} =$
$= w_4 \times \omega_4^2 \times \alpha_W \times e^{i\times\sigma\times t}$, or $\theta_M \times (-\sigma^2 + 2 \times \nu_4 \times i \times \sigma + \omega_4^2) = w_4 \times \omega_4^2 \times \alpha_W$, which real and imaginary terms, that is the complex amplitudes of each term can be indicated as vector projections on the complex plane (Fig. 12.3.1 b).

According to vector diagram on complex plane, the roll amplitude θ_M, depending on maximum angle of wave slope α_W of regular waves, equals: $\theta_M = \frac{w_4\times\omega_4^2\times\alpha_W}{\sqrt{(\omega_4^2-\sigma^2)^2+4\times\nu_4^2\times\sigma^2}}$, at the same time, the amplitude–frequency characteristic of roll motion occurring at this frequency, depending on the frequency of oncoming waves σ, is:
$a_4 = \frac{\theta_M}{\alpha_W} = \frac{w_4\times\omega_4^2}{\sqrt{(\omega_4^2-\sigma^2)^2+4\times\nu_4^2\times\sigma^2}}$.

The amplitudes of angular velocity $(\dot{\theta})_M$ and angular acceleration $(\ddot{\theta})_M$ during roll of ship in regular seas are equal: $(\dot{\theta})_M = \sigma \times \theta_M$ and $(\ddot{\theta})_M = \sigma^2 \times \theta_M$.

In addition, by use of the triangle $0AB$ one indicates on vector diagram the phase difference γ between the changing slope angle of oncoming waves: $\alpha = \alpha_W \times \sin(\sigma \times t)$ and the roll angle of ship on regular waves: $\theta = \theta_M \times \sin(\sigma \times t - \gamma_4)$. The phase difference depending on the frequency of oncoming waves (that is the phase–frequency characteristic of roll) is defined as: $tg\ \gamma_4 = \frac{2\times\nu_4\times\sigma}{\omega_4^2-\sigma^2}$.

The forms of amplitude–frequency a_5 and the phase–frequency γ_5 characteristics during pitching, when waves of frequency σ observed from the ship approach from the bow of the ship (the angular oscillations relative to the ordinate axis Gy), within the framework of the assumptions of linear equations, that is in the absence of slamming – the slaps of ship bottom on surface of water, with relatively small trim angles arising during pitching, which is typical for fairly large ships, coincide with the forms of these roll characteristics (a_4 and γ_4), if the specified conditions for pitching are met in a certain task, the

symbols of index 4, attributed to ship roll, in the formulas would be replaced by index 5 – for pitching.

Ship heaving has a slightly different form of amplitude–frequency and phase–frequency characteristics (the periodic translational motions of ship along the applicate axis Gz with the frequency of oncoming waves). For regular waves with the amplitude r_W, which is equal to half of the wave height, in two–dimensional coordinate plane of profile of the propagating wave, which ordinate of the profile varies according to harmonic law: $\zeta_W = r_W \times \sin(\sigma \times t)$, the amplitude–frequency characteristic of ship heaving with maximum amplitude ζ_M has the form: $a_3 = \frac{\zeta_M}{r_W} = \sqrt{\frac{(\omega_3^2 - f_{\mu 3} \times \sigma^2)^2 + 4 \times \nu_3^2 \times \sigma^2}{(\omega_3^2 - \sigma^2)^2 + 4 \times \nu_3^2 \times \sigma^2}}$, where: $f_{\mu 3} = \frac{\mu_3}{\Delta + \mu_3}$ – the dimensionless inertial parameter, and the phase–frequency characteristic of an isolated vertical motion of geometrically symmetrical ship motionless on the surface of the water is defined as: $tg\ \gamma_3 = \frac{2 \times (1 - f_{\mu 3}) \times \nu_3 \times \sigma^3}{(\omega_3^2 - f_{\mu 3} \times \sigma^2) \times (\omega_3^2 - \sigma^2) + 4 \times \nu_3^2 \times \sigma^2}$.

The amplitudes of translational velocity $(\dot{\zeta})_M$ and acceleration $(\ddot{\zeta})_M$ during vertical motion of ship on regular waves are equal: $(\dot{\zeta})_M = \sigma \times \zeta_M$ and $(\ddot{\zeta})_M = \sigma^2 \times \zeta_M$.

Within the assumption of the immobility of ship on the water (except ship rocking) and the assumption of ship shape symmetry relative to its diametral plane and the plane of the midship section, the characteristics of the considered types of rocking (the heaving, roll and pitching) can be determined independently of each other.

Exercise. The exercise assigns one to determine graphs of the amplitude–frequency and phase–frequency characteristics of the roll and heaving of ship stationary on the water surface, using the characteristics and particulars of design of the bunkering ship for the delivery of liquefied natural gas (www.icedesign.info, project by the International Contract Engineering Ltd, ICE Marine Design, LNG–bunkering tanker, Fig. 12.3.2). The use of liquefied natural gas as a fuel is promising direction not only in terms of economic criteria, but also from an environmental point of view, because when using natural gas as fuel the exhaust includes water vapor and carbon dioxide, which are harmless to the marine environment.

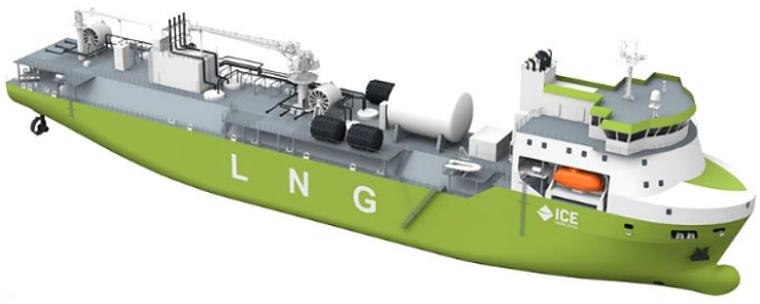


Fig. 12.3.2

The ship length over all (LOA) is 122 m, the waterline length is: L =116 m, the breadth equals: B =20 m, the side height is: H =11 m, the average draft at load waterline is: T =5.2 m, the displacement at summer load line draught is equal: Δ=9 150 ton, the cargo capacity is 8 000 cub. m of liquified natural gas, the overall completeness coefficient is: δ =0.74, the waterline area completeness coefficient equals: C_α =0.89, coefficient of completeness of immersed area of midship section is: C_β =0.96, the longitudinal acuity coefficient is: $C_\psi = \delta / C_\beta$ =0.77, the ship operation velocity is 13.6 knots (7 m/s). Other particulars of the ship are not required in the following tasks to find the indexes of ship rocking and it controlled motion.

Value of the initial transverse metacentric height in considered issue is assigned: h =2.5 м. It would be mentioned that the ratio of the ship breadth to it draft, suitable for delivering bunker fuel at relatively shallow depths, for example, in harbors, it equals to: B/T =20/5.2 = 3.85 corresponds to the upper limit of values of this relative indicator characteristic for seagoing ships, therefore assigned a rather large initial transverse metacentric height h is quite realistic.

The reduction coefficient, which methodically depends on proportionality of ship breadth B and the wave length λ_W, in the calculation is assigned equal to: w_4 =0.75. It would be mentioned that for ship, which is under regular waves impact of long length and low frequency at large and extreme wave lengths, much greater than the breadth of the ship, the coefficient w_4 under consideration is close to one, although with these data the amplitude–frequency characteristic $\left(\frac{\theta_M}{\alpha_W}\right)$ of roll, that is, the ratio of the amplitude of the roll angle to the amplitude of the wave slope angle is small. Therefore, used value of the reduction coefficient 0.75 would be perceived as close to the lower limit of values range.

Moment of inertia of the ship relative to the abscissa axis Gx is approximately equal to: $I_x \approx 0.125 \times \Delta \times B^2 = 0.125 \times 9\,150 \times 20^2 = 457\,500$, ton×m^2.

Associated (the added) moment of inertia of water relative to the abscissa axis Gx equals: $\mu_4 \approx 0.55 \times I_X = 0.55 \times 457\,500 = 251\,625$, ton×m^2.

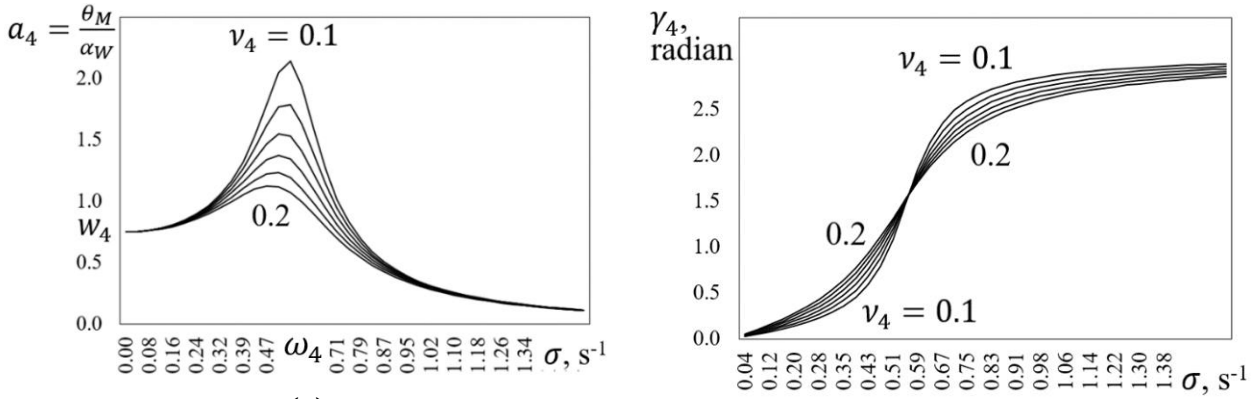
Own frequency of the ship rolling is: $\omega_4 = \sqrt{\frac{\Delta \times g \times h}{(I_x + \mu_4)}} = \sqrt{\frac{9\,150 \times 9.81 \times 2.5}{(457\,500 + 251\,625)}} =$ 0.5625, radian/s.

The determination of the damping coefficient λ_4 for the viscous resistance of water to the rocking ship and for the dissipation of the energy of the ship by the waves it generates and determination of the corresponding decrement (the multiplier in the exponent) is: $\nu_4 = \frac{\lambda_4}{2 \times (I_x + \mu_4)}$ in the solution of linear equation of ship rolling (and other kinds of rocking) is associated with methodological and calculation difficulties [14]. This issue is of non–linear nature, its theoretical solution is not known, and the use of experimental methods can be recommended.

In exercise being performed it is offered to use the periodic values of roll attenuation decrement ν_4 for calculations in the interval: 0.1, 0.12, 0.14, 0.16, 0.18 and 0.20. The found amplitude–frequency and the phase–frequency characteristics of ship rolling in regular waves are the set of curves that differ by the parameter of the attenuation decrement ν_4.

It is convenient to use the Microsoft EXCEL program for calculations.

The amplitude–frequency: $a_4 = \frac{\theta_M}{\alpha_W} = \frac{w_4 \times \omega_4^2}{\sqrt{(\omega_4^2 - \sigma^2)^2 + 4 \times \nu_4^2 \times \sigma^2}}$ and phase–frequency: $\gamma_4 = artg\left(\frac{2 \times \nu_4 \times \sigma}{\omega_4^2 - \sigma^2}\right)$ characteristics of ship rolling on regular waves are indicated on graphs (Fig. 12.3.3 a and b).



(a) (b)

Fig. 12.3.3

The added mass of water during the ship vertical heaving is equal to:

$\mu_3 \approx 0.40 \times C_\alpha \times \frac{B}{T} \times \Delta = 0.40 \times 0.89 \times \frac{20}{5.2} \times 9\,150 = 12\,529$, ton.

The dimensionless inertial parameter equals: $f_{\mu 3} = \frac{\mu_3}{\Delta + \mu_3} = \frac{12\,529}{9\,150 + 12\,529} = 0.578$.

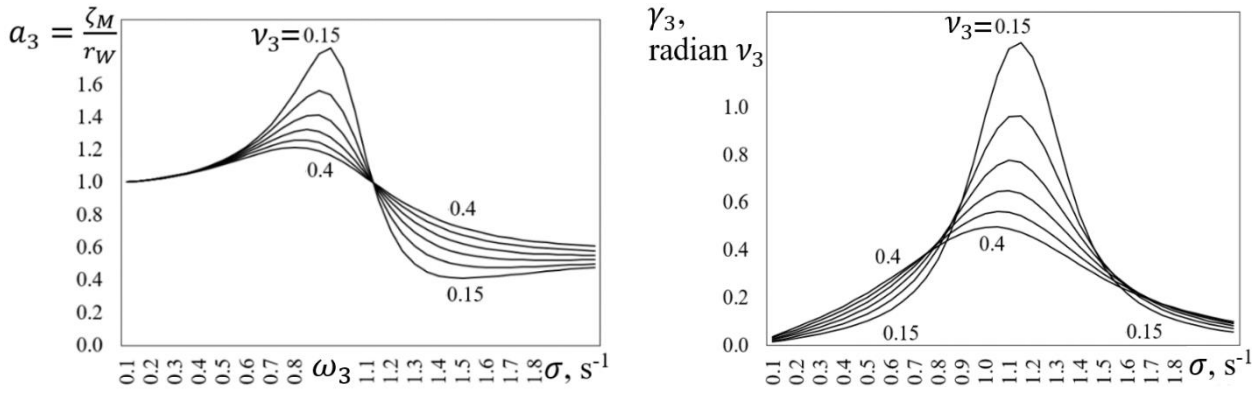
The own frequency of the ship vertical heaving is:

$\omega_3 = \sqrt{\frac{\rho \times g \times L \times B \times C_\alpha}{(\Delta + \mu_3)}} = \sqrt{\frac{1.025 \times 9.81 \times 116 \times 20 \times 0.89}{(9\,150 + 12\,529)}} = 0.979$, radian/s.

In performed exercise it is offered to use periodic values of vertical heaving attenuation decrement ν_3 in the interval for calculations: 0.15, 0.20, 0.25, 0.30, 0.35 and 0.40. Found amplitude–frequency and the phase–frequency characteristics of ship vertical heaving on regular waves are the set of curves that differ by the attenuation decrement ν_3.

The amplitude–frequency characteristic of ship vertical heaving on regular waves is: $a_3 = \frac{\zeta_M}{r_W} = \sqrt{\frac{\left(\omega_3^2 - f_{\mu 3} \times \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2}{\left(\omega_3^2 - \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2}}$ and the phase–frequency one:

$\gamma_3 = artg\left(\frac{2\times(1-f_{\mu 3})\times\nu_3\times\sigma^3}{(\omega_3^2-f_{\mu 3}\times\sigma^2)\times(\omega_3^2-\sigma^2)+4\times\nu_3^2\times\sigma^2}\right)$ are shown in the graphs (Fig. 12.3.4 a and b)



(a) (b)

Fig. 12.3.4

The maxima on the amplitude–frequency characteristics of ship rolling (Fig. 12.3.3 a) and the vertical heaving (Fig. 12.3.4 a) graphs indicate the resonances for these kinds of rocking when the frequency σ of waves impacting the stationary ship coincides with own frequencies of the ship rocking (ω_3 or ω_4).

Getting into resonance during roll of ship can pose a danger to its sustainability, and the factor of resonance of roll with oncoming waves can, in intense waves, affect significantly the sustainability (the ability ship to counteract the heeling moment and return to the equilibrium position after the heeling moment ceases), with low resistance to roll, for example, when active or passive stabilizers are not used.

Questions and comments:

12.3.1. The linearization of forces dependencies on ship motion elements in the terms of the equilibrium equations of the forces projections for the next solving the equations and determining the translational and angular displacements, velocities and the accelerations during ship motion in calm water and on waves.

12.3.2. Imaginary and complex numbers for use to solve the linear differential equations. Coordinate, exponential and the trigonometric forms of a complex number that is vector on the complex coordinate plane. Modulus of a complex number. L. Euler's transformation establishing the relationship between forms of a complex number.

12.3.3. Complex form of a particular solution of linear inhomogeneous ship rolling on regular waves equation and finding the amplitudes of angular displacement, velocity and the acceleration during roll using vector diagram on complex plane.

12.3.4. Dynamic coefficients of the main kinds of rocking on regular waves: during ship rolling – the ratio of ship rolling angle amplitude to the maximum angle of the wave slope, during pitching – the ratio of the amplitude of the trim angle to the maximum angle

of the wave slope and during ship vertical heaving – the ratio of the amplitude of ship translational motion along the axis of the applicate to the amplitude of the regular wave.

12.3.5. Amplitude–frequency characteristics of the main kinds of ship rocking as the dependence of the dynamic coefficients on frequency of periodic disturbing force or the moment.

12.3.6. Included in the prerequisites for independent determination of the amplitude–frequency and phase–frequency characteristics of the main kinds of ship rocking by the complex amplitudes method are the assumptions about the symmetry of ship relative to diametral plane and to the midship section of ship and the prerequisite of ship immobility relative to water surface.

12.3.7. Corresponding with the feasibility of experimental research, the methodological and computational difficulties in finding damping coefficients for the main kinds of ship rocking, considering the viscous resistance of water to the rocking and the dissipation of ship rocking energy by the waves it generates, which can lead to an increase in dimension of the issue of amplitude–frequency and phase–frequency characteristics determining when introducing the damping coefficient and considering ship rocking attenuation decrement as variable parameter in the linear equations.

12.3.8. The form of amplitude–frequency and phase–frequency characteristics of the main kinds of ship rocking and the influencing the resonances of the main kinds of rocking when the frequency of regular waves impacting on a stationary ship coincides with own rocking frequencies of the ship.

12.4. The characteristics of sea waves. Determination of rocking spectrum of ship in irregular waves

To determine characteristics of the main kinds of ship rocking on the surface of water, the motion of wave profile of water surface is of greatest importance, and the movement of water surface and the water on the small depths largely depends on the motion of wave profile and repeats it with large wave sizes, critical for the ship and its safety, and the movement of water at great depths usually does not affect the ship and this movement of water fades with depth.

The movement of water, considered incompressible droplet fluid in hydromechanics, in fixed coordinate system $0xyz$ is described by the Pierre–Simon de Laplace continuity equation: $\nabla\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$, where: $\vec{v} = \{v_x, v_y, v_z\}$ – fluid velocity vector, or depending on the velocity potential function $\varphi(\vec{v})$ for irrotational fluid motion, as: $\nabla^2\varphi(\vec{v}) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$, at: $\vec{v} = \left\{\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\}$.

At unlimited depth the water particles during wave motion move in circles, the radius of which decreases asymptotically with depth.

The shape of regular wave profile can be considered sinusoidal with relatively small wave height, which is twice the amplitude of the wave profile, if the wave height is compared with the wave length (or with small velocity of water motion in wave – if compare the velocity of circular motion of water on the surface with velocity of wave profile motion), which sufficiently corresponds to sea waves.

The solution of the Laplace continuity equation for finding the wave motion of water corresponds to the boundary conditions on the surface and at mathematically infinite depth, as well as on certain other initial conditions. The more in–depth presentation of waves theory could go beyond considered methods of calculating of mechanical systems in marine structures, and therefore the detailed presentation of waves theory is not given.

When describing regular waves, it is convenient to use a fixed two-dimensional Cartesian coordinate system, which abscissa axis coincides with the direction of wave profile motion, and the ordinate axis is directed upward. This two–dimensional coordinate system differs from three–dimensional systems, which coordinates ship rolling motion. Although ship rocking is described by isolated equations under the assumption of the ship symmetry relative to diametral plane and the midship–section plane. Usually within the framework of the «the plane issue» ship rocking is also considered in two–dimensional coordinate systems, for example, in the same ones in which regular waves description is given.

The velocity c_W of wave profile motion (the phase velocity of wave) is corresponded with the length of wave λ_W and the period τ_W, as: $c_W = \lambda_W/\tau_W$. The slope angle α of sinusoidal wave profile depending on the amplitude r_W (half the height of the sinusoidal wave profile), on time parameter t and on the x–axis coordinate x, directed along the wave propagation, is determined as the derivative of wave profile applicate $\frac{\partial \xi}{\partial x}$, as:
$\alpha \approx tg\ \alpha = \frac{\partial \xi}{\partial x} = k \times r_W \times \sin(\sigma \times t - k \times x)$, were: $k = \frac{2\times\pi}{\lambda_W}$ – the wave number.

To describe a stationary process of established irregular waves, an indication of the initial phase is usually not required, because the characteristics of a stationary processes do not depend on the origin of time, therefore the formula contains only arguments that depend on variables: the time t and coordinates x. For the same reason, when studying some stationary processes, instead of the function: sin, the cos function or these ones with a minus sign can be used.

Developed sea waves and ship rocking are stationary processes, which probabilistic characteristics and the elements do not depend on the reference points.

The amplitude of wave slope angle is equal to: $\alpha_W = k \times r_W = \frac{2\times\pi}{\lambda_W} \times r_W$ and the elements of the wave profile are corresponded by the relations: $\tau_W = 0.8 \times \sqrt{\lambda_W}$, s (Fig. 12.4.1 a), $\sigma = \frac{7.85}{\sqrt{\lambda_W}}$, 1/s (Fig. 12.4.1 b) and $c_W = 1.25 \times \sqrt{\lambda_W}$, m/s (Fig. 12.4.1 c).

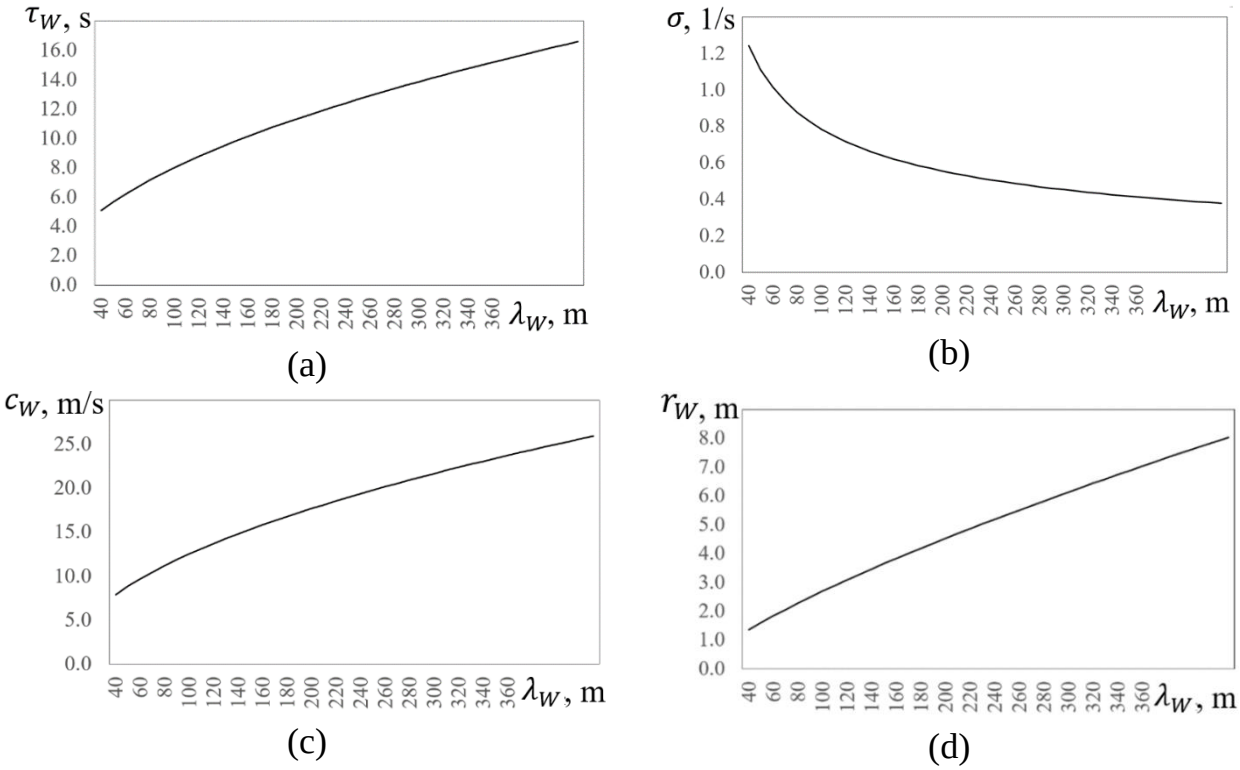


Fig. 12.4.1

One can see that for a wave of small amplitude (by correlating the ratio of the amplitude r_W, equal to half the wave height, to the wavelength), its wave elements (τ_W, σ and c_W) do not depend on its amplitude r_W, but are in certain relationships with each other since the definition of waves amplitudes under these assumptions are not covered by linear wave theory. For general approximation approach the Zimmerman's empirical formula can be used to determine the wave amplitude r_W depending on wave length λ_W (Fig. 12.4.1 d): $r_W \approx 0.085 \times \lambda_W^{3/4}$, m.

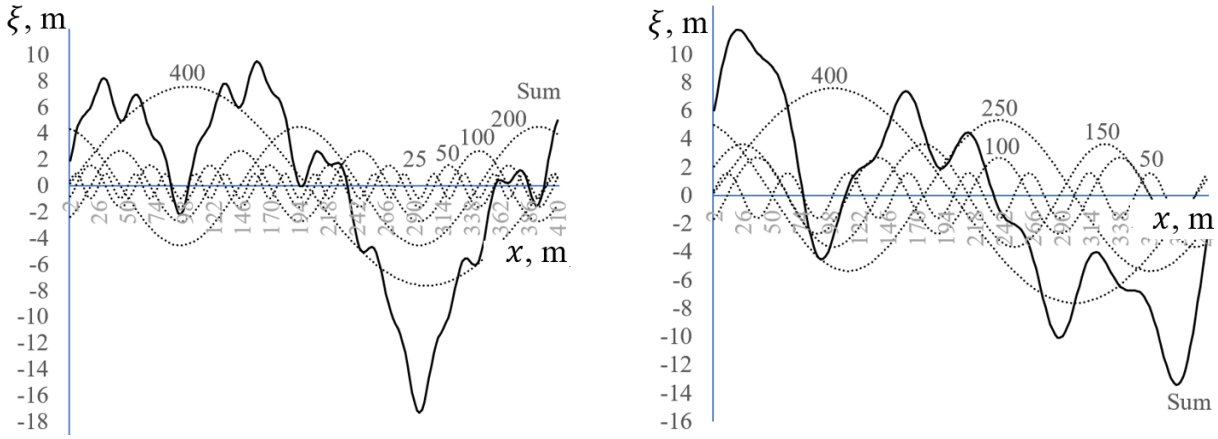
The concept of progressive regular waves, which, at an unlimited depth of water, have a sinusoidal profile on the surface, that are moving in a direction perpendicular to the lines of the crests (in contrast to standing waves, which sometimes arise in front of obstacles to wave propagation due to harmonic superposition, that is the interference of progressive and reflected waves), would be considered as an abstract model, since real sea waves are a combination of wave systems of different lengths, heights and periods, coming from and propagating to different directions (Fig. 12.4.2 a), characterized by a spectrum (which is the distribution of wave amplitudes by frequencies), although it is usually

possible to identify the predominant sea waves in the spectrum. It would be mentioned that when observing sea waves of different sizes overlapping each other in an unlimited sea, a common predominant direction of wave motion is usually visible (Fig. 12.4.2 b).

(a) (b)

Fig. 12.4.2

Illustrative examples of overlapping wave profile ordinates of different lengths are shown in the diagram for options of waves combining of multiple lengths (25, 50, 100, 200 and 400 m, Fig. 12.4.3 a) and the waves whose lengths considered correlated in proportion to Fibonacci (the Leonardus Pisanus) numbers: 0, 1, 1, 2, 3, 5, 8 etc., where each subsequent number is the sum of the two previous ones, excluding zero and first one, that is for a combination of waves' lengths whish are proportional to the Fibonacci numbers, starting with the third number: 50, 100, 150, 250 and 400 m (Fig. 12.4.3 b).



(a) (b)

Fig. 12.4.3

The examples of superposition of wave profile ordinates of different lengths and determining total wave profile ordinates are shown in the graphs under the assumptions of a superposition of different wave systems and of coincident of each single wave motion

direction, which is typical for wind waves, and that the elements of each system of regular waves (λ_W, τ_W, σ and c_W), which add up when creating the wave profile in considered example, correspond in their proportions to theoretical and the empirical values.

The spectrum of wave ordinates characterizes the profile of the resulting wave, which is the sum of sinusoidal profiles of progressive regular waves (the propagating). One can see that with a superposition of profiles of regular waves, the lengths of which are indicated in the explanations to the graph, and with the height of up to 16 m (with an amplitude of up to 8 m), the difference between the largest and smallest ordinates of resulting wave (the irregular) can exceed 25 m.

When expanded into a trigonometric series of Jean–Baptiste Joseph Fourier, for example, the function of wave ordinates $\xi(t)$ of irregular sea waves, which are illustratively shown on graphs of wave profile ordinates (Fig. 12.4.3) as a sum of wave ordinates of regular sinusoidal waves profiles of different lengths, lying in the same plane and propagating in the same direction, one can give the mathematical definition of spectral density $S(\sigma)$ of wave ordinates in the form of the Fourier transform, as:
$S(\sigma) = \frac{1}{\pi} \times \int_{-\infty}^{\infty} \xi(t) \times e^{-i \times \sigma \times t} \times dt$, or spectral density $S(\sigma)$ of wave ordinates as an element of the inverse transformation: $\xi(t) = \frac{1}{2} \times \int_{-\infty}^{\infty} S(\sigma) \times e^{i \times \sigma \times t} \times d\sigma$, that is as an element of integral equations, each of which is a solution to the other, which cover the real function of the ordinates of the wave profile $\xi(t)$ and the complex function of their spectral density $S(\sigma)$.

The profile ordinate $\xi(t)$ of irregular waves surface is considered as a random process that can be described by the Fourier–Stieltjes integral: $\xi(t) = \int_{-\infty}^{\infty} e^{i \times \sigma \times t} dG(\sigma)$.

The differential frequency increment under Fourier–Stieltjes integral sign $d\sigma$ is substitute by the increment of a random complex function $dG(\sigma)$, the argument of which is the frequency σ, while: $S(\sigma) \times d\sigma = 2 \times dG(\sigma)$, which doubled derivative is the spectral density function $S(\sigma)$.

The mathematical meaning of the Fourier–Stieltjes integral associated with the fact that the random process $\xi(t)$ is considered as a superposition of harmonic components with random complex amplitudes $dG(\sigma)$, indicating both the amplitude increments and the phase increments.

For an even function of wave profile ordinates $\xi(t)$ the Fourier transform of the formulas has the trigonometric forms:
$\xi(t) = \int_0^{\infty} S(\sigma) \times \cos(\sigma \times t) \times d\sigma$ and $S(\sigma) = \frac{2}{\pi} \times \int_0^{\infty} \xi(t) \times \cos(\sigma \times t) \times dt$.

For an odd function $\xi(t)$, the Fourier transform has other trigonometric forms:
$\xi(t) = \int_0^{\infty} S(\sigma) \times \sin(\sigma \times t) \times d\sigma$ and $S(\sigma) = \frac{2}{\pi} \times \int_0^{\infty} \xi(t) \times \sin(\sigma \times t) \times dt$.

In practical issues, the empirical function of the spectral density of irregular sea wave profile ordinates has the general form [22, P. 249]: $S_\xi(\sigma) = A \times \sigma^{-k} e^{-B \times \sigma^{-n}}$.

The values of the coefficients: A, B, k and n for the variants of the spectral density function of the wave applicates are shown in the Table 12.4.1.

Table.4.1

The spectrum shape name	A	B	k	n
II–nd International Congress on Ship Design and Strength	$0.28 \times h_W^2 \times \sigma_W^4$	$0.44 \times \sigma_W^4$	5	4
Spectrum of Yu.M. Krylov	$0.0795 \times 2 \times \pi \times h_W^2 \times \sigma_W^6$	$0.785 \times \sigma_W^4$	7	4

In addition to indicated formulas for finding the spectral density of wave ordinates the following empirical formulas are also known:

– Neumann's formula: $S_\xi(\sigma) = 3.05 \times \frac{\pi}{2} \times \sigma^{-6} e^{-2 \times (g/v_{WA})^2 \times \sigma^{-2}}$,

where $g = 9.81$, m/s^2 – the gravity acceleration, v_{WA} – wind speed, m/s;

– the formula of Pearson and Moskowitz:

$S_\xi(\sigma) = 0.162 \times g^2 \times \sigma^{-5} e^{-0.74 \times (g/v_{WA})^4 \times \sigma^{-4}}$ and other formulas are available, to one degree or another corresponding to the values of spectral density of irregular sea waves.

The average wave height h_W and the average frequency of oncoming waves σ_W one can find using empirical formulas depending on the wave height of 3 %–th probability $h_{3\%}$, which corresponds to the scale of the observed intensity of sea waves according to the Table 12.4.2: $h_W = (0.40 \div 0.65) \times h_{3\%}$, m and $\sigma_W = 1.74 \times h_{3\%}^{-0.4}$, 1/s.

Table 12.4.2

Intensity of sea waves, points								
I	II	III	IV	V	VI	VII	VIII	IX
The wave height of 3 %–th probability $h_{3\%}$, m								
0.0÷0.25	0.25÷0.75	0.75÷1.25	1.25÷2.0	2.0÷3.5	3.5÷6.0	6.0÷8.5	8.5÷11.0	>11.0

Spectrum (the spectral density) of wave profile ordinates S_ξ of II–nd International Congress on Ship Design and Strength is shown by dotted lines on the graph (Fig. 12.4.4 a) for waves of V, VI, VII and VIII points, the spectrum of Yu.M. Krylov is indicated by solid lines.

Since the frequency of sea waves corresponded with the length of the wave profile: $\sigma = \frac{7.85}{\sqrt{\lambda_W}}$ or $\lambda_W = \frac{61.62}{\sigma^2}$, and the amplitude of the wave slope angle α_W is related to the amplitude of wave ordinate r_W and the with wave number k: $\alpha_W = k \times r_W = \frac{2 \times \pi}{\lambda_W} \times r_W$, then there is a dependence on wave frequency: $\alpha_W = \frac{2 \times \pi}{61.62} \times \sigma^2 \times r = 0.102 \times \sigma^2 \times r$.

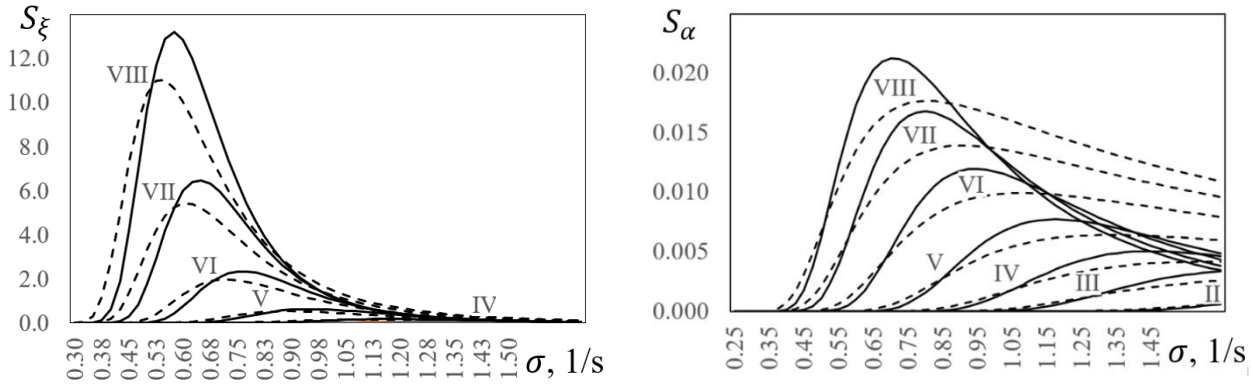


Fig. 12.4.4

If one considers the wave profile indexes: the amplitude r_W and the angle of maximum wave slope α_W as interconnected elements of the system, then the frequency–dependent proportionality index between them: $0.102 \times \sigma^2$ serves as a frequency response. Moreover, according to the theorem of Ya.I. Khinchin [22, P. 257] the ordinates of the spectral density curves of the amplitude r_W and the angle of maximum wave slope α_W are in proportion to the square of the frequency response: $S_\alpha = (0.102 \times \sigma^2)^2 \times S_\xi$, since the values of their spectral density (S_ξ and S_α) are proportional to the squares of the amplitudes of the corresponding harmonics (r_W and α_W). The spectrum of wave slope maximum angles S_α of II–nd International Congress on Ship Design and Strength is shown by dotted lines on the graph (Fig. 12.4.4 b) for waves of II, III, IV, V, VI, VII and VIII points, the spectrum of Yu.M. Krylov is done by solid lines.

Exercise. The bunkering ship for delivery of liquefied natural gas under the project, the characteristics of which are given in the previous section (item 12.3), is in conditions of strong sea waves with an intensity of VII points with a wave height 3 %–th probability $h_{3\%}$ according to observations of about 7 m (from 6.7 to 7.3 m).

The exercise assigns one to determine the characteristic spectra of roll and vertical heaving of the ship on irregular waves of the indicated intensity Using the spectrum of Yu.M. Krylov for irregular sea waves, applying the theorem of Ya.I. Khinchin and considering ship in irregular waves as a linear dynamic system, taking into account that the ordinates of the spectral density curve are proportional to the squares of the amplitudes of the corresponding harmonics, and using the amplitude–frequency characteristics of ship rocking, found by the method of complex amplitudes.

When determining the ship rocking spectra, in the exercise it is offered to assume that the damping coefficients of ship vertical heaving λ_3 and roll λ_4, and the corresponding attenuation decrements: $\nu_3 = \frac{\lambda_3}{2\times(\Delta+\mu_3)}$ and $\nu_4 = \frac{\lambda_4}{2\times(I_x+\mu_4)}$ for isolated linear equations of these types of ship rocking take the smallest values from the intervals considered above: $\nu_3 \sim 0.15$ and $\nu_4 \sim 0.1$.

In the determining of ship rolling amplitude–frequency characteristics, the characteristics are adopted for the ship of project assigned for delivery of liquefied natural gas (item 12.3): the reduction coefficient attracting the ratio of ship breadth and wave length: w_4 =0.754; the moment of ship inertia relative to x–axis Gx: $I_x \approx 457\,500$, ton×m^2; added moment of inertia relative to x–axis: $\mu_4 \approx 251\,625$, ton×m^2.

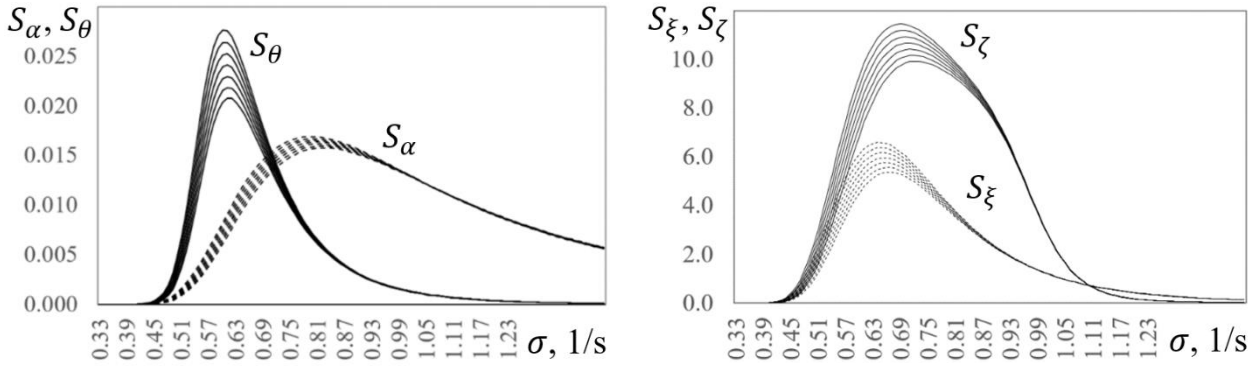
The ship own roll frequency (on calm water), which is found above, as:

$$\omega_4 = \sqrt{\frac{\Delta \times g \times h}{(I_x + \mu_4)}} = \sqrt{\frac{9\,150 \times 9.81 \times 2.5}{(457\,500 + 251\,625)}} = 0.563, \text{ radian/s.}$$

The amplitude–frequency characteristic of ship rolling by the complex amplitudes method is defined above in the form: $\frac{\theta_M}{\alpha_W} = \frac{w_4 \times \omega_4^2}{\sqrt{(\omega_4^2 - \sigma^2)^2 + 4 \times \nu_4^2 \times \sigma^2}}$.

Spectral density of the amplitudes S_θ of the roll angle during ship rolling on irregular sea waves of the intensity of VII points depending on the spectral density of the wave slope angles S_α in accordance with the theorem of Ya.I. Khinchin is determined proportionally to the square of amplitude–frequency characteristic $\left(\frac{\theta_M}{\alpha_W}\right)$, as (Fig. 12.4.5 a):

$S_\theta = \left(\frac{\theta_M}{\alpha_W}\right)^2 \times S_\alpha = \frac{w_4^2 \times \omega_4^4}{(\omega_4^2 - \sigma^2)^2 + 4 \times \nu_4^2 \times \sigma^2} \times S_\alpha$, and the spectra of sea wave ordinates and wave slope angles are determined for irregular sea waves with an intensity of VII points at a wave height of 3 %–th probability $h_{3\%}$ of about 7 m (from 6.7 to 7.3 m).



(a) (b)

Fig. 12.4.5

The dimensionless inertial coefficient was defined previously (item 12.3), as:

$$f_{\mu 3} = \frac{\mu_3}{\Delta + \mu_3} = \frac{12\,529}{9\,150 + 12\,529} = 0.578.$$

The own frequency of considered ship heave (on calm water), found above, is:

$$\omega_3 = \sqrt{\frac{\rho \times g \times L \times B \times C_\alpha}{(\Delta + \mu_3)}} = \sqrt{\frac{1.025 \times 9.81 \times 116 \times 20 \times 0.89}{(9\,150 + 12\,529)}} = 0.979, \text{ radian/s.}$$

The amplitude–frequency characteristic of the ship heave is determined above by complex amplitudes method in the form: $\frac{\zeta_M}{r_W} = \sqrt{\frac{\left(\omega_3^2 - f_{\mu 3} \times \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2}{\left(\omega_3^2 - \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2}}$.

Spectral density of heave amplitudes of ship S_ζ, depending on the spectral density of wave profile ordinates S_ξ, in accordance with the theorem of Ya.I. Khinchin is determined proportionally to the square of the amplitude–frequency characteristic $\left(\frac{\zeta_M}{r_W}\right)$, as (Fig. 12.4.5 b): $S_\zeta = \left(\frac{\zeta_M}{r_W}\right)^2 \times S_\xi = \frac{\left(\omega_3^2 - f_{\mu 3} \times \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2}{\left(\omega_3^2 - \sigma^2\right)^2 + 4 \times \nu_3^2 \times \sigma^2} \times S_\xi$.

The spectra of irregular sea waves and of the main kinds of ship rocking can be used to assess the possibility of fatigue failure of structures experiencing loads associated with the action of waves, as well as to improve the criteria for sustainability of a ship in waves, in particular, to unify the approaches to assessing sustainability used by the Maritime Classification Societies, expanding the use of physical calculations as free as possible from conventions, for example, if the ship is considered as a dynamic system that receives the energy of incoming waves, accumulates this energy and spends it when damping the roll due to the viscosity of water and due to dissipation of the energy by diverging waves, generated during rolling motion of ship.

Questions and comments:

12.4.1. The influence of the characteristics of large waves profile and the motion elements of this profile on ship roll, considering the influence of wave profile also on the movement of water at relatively small depths within the influence of the wave on ship.

12.4.2. Information from the hydromechanics on shape of the lines of motion of the wave profile and water under the surface of the wave profile when solving the continuity equation of Laplace for the incompressible fluid, considering the boundary conditions on the surface, at infinite depth and taking into account the initial conditions.

12.4.3. The sinusoidal profile of a regular wave of relatively small amplitude (based on the ratio of height and wave length) for theoretical infinite depths.

12.4.4. The relationships of wave length, it period and profile velocity, the amplitude of a sine wave relation to the maximum angle of the wave slope, the wave number.

12.4.5. The independence of stationary processes of established irregular waves on the time count beginning and on waves initial phases.

12.4.6. Nonlinearity of the dependence of the amplitude of moving wave profile on other elements of wave: on the period, frequency and the phase velocity of wave profile. Empirical Zimmerman’s formula for finding wave height depending on its length.

12.4.7. Irregular sea waves characteristic by a spectrum (the amplitudes distribution over frequencies), which is a superposition of wave profiles having different elements (the length, period, frequency and the phase velocity of wave profile), spectral characteristics of which do not depend on time, provided the considered waves are stationary process.

12.4.8. Mathematical definition of the spectral density function (the spectrum) in the form of the Fourier transform or the inverse transform, trigonometric forms of the Fourier transform for even and odd functions of wave profile ordinates.

12.4.9. General view of the empirical function of the spectral density of sea wave ordinates. The data required to determine spectrum of wave ordinates of the spectra (the spectrum of Yu.M. Krylov, the one of II–nd International Congress on Ship Design and Strength, the spectra of: Neumann, Pearson and Moskowitz, other empirical forms of the spectra): the average wave height, the wave height of 3 %–th probability, depending on the intensity of sea waves and affecting the average wave height, the wind speed, etc.

12.4.10. Transformation of spectral density functions using the theorem of Ya.I. Khinchin in the proportion of the square of the frequency response for the interconnected elements of linear dynamic systems: the amplitude and maximum angle of the wave slope of irregular waves, as well as the ordinates of wave profile and the displacements of the ship during it vertical heaving in irregular waves and the amplitudes of the wave slope angle and amplitudes of ship rolling angle, considering that the spectral density of the listed harmonics are proportional to the squares of their amplitudes.

12.4.11. The possibilities of spectra use of ship motion in irregular sea waves to analyze the conditions of fatigue failure of structures experiencing loads associated with wave action.

12.4.12. Options for improving approaches of evaluation of ships stability, which can be considered as dynamic systems that receive the incoming waves energy, accumulate this energy and spend it when damping the motion due to the viscosity of water and due to dissipation of the energy by diverging waves, generated during ships motions on incoming waves.

12.4.13. Promoting the safety of navigation and the quality of inspection assessment of ships seaworthiness, deepening the uniformity of approaches of evaluation ships stability by Maritime Classification Societies, subject to joint desire of these societies to determine stability on the basis of its physical calculations while reducing the conventionality of rules.

12.5. The controllability of ship in water and equations of ship motion

Controllability of ship methodically means its performance to follow given trajectory in water in accordance with control signals specified by the actions of the ship control system – an automatic system or a navigator. The property of ship to move along the straight line in water means its sustainability on the course [19 and 28], and the ability to move along controlled curved trajectory means the ship turning ability.

In some cases, the ship stainability on course and it turning ability may come into conflict, that is, elements of the ship that improve its stainability on course, for example, the ratio of the length of the ship to the breadth increase or installing the fins in front of the rudders improving stability on course at the same time can impair turning ability. And it is also possible the reverse interdependent change in these performances, that is, a ship that is not stable enough on course, in particular, due to certain elements of the shape or in certain modes of motion, may have good turning ability.

According to evaluation of uniformly opposite influence on ship stability on course and it turning ability of elements or the motion modes, the exceptions are possible, in particular, an increase of stern rudder area, which also serves as the ship stabilizer on course, within certain limits can simultaneously contribute to both improving the turning ability, and the ship motion stability on course, although an increase in the rudder area also increase its mass and cost, as well as the rudder hydrodynamic resistance.

For cargo ships, which ae moving mostly in a straight course, the stability on course could be more important, but if there is an overly effective control, the yaw increases, which leads to velocity drop–in, control system failures and helmsman fatigue. For coastal and harbor ships operating in a limited aquatic areas, the turning ability is usually more important.

The property of ship controllability methodically also covers the maneuverability, which means the ship opportunity to change movement elements during certain velocity. A maneuver of a ship is an element of controlled movement trajectory, and sequence of the maneuvers is maneuvering. The controllability of ship serves as a more general methodological concept than the maneuverability. In contrast to maneuverability the controllability is achieved through control operation that must correspond to the control objectives and therefore this requires feedback in the system, including the navigator or autopilot and the moving ship equipped with the controls.

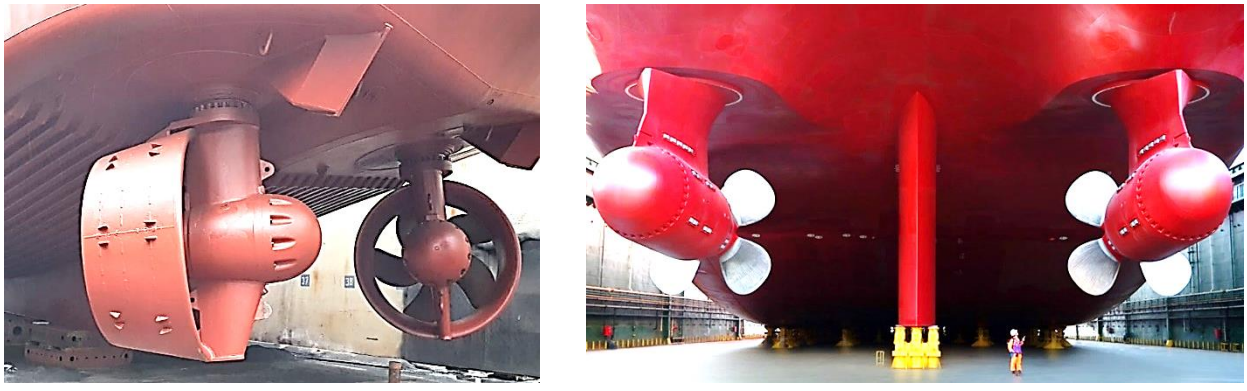
The characteristics of ship maneuverability do not depend on the presence of feedback. At the same time, one can assume that maneuverability methodically covers the property of turning ability of a ship and, in a particular case, the property of ship motion stability on course. In other words, the controllability, maneuverability and ship turning ability could usually be considered as the system of nested concepts.

The structure of the equations for the controlled motion of ship in water is inextricably linked with physical foundations of the phenomena described by these equations. The prospects for mathematical and numerical methods for studying ship controllability are largely due to the fact that the properties of a ship in water as an object when studying its controllability, including a feedback control system, can differ significantly from the properties of the traditionally considered physical model of a ship. A ship equipped with controls and controlled by a navigator or by automatic control system is capable of reacting in various ways to external influences, and the patterns of ship movement are described not only by the mechanics of ship physical model.

The equipment and machinery of ship control in water are methodically distributed into two groups: the main and auxiliary ones. There is also a group of limiting controls, which include anchor holding systems, the dynamic stabilization systems, the braking devices, etc.

When using the main controls, the operation of the propulsors or the availability of ship translational velocity is required. The main devices of ships control include their steering devices, as well as the propulsors and propulsion–steering ones, for example, the rotary propulsion–steering columns and Azipods (the electric propulsors placed in rotating gondolas in the water: «azimuth» and «pod» – registered trademark of Asea Brown Boveri). The several propulsors installed on ship can be used to maneuver it both the velocity – by adjusting the thrust of the propulsors, and the course of ship – by changing the balance of thrust and the direction of action of the propulsors installed on opposite sides.

The propulsion and steering devices include: a complex of the propeller and a rotary nozzle with a stabilizer, a rotary steering column (Fig. 12.5.1 a), rotary propulsion Azipod (Azipods on the stern of the ice-swimming tanker «Vasily Dinkov» are shown in the picture, https://new.abb.com/, Fig. 12.5.1 b), the wing propulsor etc.



(a) (b)

Fig. 12.5.1

The group of ship steering devices includes: the rudders, multi–blade steering complexes, rotary steering devices, reversible steering devices of water–jet propulsors, etc.

The rudders are used – the balanced, unbalanced and semi–balanced, the simple – with several supports on the sternpost and rudder post, semi–suspended ones – supported on a stock and bracket, and the outboard rudders – supported on stocks. Streamlined and the flat rudder profiles are used. On most ships, one rudder is used in stern; in some variants, two or three rudders are used. The most effective rudders are behind the propellers. When turning the rudder blade (which is when shifting) during ship motion, the transverse force arises on the blade, leading to drift and rotation of the ship.

Auxiliary devices of ship control without running or during low velocity are driven by its own engines and do not require the operation of the ship's propulsors and its translational velocity. As the ship velocity increases, the effectiveness of the auxiliary controls usually reduces.

The motion of a ship on still water, can usually be described by a system of equations in three projections, considering the necessary assumptions, as:

$$(\Delta + \mu_{11}) \times \dot{v}_x - (\Delta + \mu_{22}) \times v_y \times \omega_z - \mu_{26} \times \omega_z^2 = N_x,$$

$$(\Delta + \mu_{22}) \times \dot{v}_y + (\Delta + \mu_{11}) \times v_x \times \omega_z + \mu_{26} \times \dot{\omega}_z = N_y,$$

$$(I_Z + \mu_{66}) \times \dot{\omega}_z + \mu_{26} \times \dot{v}_y - (\mu_{22} - \mu_{11}) \times v_x \times v_y + \mu_{26} \times v_x \times \omega_z = M_z,$$

where: $v_x = \dot{\xi}$, $v_y = \dot{\eta}$ and $\omega_z = \dot{\chi}$ – the projections of ship translational velocity onto the abscissa axes Gx, the ordinate Gy and the angular velocity of rotation relative to the applicate axis Gz associated with the ship; N_x, N_y and M_z – the projections of control efforts on axes of the coordinate system associated with the ship $Gxyz$, including the force acting on rudder of any type (the regular rudder, nozzle, rotary steering column, Azipod or the other) in projections on the axis and the projection of moment.

In the considered equations it is assumed, that the energy of ship motion is replenished by the working propeller while maintaining the velocity v_x, or a relatively short period is considered during which the energy of ship motion by inertia without the propeller work is not noticeably attenuated, so damping forces or moments are not included in the equations.

Integration of equations draw up in specified coordinate system and the determination of angular velocity ω_Z and projections of the translational velocity (v_x and v_y) of ship motion in water are carried out in the associated with ship coordinate system $Gxyz$, and then the angular and translational motions of ship are integrated to find motion coordinates in the fixed coordinate system $0\xi\eta\zeta$.

The left side of considered equations includes the projections of reactive forces and moments: the inertial, that is proportional to translational and angular accelerations, as well as their bilinear combinations (the products) of translational and angular velocities (the definite analogues of damping and the Coriolis forces), and on the right side – the projections of external and driving forces and moments (the control efforts).

Despite certain similarities in the designation of acting forces and moments in the issues about ship rolling and about it controllability, there are also the differences, as mentioned. In calculating of ship controllability indexes, the following classification of acting forces and moments is common, which are classified into the categories: the driving – from the action of the propulsion of the ship and controls (the control efforts), the external ones, which are caused outside the ship, and reactive forces and moments – the inertial and damping, these are discussed in more detail above (item 12.1).

The assumptions of the ship symmetry relative to the diametral plane and the plane of the midship section, of the invariance in time of the velocity v_X projection on the associated with ship abscissa axis Gx, as well assumption of smallness of ship breadth compared to its length allow to simplify the system of equations describing the motion of ship in stationary water.

After considering the simplifying assumptions in the equations and excluding double indexation of the coefficients of added masses and moments of inertia of water, the system of equations takes the form:

$(\Delta + \mu_2) \times \dot{v}_y + (\Delta + \mu_1) \times v_x \times \omega_z = N_y,$

$(I_z + \mu_6) \times \dot{\omega}_z - (\mu_2 - \mu_1) \times v_x \times v_y = M_z.$

To solve these equations system, it is necessary to transform the first equation: $\dot{v}_y = \frac{1}{\Delta+\mu_2} \times (N_y - (\Delta + \mu_1) \times v_x \times \omega_z)$, to differentiate the second equation with respect to time and consider the assumption that the projection of the velocity v_x onto the associated longitudinal axis Gx is constant, that is: $\dot{v}_x = 0$, after the differentiation, the second equation of the system has the form: $(I_Z + \mu_6) \times \ddot{\omega}_z - (\mu_2 - \mu_1) \times v_x \times \dot{v}_y = \dot{M}_z$, and substitute of transformed first equation from the system into the second one, that is:

$$\ddot{\omega}_z + \omega_z \times \frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(I_z+\mu_6)\times(\Delta+\mu_2)} \times v_x^2 = \frac{\dot{M}_z}{I_z+\mu_6} + \frac{(\mu_2-\mu_1)\times v_x \times N_y}{(I_z+\mu_6)\times(\Delta+\mu_2)}.$$

The solution of resulting linear inhomogeneous equation can be found in the form of the sum: $\omega_Z = \omega_{Z1} + \omega_{Z2}$, including the general solution ω_{Z1} of the corresponding homogeneous equation: $\ddot{\omega}_Z + \omega_Z \times p^2 = 0$, assuming that ship is moving without control, those at zero control inputs: $N_y = 0$ and $M_z = 0$, the roots of the characteristic equation are: $p = \pm v_x \times \sqrt{\frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(I_z+\mu_6)\times(\Delta+\mu_2)}}$, and the sought particular solution ω_{Z2} of the original inhomogeneous equation, the form of which is in the same in which the control actions: N_y and M_z are to be specified, which are not zeros, as this is to be for general solution of homogeneous equation.

The general solution of a homogeneous equation is added to a particular solution of an inhomogeneous one within the framework of the linear theory of controllability in compliance with the principle of their independence – the superposition.

A particular solution ω_{z2} is sought in the form of same function on time as the right side of the inhomogeneous equation. For this reason, the shape of ship motion trajectory could be diverse and depends on the control efforts action, that is on moment M_z function and on the lateral force on the rudder N_y and on their functional dependencies on the time parameter t, the particular solution cannot be found in a general form, and the noted feature represents some methodological complexity.

Considering that the root p of characteristic equation can be a real number or the imaginary one, depending on value of the factor under the root: $(\mu_2 - \mu_1)$, then the following options for the general solution of the homogeneous equation are possible:
$\omega_{z1} = C_1 \times e^{p \times t} + C_2 \times e^{-p \times t}$ at $p^2 > 0$
and $\omega_{z1} = C_1 \times \cos(p \times t) + C_2 \times \sin(p \times t)$ at $p^2 < 0$,
where: C_1 and C_2 – the integration constants determined due to initial conditions.

Given the initial conditions: $\omega_Z = \omega_{Z0}$ and $\dot{\omega}_Z = \dot{\omega}_{Z0}$ at $t = 0$ in the absence of driving forces and moments (the efforts of ship's propulsors and controls): $N_Y = 0$ and $M_Z = 0$ the general solution of the homogeneous equation can be found in form:
$\omega_{z1} = ((\omega_{z0} + \dot{\omega}_{z0}/p) \times e^{p \times t} + (\omega_{z0} - \dot{\omega}_{z0}/p) \times e^{-p \times t})/2$ at $p^2 > 0$
and $\omega_{z1} = \omega_{z0} \times \cos(p \times t) + (\dot{\omega}_{z0}/p) \times \sin(p \times t)$ at $p^2 < 0$.

Without control of ship, when controlling efforts – the turning moment and transverse force are equal to zero: $M_z = 0$ and $N_y = 0$, if the added mass μ_2, characterizing the inertia of water impact when the ship moves with acceleration in transverse direction to the side – along the ordinate axis Gy, is greater than the added one mass μ_1, characterizing inertia in water when ship moves with acceleration along the abscissa axis Gx: $\mu_2 > \mu_1$, that is for the negative or positive real roots (the real number values) of characteristic equation, at: $p^2 = v_x^2 \times \frac{(\mu_2 - \mu_1) \times (\Delta + \mu_1)}{(I_z + \mu_6) \times (\Delta + \mu_2)} > 0$ the solution of the homogeneous equation considering the initial conditions: $\omega_{z1} = \omega_{z0}$ at $t = 0$ in the form of a function of the angular velocity of ship rotation ω_{z1} (according to its relative value: ω_{z1}/ω_{z0}) has an exponential form: $\frac{\omega_{z1}}{\omega_{z0}} = \frac{1}{2} \times \left(\left(1 + \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}}\right) \times e^{p \times t} + \left(1 - \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}}\right) \times e^{-p \times t} \right)$, which can be converted to the sum of the hyperbolic cosine and sine functions (ch and sh):
$\frac{\omega_{z1}}{\omega_{z0}} = \frac{e^{p \times t} + e^{-p \times t}}{2} + \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \frac{e^{p \times t} - e^{-p \times t}}{2}$, and: $\frac{\omega_{z1}}{\omega_{z0}} = ch(p \times t) + \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times sh(p \times t)$,
where, the hyperbolic sh and ch functions are: $sh\, x = \frac{e^x - e^{-x}}{2}$ and $ch\, x = \frac{e^x + e^{-x}}{2}$.

Under the assumption that initial angular velocity ω_{z0} and initial angular acceleration $\dot{\omega}_{z0}$ are in proportion to the modulus of real or imaginary root of the characteristic equation p: $\dot{\omega}_{z0} = p \times \omega_{z0}$, which is possible for ship moving without control, the general solution of the homogeneous equation describing the motion of ship and its own properties as a physical body with not control can be simplified: $\frac{\omega_{z1}}{\omega_{z0}} = ch(p \times t) + sh(p \times t)$.

Exercise. It is assigned to determine solution of homogeneous equations of motion without control in the form of function of the angular velocity of rotation ω_{z1} (according to its relative value: ω_{z1}/ω_{z0}) of considered bunkering ship for the delivery of liquefied natural gas, according to the project, which characteristics are specified in previous section (item 12.3) and are equal to: the length along the waterline: L =116 m, the breadth: B =20 m, the side height: H =11 m, the average draught at loaded waterline: T =5.2 m, ship displacement at summer draught load line: Δ=9 150 ton, the general completeness coefficient: δ =0.74, the waterline area completeness coefficient: C_α =0.89, the operating velocity of ship 13.6 knots (7 m/s).

Initial angular rate of ship turn is assigned: $\omega_{z0} = 2 \times \pi/60$ =0.1047, 1/s, that is with an initial angular velocity ω_{z0} the ship has to make a complete turn ($2 \times \pi$) for 60 s. For given initial conditions, it is assumed that the initial angular velocity ω_{z0} and the initial angular acceleration $\dot{\omega}_{z0}$ are in proportion to the modulus of the real or imaginary characteristic number p: $\dot{\omega}_{z0} = p \times \omega_{z0}$, which corresponds to ship moving without control.

The own moment of inertia of the ship I_z, when rotating with acceleration relative to the applicate axis Gz, is commensurate (equal approximately) to the moment of inertia relative to the ordinate axis Gy and amounts to (item 12.2):

$I_z \approx I_y \approx 0.065 \times C_\alpha \times \Delta \times L^2 = 0.065 \times 0.89 \times 9\,150 \times 116^2 = 7\,122\,631$, ton×m^2.

The added mass of water μ_2 when ship moving with translational acceleration along the ordinate axis Gy can be defined as part (presumably half, that is about 50%) of the added mass μ_3 when ship moving with translational acceleration along the axis of the applicate Gz (item 12.2): $\mu_2 \approx 0.5 \times \mu_3 = 0.5 \times \Delta \times \left(1.563 \times \frac{C_\alpha}{\delta} - 1\right) =$

$= 0.5 \times 9\,150 \times \left(1.563 \times \frac{0.89}{0.74} - 1\right) = 4\,009$, ton.

The attached moment of inertia μ_6 in water, when the ship rotated with the acceleration relative to applicate axis Gz can be determined as part (presumably half, that is about 50 %) of the added moment of inertia μ_5 when ship rotated with acceleration relative to the ordinate axis Gy (item 12.2):

$\mu_6 \approx 0.5 \times \mu_5 = 0.5 \times \Delta \times L^2 \times C_\alpha \times \left(\frac{0.098}{\delta} - 0.065\right) =$

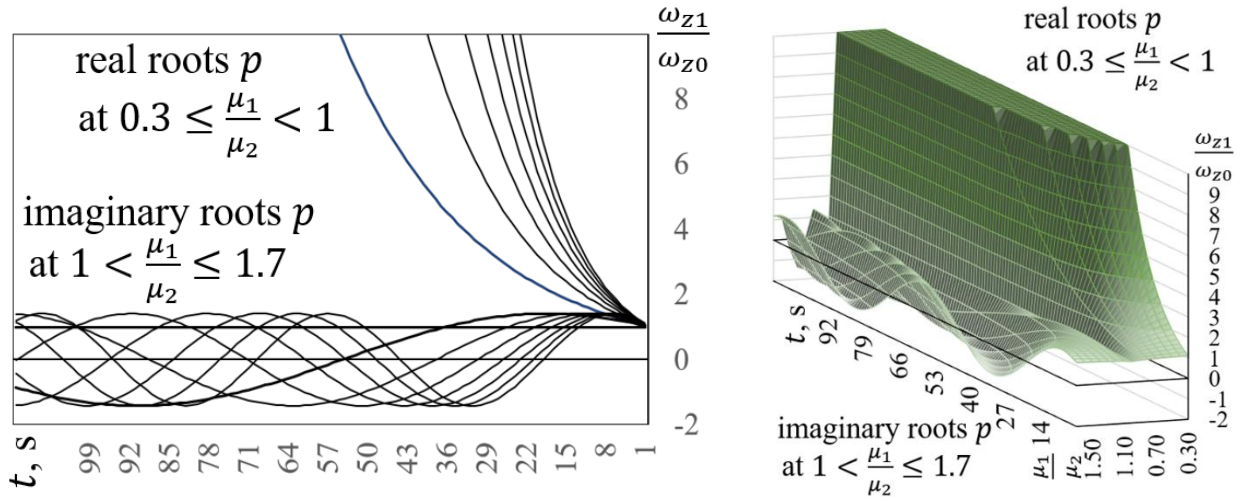
$= 0.5 \times 9\,150 \times 116^2 \times 0.89 \times \left(\frac{0.098}{0.74} - 0.065\right)$ =3 639 798, ton×m^2.

The solution to homogeneous equations of motion without control in the form of a function of the angular velocity of rotation of considered ship ω_{z1} (according to its relative value: ω_{z1}/ω_{z0}) can be graphically performed as the functional surface (Fig. 12.5.2) depending on two arguments: on the time parameter t and on those of the added masses (μ_1 and μ_2) of the ship characterizing it inertia in water when moving with acceleration along the axes: the abscissa one Gx and the ordinate Gy in the form of their ratio (μ_1/μ_2), on

which depend the real roots of the characteristic equation: $p = \pm v_x \times \sqrt{\frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(I_z+\mu_6)\times(\Delta+\mu_2)}}$ (at: $\mu_1/\mu_2 < 1$) and imaginary ones (at: $\mu_1/\mu_2 > 1$).

The calculations are conveniently to do in the Microsoft EXCEL.

The indicated two graphical forms got of one functional surface of the dependence of the angular velocity of the bunkering ship ω_{z1} rotation (according to its relative value: ω_{z1}/ω_{z0}, Fig. 12.5.2 a and b) on the time parameter t and on the ratio of added masses (μ_1/μ_2) are together given for methodological purposes for clarity of the graph of the function of the angular velocity of ship turn and the relationship of its arguments (t and μ_1/μ_2) both for imaginary and real roots of the characteristic equation.



(a) (b)

Fig. 12.5.2

The characteristic equation has a zero root: $p = 0$ at: $\mu_1 = \mu_2$.

The variable values of the ratio of ship in water added masses (μ_1/μ_2) used to determine the graph are shown in the Table 12.5.

These values of the added masses of water, as well as the rest of the initial data, correspond to the values of the moduli of the characteristic equation roots p, also shown in this Table. The time interval when determining the graphs are taken in the range from 1 to 112 s with a calculation step of one second.

For imaginary roots of the characteristic equation: $p^2 < 0$ (for example, if the ship breadth is large, which corresponds with the parameter: μ_1/μ_2) the general solution of the homogeneous equation regarding the angular velocity of the ship without control has a trigonometric form: $\frac{\omega_{z1}}{\omega_{z0}} = \cos(p \times t) + \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \sin(p \times t)$, and under the assumption that the initial angular velocity ω_{z0} and the initial angular acceleration $\dot{\omega}_{z0}$ for a ship moving without control are in proportion to the modulus of the root of the characteristic

equation p: $\dot{\omega}_{z0} = p \times \omega_{z0}$, the general solution of the homogeneous equation can be transformed to the form: $\frac{\omega_{z1}}{\omega_{z0}} = \cos(p \times t) + \sin(p \times t)$.

Table 12.5

The ratio of added masses of ship (μ_1/μ_2)													
0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70
Values of the added mass (μ_1, ton), characterizing inertia along the abscissa axis Gx														
1 203	1 603	2 004	2 405	2 806	3 207	3 608	4 009	4 410	4 810	5 211	5 612	6 013	6 414	6 815
Modulus of the root p of the characteristic equation														
Real root								Imaginary root						
0.1001	0.0945	0.0878	0.0800	0.0704	0.0585	0.0420	0.0	0.0433	0.0621	0.0772	0.0904	0.1024	0.1137	0.1243

The function of general solution of the homogeneous equation of ship motion without control in the form of angular velocity in its relative value $\left(\frac{\omega_{z1}}{\omega_{z0}}\right)$ on the set of functional lines (Fig. 12.5.2 a) and on the graphic surface (Fig. 12.5.2 b) shows an infinitely increasing amplitude of the angular velocity and loss of motion sustainability – with the real values of roots of the characteristic equation (at: $\mu_1/\mu_2 < 1$) and the self–oscillation of ship turn angular velocity– with the imaginary roots of the characteristic equation ($\mu_1/\mu_2 < 1$).

It would be mentioned that roots of the characteristic equation, both positive and negative: $p = \pm\sqrt{p^2}$, are real if the added mass during ship motion with acceleration in the direction of the side (along the ordinate axis Gy) is greater than the added mass during acceleration of ship in the longitudinal direction (along the abscissa axis Gx): $\mu_2 > \mu_1$.

Under this condition, ship moving without control leaves a limited trajectory and turns relative to the motion direction at an angle at which: $\mu_2 > \mu_1$, and in this direction of motion, if one redefines the coordinate axes, connecting them with the direction of motion, and find the corresponding roots of the characteristic equations that at: $\mu_2 > \mu_1$ are imaginary, the ship continues to move, performing self–oscillations, which correspond to the trigonometric (cosine and sinusoidal) dependences of the angular velocity of ship turn without control in relation to its initial angular velocity $\left(\frac{\omega_{z1}}{\omega_{z0}}\right)$ on time parameter t.

The drift of the ship towards the side (which is along the transverse axis Gy by inertia or under the influence of one or another external force), provided that the motion

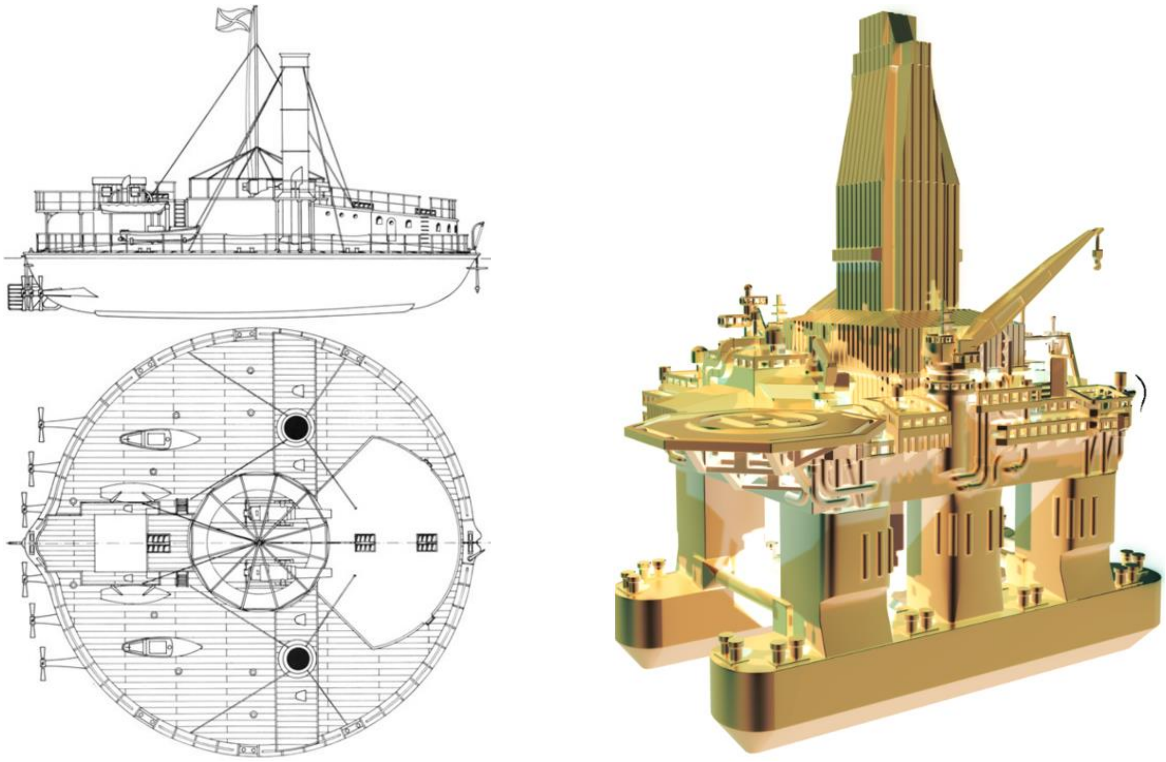
of the ship along the abscissa axis Gx, is stopped, in particular, when the propulsors are not working, occurs with a separation of the boundary layer of the water flow alternately with each extremity of the ship in the form of vortices, the pattern of formation of which in hydromechanics is characterized by their dependence on the Strouhal number: $Sh = p \times L/v_x$. In this case, the modulus of the root p of the characteristic equation serves as the frequency of self–oscillations and separation of boundary layer vortices. The flow around a ship when drifting towards the side with periodic separation of the boundary layer corresponds to its movement without control in the form of self–oscillations with imaginary roots of the characteristic equation.

Most ships do not have theoretical stability on course. To keep them on course, an automatic feedback control system can be used, operating on sensors of heading, or angular velocity, or angular acceleration, or sensors of several elements of motion, or the ship is to be controlled and kept on course by the navigator based on observations of these elements of motion. Attenuation of the amplitudes of the elements of motion without control of ship in water (the angular velocity of turn ω_{z1} and the transverse translational velocity v_{x1}) as the input velocity decreases when finding the root of the characteristic equation of the translational velocity of ship when it moves by inertia, which is when the kinetic energy of ship moving by inertia is gradually spent.

At zero root of the characteristic equation ($p = 0$, when: $\mu_1/\mu_2 \approx 1$) the amplitude of the angular velocity of ship turn does not change and in option without the control the ship elements of motion remain the initial: $\omega_{z1} = \omega_{z0}$. Considering the peculiarity of the shape of the submerged hull, some ships may have the property of indifferently stable motion on water while maintaining the initial elements of movement, at: $\mu_1 \approx \mu_2$. These include a series of historical battleships (the artillery ships) of the «Novgorod» project (Fig. 12.5.3 a), which shape of hulls was round in plan. The construction of these ships began in 1871. The length and breadth were 31 m, the draft 4.11 m, the displacement was about 2 700 ton, total power of steam engines were 2 510 kW.

Some multi–hull ships, for example, the self–propelled floating semi–submersible drilling platform «Polar Star» (2012, Fig. 12.5.3 b, https://cults3d.com) may also have a similar ratio of added masses ($\mu_1 \sim \mu_2$, $\mu_1/\mu_2 \approx 1$). This floating semi–submersible platform of the Moss Maritime CS50 Mk II project, designed for offshore prospecting, evaluating, exploration and production drilling of wells up to 7 500 m deep for oil and gas with an underwater wellhead location at sea depths from 70 to 500 m. It has the following characteristics and particulars: the displacement of up to 59 300 ton, the gross registered tonnage (GRT) – 54 450, the length – 122.54 m, the breadth 91.97 m, the height to the top of the tower – 128 m, the height to the upper deck – 45 m, the draught when moving on a tug or under its own power – 9.85 m, the survival draught in critical conditions – 21.5 m,

the operational draught – 23.5 m, the engine power of 6 pcs. – 5417 kW each, the propulsors – Azipod, 4 pcs., self–velocity – 8 knots.



(a) (b)

Fig. 12.5.3

It would be mentioned that in addition to inertia in water when moving with accelerations along the axes: abscissa Gx and the ordinate Gx, characterized by added masses: μ_1 and μ_2 for ships such as the floating semi–submersible drilling platform «Polar Star» there are possibly other directions of acceleration, for example, some diagonal direction for which the inertia characteristic – the added mass of water μ_{12} would be greater than the added masses: μ_1 and μ_2 when moving with acceleration along the axes: abscissa Gx and the ordinate Gx. But this, apparently, does not change the conclusion about indifferently stable motion on water assuming the preservation of the initial elements of the platform's motion, since in this case there is a second diagonal direction and if the coordinate axes are directed along these diagonals, then the equality of the added masses: $\mu_{12} = \mu_{21}$ could be preserved, despite the fact that in the diagonal directions of acceleration these added masses could be greater than when moving with acceleration along the axes: abscissa Gx and the ordinate Gx: $\mu_1 \sim \mu_2$.

If root of the characteristic equation is equal to zero: $p = 0$, at: $\mu_2 = \mu_1$ and the initial angular acceleration relative to the applicate axis Gz is equal to zero: $\dot{\omega}_{Z0} = 0$, then the general solution of the homogeneous equation is a constant number, this means that when a ship moves in water it continuous rotation about the associated vertical axis Gz with constant angular velocity ω_{z0}.

And if the root of the characteristic equation is equal to zero: $p = 0$ and at the same time the initial angular speed of rotation of the ship relative to the vertical axis Gz associated with it is too equal to zero: $\omega_{z0} = 0$, then the general solution of the homogeneous equation is identical to zero: $\omega_{z1} \equiv 0$, and the ship without work the ship's propulsors and controls move by inertia in a straight line without turning.

To understand the mechanical processes of ship motion in the water, it is also of interest to analyze the initial time interval of motion (several seconds after the moment of initial conditions), when the graphs of ship rotation angular velocity changes without control in relative magnitude practically coincide as for an unstable ship on course (when the positive and negative roots of the characteristic equation are real), and for a ship motion in the self–oscillation mode (with the imaginary roots of the characteristic equation), as well as for a characteristic equation of ship that is indifferently stable at zero root, moving in the direction, corresponding the initial conditions.

On the graphical functional surface (Fig. 12.5.4) for a short period – about 5 seconds after the moment of initial conditions, the following are indicated: the similarity of the graphs of the angular velocity of ship turn without control and the bifurcation line at zero values of root of the characteristic equation: $p = 0$, when $\mu_1/\mu_2 \approx 1$.

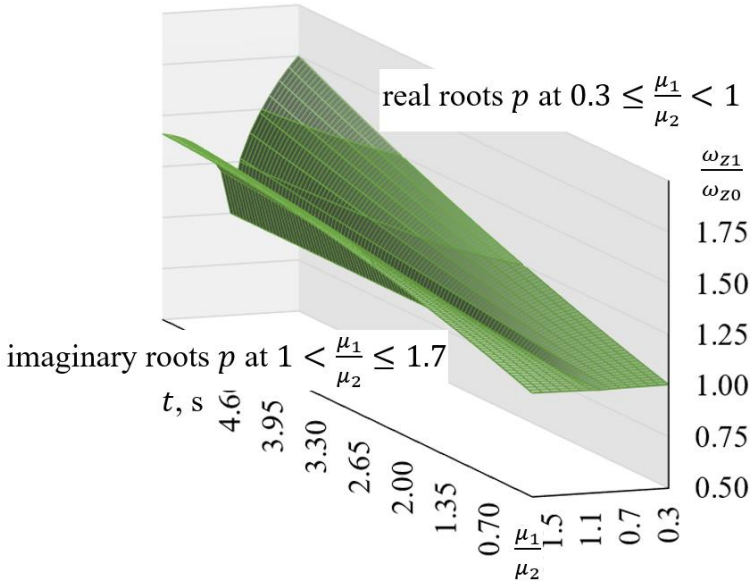


Fig. 12.5.4

The graphs under these conditions are essentially symmetrical for real ($p^2 > 0$ at: $\mu_2 > \mu_1$) and imaginary ($p^2 < 0$ at: $\mu_2 < \mu_1$) roots of the characteristic equation in a sufficiently small vicinity of the bifurcation line, which is the line at: $p \times t \sim 0$ and $\frac{\mu_1}{\mu_2} \sim 1$, which distinguishes the premises on which the real (at: $\mu_1/\mu_2 < 1$) and imaginary (at: $\mu_1/\mu_2 > 1$) roots of the characteristic equation depend.

The difference of graphical functional surface for these values of the added masses of water and the roots of the characteristic equation are visible with a longer interval of change in the time parameter t (Fig. 12.5.4), commensurate with a large fraction of the self–oscillation period: $2 \times \pi/p$ for imaginary roots of the characteristic equation: $p^2 < 0$ at $\mu_2 < \mu_1$.

If with a real root of the characteristic equation: $p^2 > 0$ for $\mu_2 > \mu_1$ the integration constants are not equal to zero:
$C_1 = (\omega_{z0} + \dot{\omega}_{z0}/p)/2 \neq 0$ and $C_2 = (\omega_{z0} - \dot{\omega}_{z0}/p)/2 \neq 0$ (that is at: $\omega_{Z0} \neq 0$ and/or $\dot{\omega}_{Z0}/p \neq 0$), then the trajectory of ship is unstable. In this case, the ship, left to itself moving without control, would turn across the direction of the initial motion and would ultimately move by inertia in the direction of the side and would self–oscillate across the direction of motion.

After solving the system of equations and finding the angular velocity ω_z of ship turn relative to the Gz axis the other functions of motion elements: ship translational velocity, acceleration and the displacement of ship in direction of the Gy axis are determined in the same way as the functions of angular velocity, angular displacement and rotation relative to the Gz axis after substitute the angular velocity function into these equations. The reverse sequence of finding the angular velocity ω_z relative to the applicate axis Gz is also possible if the translational velocity v_y is initially determined in projection onto the ordinate axis Gy associated with the ship.

Questions and comments:

12.5.1. Seaworthy property of ship controllability, covering it straight–line motion stability, its turning ability under the influence of the control system and the maneuverability (not related to the control system the ability of the physical ship to perform maneuvers under certain velocity).

12.5.2. Elements of ship shape having uniformly an opposite effect on stability on course and on ship turning ability, when, for example, the ship motion stability on course improve: by increasing the ratio of the length of the ship to breadth or installing the fins in front of the rudders at the same time can proportionally worsen the ship turning ability.

12.5.3. The property of maneuverability of a physical ship, independent on control system and on the presence of feedback in it, is the ship ability to perform maneuvers with definite velocity.

12.5.4. The main controls operating while the ship is translationally moving or under propulsors running, the auxiliary controls operating without the ship motion or at low velocity, the increase of main controls efficiency with increasing of ship velocity and reducing the effectiveness of ship auxiliary controls.

12.5.5. Types of steering devices of ships: the rudders, multi–blade steering complexes, the rotary steering devices, the reversible steering devices of water–jet ships, etc.

12.5.6. Types of propulsion and steering devices: the propeller and rotary nozzle in combination with the stabilizer, the rotary steering column and the Azipod, wing propulsion, etc.

12.5.7. A system of equations in three projections describing the plane motion of ship on the surface of still water. Integrating the equations in the coordinate system associated with ship when determining the velocities (translational and the angular) and the accelerations of ship and finding the translational and angular displacements in fixed coordinate system.

12.5.8. Assumptions of ship symmetry relative to the diametral plane and the midship section plane, of time invariance of the velocity projection on the abscissa axis associated with ship, as well as of ship breadth smallness compared with its length and the simplification of system of equations describing the ship motion in still water.

12.5.9. Transformation of system of two first–order linear equations into one second–order linear equation, which serves the solution of equations system describing the ship motion in still water.

12.5.10. Finding the angular velocity of ship rotation relative to the vertical axis of the applicate when it moves in still water by solving linear inhomogeneous equation in the form of the sum of the general solution of corresponding homogeneous equation and a of the original inhomogeneous equation particular solution, which is sought in the same functional form as and the right side of the inhomogeneous equation.

12.5.11. The roots of characteristic equation for the general solution of a homogeneous equation depending on real or the imaginary form of the roots, when determining the function of the angular velocity of rotation of ship moving by inertia without the work of propulsors and controls, the exponential and trigonometric forms of solution and an assessment of ship motion stability, depended on the characteristic equation roots number form: the real or imaginary.

12.5.12. An exponential form of general solution of the homogeneous equation, if the root of the characteristic equation is real number, when the added mass during the ship motion in water with acceleration along the abscissa is less than the added mass of water when moving with acceleration along the ordinate axis, the unstable motion of the ship and its tendency to turn across the motion direction and then move towards the side, performing the self–oscillations.

12.5.13. Trigonometric form of general solution of the homogeneous equation and the self–oscillation of elements of ship motion if root of the characteristic equation is imaginary number, if the added mass when ship moving in water with acceleration along

abscissa is greater than the added mass when moving with acceleration along the ordinate axis and self–oscillations of ship motion elements.

12.5.14. Ship motion indifferent stability with zero root of the characteristic equation and continuation is of ship rotation with initial angular velocity relative to vertical axis (the applicate).

12.5.15. The vicinity of the bifurcation line corresponding to zero values of root of the characteristic equation, which delimits the prerequisites when the added mass during the ship motion in water with acceleration along the ordinate axis is less than the added mass during the ship acceleration along the abscissa axis, and the root of the characteristic equation is imaginary number, and vice versa, when the added mass is greater when ship moves with acceleration along the ordinate axis, and root of the characteristic equation is real number.

12.5.16. Attenuation of amplitudes of elements of ship motion (the angular velocity of turn and transverse translational velocity), when the characteristic equation input factor – the ship translational velocity decreases along the abscissa axis when the ship moves by inertia, that is when the kinetic energy is spent gradually by ship inertia motion.

12.6. Self–oscillations of a moving ship without control

The purpose of research of ship motion without control is corresponded with the study of moving ship own properties as the mechanical system, as well as with improvement of the control systems – the algorithms for ship keeping on course or the maneuvering, including the nonlinear algorithms with sensors for ship motion elements monitoring and the feedback use.

Movement in form of the self–oscillations of ship without control occurs at imaginary roots of the characteristic equation (item 12.5), when: $p^2 < 0$, if the added mass during the ship motion in water with acceleration along the ordinate axis Gy is less than the added mass during the ship acceleration along the abscissa axis Gx: $\mu_2 < \mu_1$.

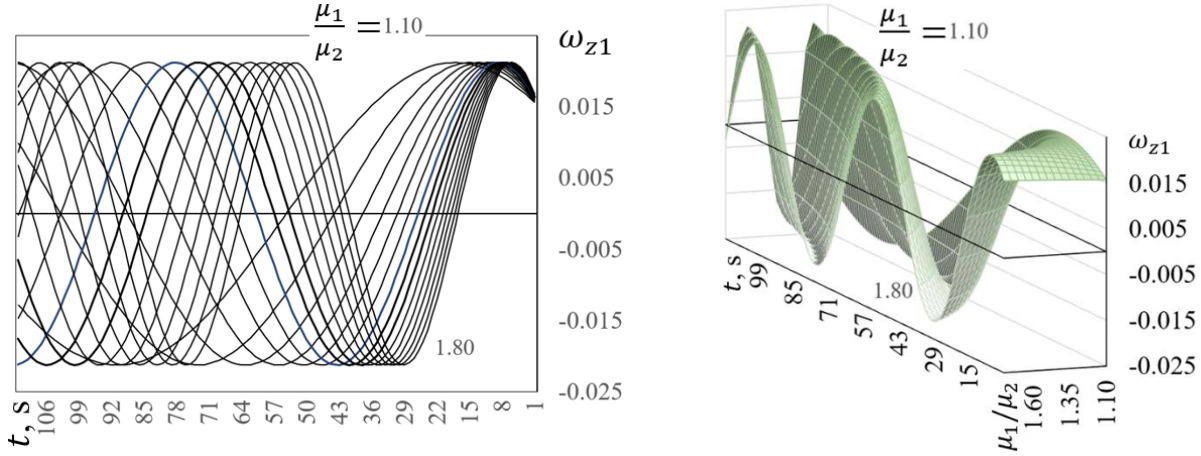
The availability of added masses ratio: $\mu_1/\mu_2 > 1$ methodically corresponds with change the axes of associated coordinate system: Gx to Gy (for the left coordinate system Gx – to port side, Gy – to the bow) in accordance with change of ship motion direction, since for real roots of the characteristic equation, when: $p^2 > 0$ and $\mu_2 > \mu_1$ the ship motion without control is unstable on course and goes into stable motion in the form of self–oscillations in the transverse direction, while: $p^2 < 0$, $\mu_2 < \mu_1$, if the direction of the abscissa axis is aligned with the direction of the ship motion in this mode.

The assumption that the velocity v_x of the translational motion of the ship along the abscissa axis Gx is initially used to find the elements of ship motion (ω_{z1}, v_{y1}, CR_1 and y_1) without control in the self–oscillation mode, which are solutions to the homogeneous

equation (item 12.4, with zero control efforts: $N_y = 0$ and $M_z = 0$), corresponding to the original system equations in projections onto the axes associated with the ship: the forces on the ordinate axis Gy and the moments on applicate axis Gz. This assumption is of a conventional nature and the forward velocity of ship v_x along the original direction when the ship is deviating, unstable on course, without control could be considered constant for a short time, or that the ship is towed at a constant velocity v_x, and the fastening of a long towing cable is made at a point located close to the ship center of gravity, so that the ship of large breadth or attached to the cable in area of the center of gravity is anchored in current conditions, as well as when the ship is drifting under the influence of wind, etc.

In the coordinate system $Gxyz$, associated with the ship, the following elements of ship motion without control are determined, which are the solutions of homogeneous equation: the angular velocity of the ship rotation relative to the applicate axis ω_{z1} and the translational velocity of the ship transverse motion v_{y1}.

To find the solution – the turning velocity of ship motion without control in the self–oscillation mode, in graphical form (Fig. 12.6.1 a and b), determined in the Microsoft EXCEL program, depending on the variables: t and $\frac{\mu_1}{\mu_2}$ the initial angular velocity is taken equal to: $\omega_{z0} = 0.015$ 1/s and the initial angular acceleration $\dot{\omega}_{z0}$ are in proportion to the modulus of the root of the characteristic equation p: $\dot{\omega}_{z0} = p \times \omega_{z0}$, what is possible for a ship moving without control: $\omega_{z1} = \omega_{z0} \times \cos(p \times t) + (\dot{\omega}_{z0}/p) \times \sin(p \times t)$, for imaginary roots of the characteristic equation: $p^2 = v_x^2 \times \frac{(\mu_2 - \mu_1) \times (\Delta + \mu_1)}{(I_z + \mu_6) \times (\Delta + \mu_2)} < 0$.



(a) (b)

Fig. 12.6.1

Two graphical forms of one functional dependence of angular velocity of ship turn: ω_{z1} on time parameter t and on ratio of added masses $\frac{\mu_1}{\mu_2}$ with imaginary roots of the

characteristic equation: $p^2 < 0$ are given together for clarity of the function graph of angular velocity of ship turn and it arguments relationship.

Lines on the graphs correspond with values of added masses: the mass μ_1 – with acceleration of ship along the abscissa axis Gx and the mass μ_2 when the ship moves with acceleration along the ordinate axis Gy, that is the ratio of added masses μ_1/μ_2 with an initial value of 1.10 and in increments of 0.05 to 1.80 and values of modulus of imaginary root p of the characteristic equation: 0.0433, 0.0534, 0.0621, 0.0700, 0.0772, 0.0840, 0.0904, 0.0965, 0.1024, 0.1081, 0.1137, 0.1191, 0.1243, 0.1295 and 0.1346. Time period of ship motion for determining the graph is taken from 0 to 112 s with an interval of one.

When transforming a system of two equations in the projections of forces on the ordinate axis Gy and moments on the axis Gz of applicate to one equation (item 12.5): $\dot{v}_y = \frac{1}{\Delta+\mu_2} \times (N_y - (\Delta + \mu_1) \times v_x \times \omega_z)$, as part of solution, which is: $v_y = v_{y1} + v_{y2}$ the general solution v_{y1} of the corresponding homogeneous equation (for a moving ship without control, when the control efforts – the turning moment and transverse force are equal to zero: $M_z = 0$ and $N_y = 0$), is the integral of appropriate derivative:

$\dot{v}_{y1} = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z1}$.

The transverse translational velocity of ship motion without control with imaginary roots of the characteristic equation: $p^2 < 0$, is determined by solving the equation:

$$v_{y1} = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \int_0^t \omega_{z1} \times dt + v_{y0} =$$

$$= v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p} \times \sin(p \times t) - \frac{1}{p^2} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \cos(p \times t)\right) + C_1 + v_{y0}.$$

When set of the initial conditions ($v_{y1} = v_{y0}$ at $t = 0$) in the considered formula:

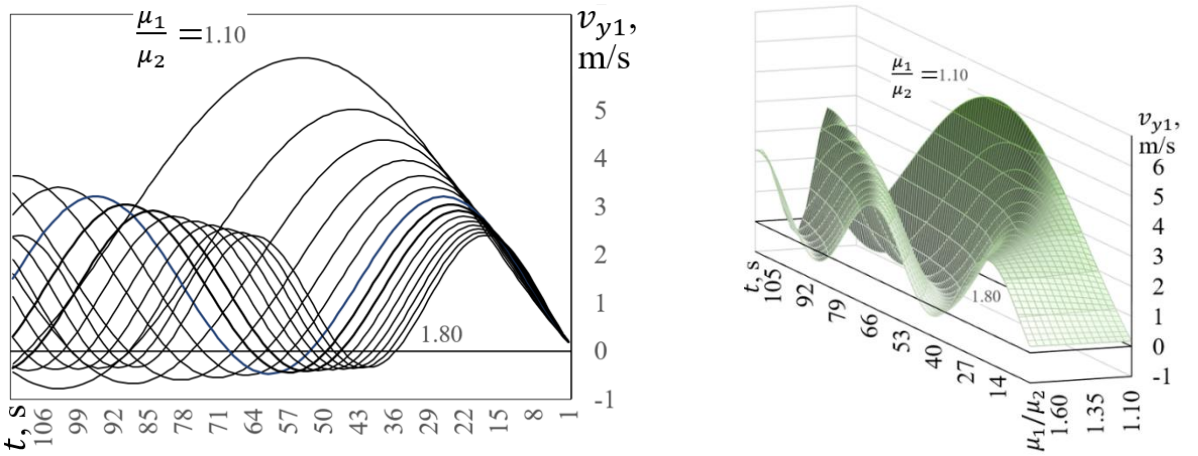
$$v_{y0} = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p} \times \sin(0) - \frac{1}{p^2} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \cos(0)\right) + C_1 + v_{y0}$$

one can find the integration constant: $C_1 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \frac{\dot{\omega}_{z0}}{p^2}$.

With initial drift angle of the ship without control, which in considered issue is taken equal to 0.574 degrees: $\beta_{D0} = 0.01$ radian and assuming constant velocity of the ship along the x–axis: v_x =7.0 m/s the initial transverse velocity is:

$v_{y0} = v_x \times \beta_{D0} = 7.0 \times 0.01 = 0.07$ m/s.

For specified data the calculations are performed in the Microsoft EXCEL program and the results – the translational transverse velocity of ship motion in the self–oscillation mode with imaginary roots of the characteristic equation: $p^2 < 0$ at $\mu_2 < \mu_1$ are indicated in the form of graphical dependence on the variables: t and $\frac{\mu_1}{\mu_2}$ (Fig. 12.6.2 a and b).



(a) (b)

Fig. 12.6.2

In the fixed coordinate system $0\xi\eta\zeta$ the following elements of ship motion without control are determined (at: $p^2 < 0$), which are solutions of the homogeneous equation: course CR_1 and the lateral movement of the ship y_1.

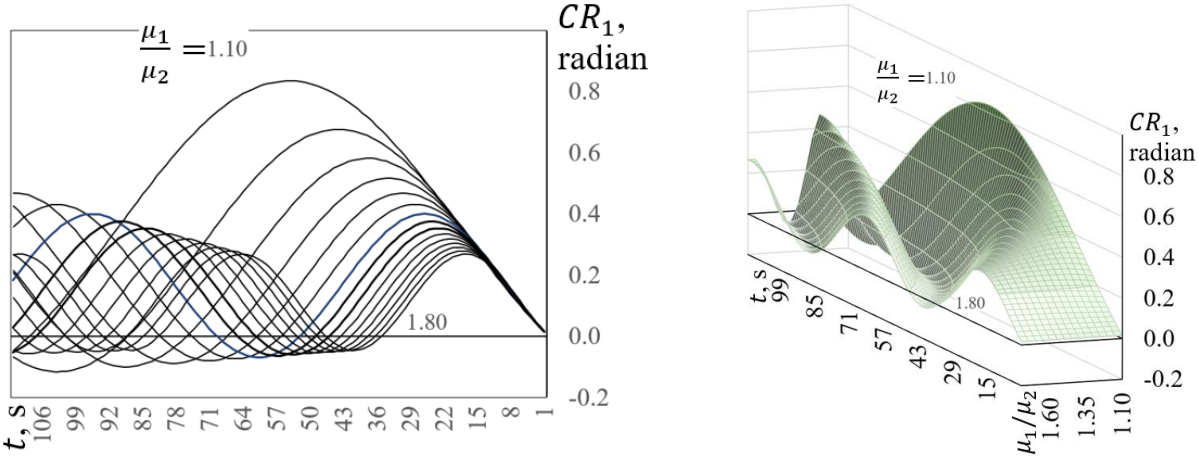
Course of ship is the integral function of the angular velocity of its turn ω_{z1}:

$CR_1 = \int_0^t \omega_{z1} \times dt + CR_0$, that is:

$CR_1 = \omega_{z0} \times \frac{1}{p} \times \sin(p \times t) - \frac{\dot{\omega}_{z0}}{p^2} \times \cos(p \times t) + C_2 + CR_0$, and the integration constant for the initial conditions: $CR_1 = CR_0$ at $t = 0$ is equal to:

$CR_0 = \omega_{z0} \times \frac{1}{p} \times \sin(p \times 0) - \frac{\dot{\omega}_{z0}}{p^2} \times \cos(p \times 0) + C_2 + CR_0$ and $C_2 = \frac{\dot{\omega}_{z0}}{p^2}$.

If one sets the initial course of the ship equal to zero: $CR_0 = 0.0$ in the fixed coordinate system $0\xi\eta\zeta$, then the course of ship motion without control in the self–oscillation mode with imaginary roots of the characteristic equation: $p^2 < 0$, at $\mu_2 < \mu_1$ is indicated in the form of a graphical dependence on the variables: t and $\frac{\mu_1}{\mu_2}$ (Fig. 12.6.3 a and b).



(a) (b)

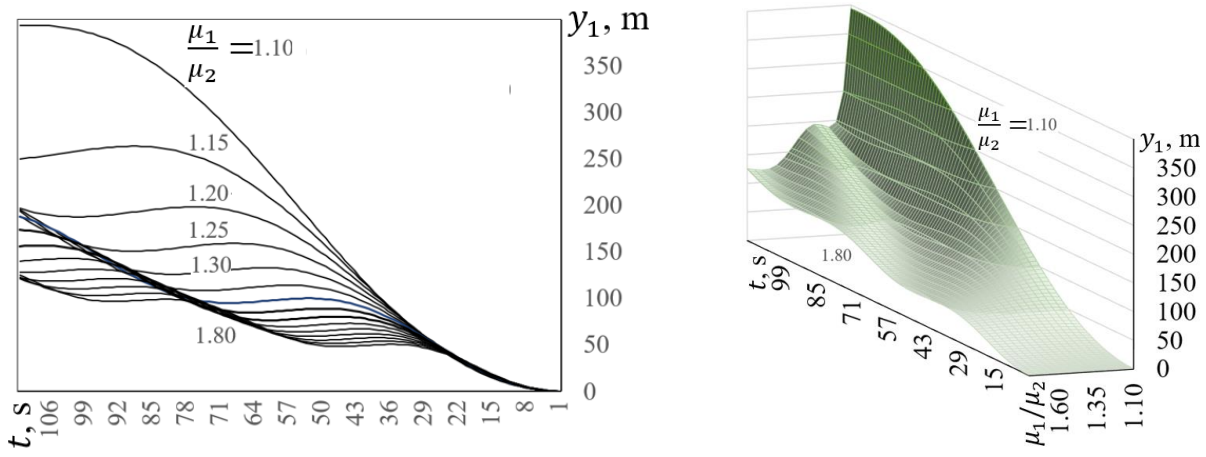
Fig. 12.6.3

The transverse displacement y_1 of ship, determined in a fixed coordinate system $0\xi\eta\zeta$, is an integral function with respect to the ship transverse translational velocity v_{y1}: $y_1 = \int_0^t v_{y1} \times dt + y_0$, that is for ship moving in the mode of self–oscillation without control with imaginary roots of the characteristic equation: $p^2 < 0$ with $\mu_2 < \mu_1$ the transverse displacement can be determined, as:

$$y_1 = -v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p^2} \times \cos(p \times t) + \frac{1}{p^3} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \sin(p \times t)\right) +$$
$$+(C_1 + v_{y0}) \times t + C_3 + y_0.$$

By setting the initial conditions: $y_1 = y_0$ at $t = 0$ one can determine the integration constant: $y_0 = -v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p^2} \times \cos(p \times 0) + \frac{1}{p^3} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times \sin(p \times 0)\right) +$
$+\left(C_1 + v_{y0}\right) \times 0 + C_3 + y_0$ and $C_3 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \frac{1}{p^2}$.

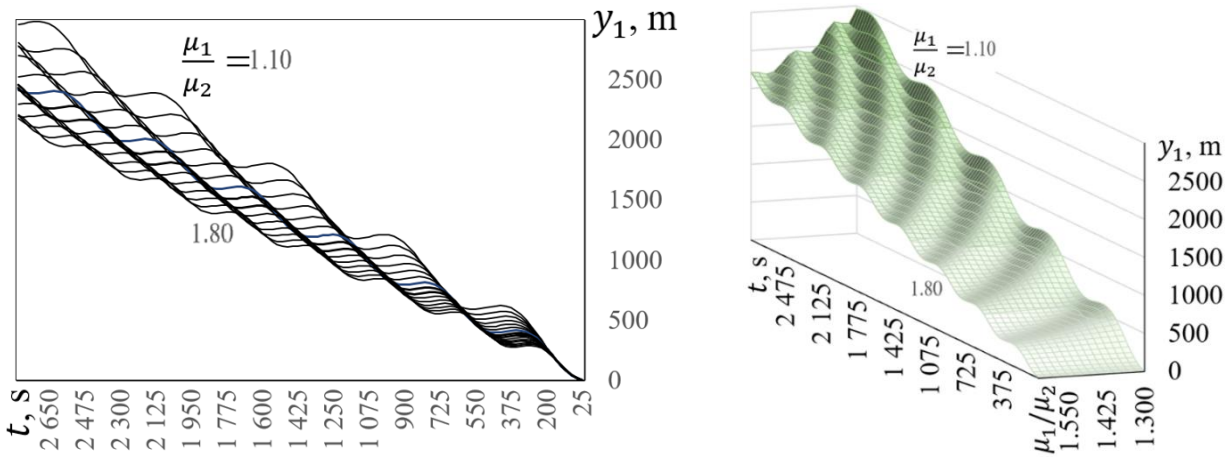
If one sets the initial lateral displacement of ship moving without control equal to zero: $y_0 = 0$, then the lateral displacement of ship moving without control in the self–oscillation mode with imaginary roots of the characteristic equation: $p^2 < 0$, $\mu_2 < \mu_1$ is indicated in form of the graphical dependence on variables: t and $\frac{\mu_1}{\mu_2}$ (Fig. 12.6.4 a and b).



(a) (b)

Fig. 12.6.4

A longer process during 2800 s of ship motion without control, for example, the drift of ship with the velocity: v_x =1 m/s for a slightly narrower interval of variation of the parameter of added masses μ_1/μ_2 is shown in the form of graphical dependence on the variables: t and μ_1/μ_2 (Fig. 12.6.5 a and b).



(a) (b)

Fig. 12.6.5

The graphical functional surface shows that in addition to self–oscillations, the initial conditions (ω_{z0}, $\dot{\omega}_{z0}$, v_{y0}, CR_0 and y_0) impact on ship motion without control throughout the entire time, if the ship motion without control does not fade, and for this it would be necessary to restore the energy spent on damping the movement due to the viscosity of water and the dissipation of energy in the water space and on its surface, in particular, the dissipation of waves energy generated by the ship, diverging from the ship.

This conclusion also follows from solution of the homogeneous equation of motion of ship without control, when the control efforts are zeroes – the turning moment and the transverse force: $M_z = 0$ and $N_y = 0$, and if root of the characteristic equation is equal to zero: $p = 0$ at $\mu_2 = \mu_1$ the ship without control has the «indifferent» motion stability and moves with preservation of the initial elements of motion (ω_{z0} and v_{y0}) in the coordinate system, associated with the ship $Gxyz$: $v_{y1} = v_{y0}$ and $\omega_{z1} = \omega_{z0}$, the angular acceleration of ship depends on angular velocity of ship motion without control, at zero root the characteristic equation also has zero value: $\dot{\omega}_{z1} = \dot{\omega}_{z0} = 0$ when control efforts are zeroes.

Thus, influence of the initial elements of movement throughout its entire duration is preserved under any modes of ship motion without control, both with a zero value of the root of the characteristic equation (when: $\mu_2 = \mu_1$), and with an imaginary root (at: $\mu_2 < \mu_1$), and with a real number root of the characteristic equation ($\mu_2 > \mu_1$) it is impossible to establish the influence of the initial elements of ship motion if it is of sufficient duration (with a duration greater than the fraction of the parameter: $2 \times \pi/p$), because during unstable motion its elements increase indefinitely in magnitudes and the ship goes into the above–mentioned mode of self–oscillatory motion ($\mu_2 < \mu_1$).

At a zero value of root of the characteristic equation, the attenuation of the initial elements of motion (ω_{z0} and v_{y0}), determined in the $Gxyz$ coordinate system, associated with the ship, occurs during the translational motion attenuates, if the ship energy spent during movement is not restored.

For real roots of the characteristic equation: $p^2 > 0$, $\mu_1/\mu_2 < 1$ the results of integrating a homogeneous equation describing ship motion without control, when the control efforts – turning moment of the ship and the transverse force are equal to zero: $M_z = 0$ and $N_y = 0$, considering the initial conditions (ω_{z0}, $\dot{\omega}_{z0}$, v_{y0}, CR_0 and y_0 at: $t = 0$) the elements of the ship motion (ω_{z1}, v_{y1}, CR_1 and y_1) can be determined by integrating the equations as follows.

In the coordinate system $Gxyz$, associated with ship, the angular velocity ω_{z1} of ship rotation relative to the applicate axis Gz without control as functional dependence on time t and on initial conditions (ω_{z0} and $\dot{\omega}_{z0}$) is equal:

$\omega_{z1} = ((\omega_{z0} + \dot{\omega}_{z0}/p) \times e^{p \times t} + (\omega_{z0} - \dot{\omega}_{z0}/p) \times e^{-p \times t})/2,$

or $\omega_{z1} = \omega_{z0} \times ch(p \times t) + \frac{1}{p} \times \dot{\omega}_{z0} \times sh(p \times t),$

and the translational transverse velocity v_{y1} along the ordinate axis Gy is defined as:

$\dot{v}_{y1} = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times (ch(p \times t) + \frac{1}{p} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times sh(p \times t)),$

$v_{y1} = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p} \times sh(p \times t) + \frac{1}{p^2} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times ch(p \times t)\right) + C_4 + v_{y0},$

when set initial conditions ($v_{y1} = v_{y0}$ at $t = 0$) into the formula, the integration constant is determined, as: $C_4 = -v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \frac{\dot{\omega}_{z0}}{p^2}$.

In a fixed coordinate system $0\xi\eta\zeta$ the ship course as the functional dependence on time t and on initial conditions can be determined by integrating the angular velocity ω_{z1} relative to the applicate axis, as: $CR_1 = \int_0^t \omega_{z1} \times dt + CR_0,$

that is: $CR_1 = \omega_{z0} \times \frac{1}{t} \times sh(p \times t) + \frac{\dot{\omega}_{z0}}{p^2} \times ch(p \times t) + C_5 + CR_0$ for real roots of the characteristic equation: $p^2 > 0$, $\mu_1/\mu_2 < 1$, the integration constant is sought by set of the initial conditions ($CR_1 = CR_0$ with $t = 0$):

$CR_0 = \omega_{z0} \times \frac{1}{t} \times sh(p \times 0) + \frac{\dot{\omega}_{z0}}{p^2} \times ch(p \times 0) + C_5 + CR_0$ and $C_5 = -\frac{\dot{\omega}_{z0}}{p^2}$.

The transverse displacement of the ship as a functional dependence on time t and on initial conditions can be determined by integrating the angular velocity relative to the applicate axis, as: $y_1 = \int_0^t v_{y1} \times dt + C_6 + y_0,$

that is: $y_1 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p^2} \times ch(p \times t) + \frac{1}{p^3} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times sh(p \times t)\right) +$

$+(C_4 + v_{y0}) \times t + C_6 + y_0,$

the integration constant can be found by set of the initial conditions ($y_1 = y_0$ at $t = 0$), as: $y_0 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \omega_{z0} \times \left(\frac{1}{p^2} \times ch(p \times 0) + \frac{1}{p^3} \times \frac{\dot{\omega}_{z0}}{\omega_{z0}} \times sh(p \times 0)\right) +$

$+(C_4 + v_{y0}) \times 0 + C_6$ and $C_6 = y_0 - v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \frac{\omega_{z0}}{p^2}$.

The got functions of the elements of motion of the ship (ω_{z1}, v_{y1}, CR_1 and y_1) with real roots of the characteristic equation: $p^2 > 0$ at $\mu_1/\mu_2 < 1$ indicate that the ship motion without control is unstable, and values of motion elements increase indefinitely (in particular, this is indicated on the graphical dependence of angular velocity of ship turn Fig. 12.5.2, including, among other things, the solutions for real roots of the characteristic equation: $p^2 > 0$ at $\mu_1/\mu_2 > 1$), and the ship, deviating from the original course, tends to lie on a trajectory with self–oscillations, which is characteristic for imaginary roots of the characteristic equation: $p^2 < 0$, $\mu_1/\mu_2 < 1$. Therefore, this item does not provide graphical functional dependences of ship motion elements on the variables: t and μ_1/μ_2 for real roots of the characteristic equation.

It would be mentioned that these properties of ship without control (which are properties of a ship physical model) characterize its motion as mechanical system and to a lesser extent affect the patterns of water motion, considered in the hydromechanics, and rather analogical patterns are characteristic, for example, the vehicles moving on slippery surface of slope or by inertia, which is clearly noticeable when braking under these conditions – there is a vehicle tendency to turn across the original direc-tion of motion and skid on a slippery surface.

Questions and comments:

12.6.1. Study of ship motion elements as the mechanical system without control by use of the homogeneous equations, aimed at properties of a moving ship research and improving control systems – the algorithms for keeping a ship on course or maneuvering.

12.6.2. The motion in the form of self–oscillations of a ship without control with imaginary roots of the characteristic equation, when the added mass during the ship motion in water with acceleration along the ordinate axis is less than the added mass during acceleration of the ship along the longitudinal abscissa axis.

12.6.3. Determining the functions of the ship rotation velocity relative the vertical applicate axis and the translational velocity of the ship transverse motion in the coordinate system, associated with the ship, and finding the ship without control course of transverse movement in a fixed coordinate system when solving homogeneous equations of ship motion, if the control efforts – the turning moment of ship and the transverse force are equal to zeroes, and the integration constants of homogeneous equations are determined by the initial values of the ship elements of motion.

12.6.4. The continuous influence of initial conditions on ship motion in the mode of self–oscillation without control, if this movement does not attenuate within the framework of the abstract assumption of the absence of resistance to movement for a short

duration of movement or within the framework of the assumption that the energy spent on damping is restored as it is spent.

12.6.5. Characteristic of self–oscillations of ship motion without control, which inertia in water in the longitudinal direction is greater than in the transverse one, or with other ratio of inertia in these directions, the ship as the mechanical system unstable motion with an unlimited increase of the motion elements magnitudes (the angular velocity and transverse translational velocity) in the coordinate system, associated with the ship and further transition of motion mode from the unstable to self–oscillatory.

12.7. The numerical integration of the equations of ship motion

Numerical integration is an effective and practically appropriate way to solve the equations of ship motion. In addition, an analytical solution of some equations, for example, the nonlinear ones cannot be found, but they can be solved numerically in the intervals of the arguments in which these solutions exist.

There are other methods of step–by–step numerical integration that can be used to sufficiently approximate the value of the integral of a function or the value of a function from its derivative. Among these methods are known: the Euler's method, in which the value of the antiderivative for each step is equal to the sum of its value for the previous step and the increment for this step, equal to the product of the value of the derivative for the previous step by magnitude of this step (the argument increment), the methods of rectangles and trapezoids, which are methodologically close to Euler's one, with the difference that the value of the derivative can be taken as the average within a step or as a generalized value in one or another ratio of derivatives within a step, the method of Runge–Kutta, which differs from the Euler's method in that within a step for certain values of the argument several derivatives are sought, which are used to determine the increment of the integral, and the other methods are available.

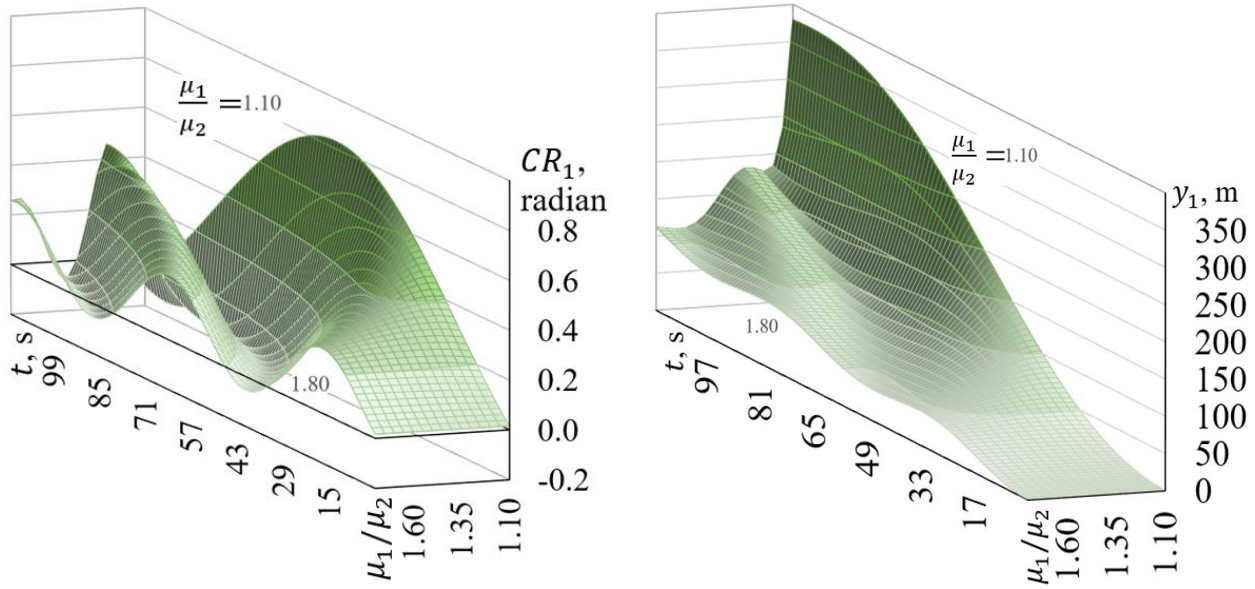
Exercise. It is necessary to determine by numerical integration in a fixed coordinate system $0\xi\eta\zeta$ of course CR_1 of the ship, moving without control (at zeroes control efforts – the moment and transverse force: $M_z = 0$ and $N_y = 0$), and its lateral displacement y_1 under given initial conditions: $\omega_{z0} = 0.015$ 1/s, $\dot{\omega}_{z0} = \omega_{z0} \times p$ 1/s^2, $v_{y0} = 0.07$ m/s, $y_0 = 0$ and $CR_0 = 0$ at $t = 0$ and for imaginary roots of the characteristic equation: $p^2 < 0$ at $\mu_2 < \mu_1$, when the angular velocity of the ship rotation relative to the applicate axis Gz is equal: $\omega_{z1} = \omega_{z0} \times \cos(p \times t) + (\dot{\omega}_{z0}/p) \times \sin(p \times t)$.

The course of ship is in integral dependence on velocity of the ship rotation ω_{z1} relative to the applicate axis: $CR_1 = \int_0^t \omega_{z1} \times dt + CR_0$, and the lateral displacement is in integral dependence on the transverse velocity v_{y1}:

$y_1 = \int_0^t v_{y1} \times dt + y_0 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \int_0^t \int_0^t \omega_{z1} \times dt^2 + v_{y0} \times t + y_0.$

Numerical integration of the function f' by the Euler's method and finding the antiderivative f is performed with uniform integration steps Δx in the sequence:
$x_i = x_{i-1} + \Delta x, f(x_i) = f_{i-1} + \Delta x \times f'(x_{i-1}), i = 1, 2, 3, \dots, n.$

By this sequence: $t_i = t_{i-1} + \Delta t$, $CR_1(t_i) = CR_{1_{i-1}} + \Delta t \times \omega_{z1}(t_{i-1})$ in the Microsoft EXCEL by numerical integration with time parameter step: Δt=1 s in the interval from 0 to 112 s the dependence of the ship course CR_1, moving without control, on the variables: t and μ_1/μ_2 was found (Fig. 12.7.1 a). In this case, for numerical integration, an analytical determination of the antiderivative of function of the angular velocity ω_{z1} of ship rotation relative to applicate axis Gz is not performed.



(a) (b)

Fig. 12.7.1

To determine using the Euler's method of the double integral finding: $\int_0^t \int_0^t \omega_{z1} \times dt^2$ in the formula of ship lateral displacement y_1 one can use the integrand in repeated integral: $\int_0^t \omega_{z1} \times dt$ to determine through the function of ship course CR_1, found above by numerical integration, as: $\int_0^t \omega_{z1} \times dt = CR_1 - CR_0$, then the lateral displacement of the ship moving without control is equal to:
$y_1 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \int_0^t (CR_1 - CR_0) \times dt + v_{y0} \times t + y_0.$

If one takes into account the given initial conditions, then the lateral displacement, using the calculated heading CR_1, one can find in the form of a graphical functional surface (Fig. 12.7.1 b) by numerical integration using the formula:
$y_1 = v_x \times \frac{\Delta+\mu_1}{\Delta+\mu_2} \times \int_0^t CR_1 \times dt + 0.07 \times t.$

In this case, the analytical determination of the antiderivative of the function of the angular velocity of rotation of ship ω_{z1} relative to the applicate axis Gz is also not performed: $\int_0^t \int_0^t \omega_{z1} \times dt^2$.

In order to judge the accuracy of the numerical determination of the integral and the double integral by the Euler's method when finding the elements of ship motion without control (CR_1 and y_1) one can compare the obtained functional dependencies (Fig. 12.7.1 a and b) with the dependencies of these elements of the ship motion got by the analytical determination of the integrals (item 12.6): the course of ship CR_1 (Fig. 12.6.3 b) and its lateral displacement y_1 (Fig. 12.6.4 b). It is visible that the graphical dependencies in the form of functional surfaces got by numerical integration using the Euler's method (with an integration step of 1 s in the range from 0 to 112 s) and found by the analytical definition of antiderivatives are indistinguishable in detail, which indicates the high accuracy of the numerical integration of the angular velocity function ω_{z1} of rotation ship without control relative to the applicate axis Gz when finding the elements of ship motion (CR_1 and y_1).

Among the common methods of numerical integration, when and solving differential equations and in the other issues, the Runge–Kutta method is also known, which is considered more accurate than, for example, the integration by the Euler's method. When numerical integration by the Runge–Kutta one is performed, an analytical determination of antiderivatives is also not required, and the numerical values of the antiderivatives are found by computational operations with the derivatives themselves. When integrating using the Runge–Kutta method, each calculation step is divided in half and, when the result of integration is found, the additional intermediate calculations are performed at each step.

The required function f to apply numerical integration by the Runge–Kutta method must be presented in the form of the derivative $f'(x, f)$ and the initial value of the function has to be specified: $f(x_0) = f_0$, then the remaining values of the function f are calculated by steps Δx, as: $f_i = f_{i-1} + \frac{\Delta x}{6} \times (K_1 + 2 \times K_2 + 2 \times K_3 + K_4)$, where the values of the terms: K_1, K_2, K_3 and K_4 are equal to:

$K_1 = f'(x_{i-1}, f_{i-1})$, $K_2 = f'(x_{i-1} + \frac{\Delta x}{2}, f_{i-1} + \frac{\Delta x}{2} \times K_1)$,

$K_3 = f'(x_{i-1} + \frac{\Delta x}{2}, f_{i-1} + \frac{\Delta x}{2} \times K_2)$ and $K_4 = f'(x_{i-1} + \Delta x, f_{i-1} + \Delta x \times K_3)$.

For example, the time derivative of integral ship course: $CR_1 = \int_0^t \omega_{z1} \times dt + CR_0$, moving without control (with zero control efforts – the moment and transverse force: $M_z = 0$ and $N_y = 0$) is equal: $CR_1' = \omega_{z1}$. The zero value of the ship's heading corresponds to the initial conditions: $CR_1(t_0) = CR_0$. Since the angular velocity ω_{z1} of the ship rotation relative to applicate axis Gz, which is the derivative of the ship course CR_1 sought

according to variable time parameter t does not depend on the ship course as on variable, then the remaining values CR_1 are calculated as follows:

$CR_{1_i} = CR_{1_{i-1}} + \frac{\Delta t}{6} \times (K_1 + 2 \times K_2 + 2 \times K_3 + K_4)$,

where: $K_1 = \omega_{z1}(t_{i-1})$, $K_2 = K_3 = \omega_{z1}(t_{i-1} + \Delta t/2)$ and $K_4 = \omega_{z1}(t_i)$.

The transformations of the formula for calculating the integral using the Runge–Kutta method show that its higher accuracy than the Euler's method in the problem of determining the elements of ship motion is due to the fact that for each step in the numerical integration using the Runge–Kutta method the derivative value is used as at the beginning and at the end of each step with a weight coefficient of $\frac{1}{6}$ for these values in the final increment of the integral, as well as the value of the derivative in the middle of the step with a weight coefficient of $\frac{2}{3}$. The numerical integration step when using this method becomes, in essence, two times smaller, and the number of derivative calculations correspondingly doubles.

It would be mentioned that the accuracy of calculations by other methods, for example, by the Euler's method, also increases (and the error decreases), and the number of calculations also increases if the integration step is reduced. Apparently the simple numerical integration algorithms can be given the preference for practical purposes, when solving problems of mechanics and in numerical integration in cases where the use of a simpler computational method provides sufficient accuracy in finding the value of integral without its analytical definition, and if computing capabilities allow one to obtain any necessary accuracy of the value sought by rational choice of the integration step magnitude.

Questions and comments:

12.7.1. Numerical integration using the Euler's method, in which value of the antiderivative for each step is equal to the sum of its value for previous step and the increment for this step, equal to product of the value of derivative for the previous step by the value of this step.

12.7.2. The Runge–Kutta method for numerical integration, which differs from the Euler's method in that within a step, for certain values of the argument, the several derivatives are searched for, which are used to determine the increment of the integral.

12.7.3. Integral dependence of ship course on its turning velocity and the integral dependence of the lateral displacement of a moving ship on its transverse velocity.

12.7.4. The judgment about accuracy of numerical integration by the Euler's method based on a comparison of graphical functional surfaces got by the Euler's method and those obtained by the analytical definition of integral.

12.7.5. The form of the derivative which is required to use in the Runge–Kutta method and the arguments for the derivative. Rules for determining increment terms during numerical integration using the Runge–Kutta method.

12.8. Numerical solution of ship rolling nonlinear equations on regular waves

The methods of numerical integration are of great importance in solving nonlinear differential equations, which analytical solution of is unknown, except for certain types of nonlinear equations, and the problem of analytical insolvability of nonlinear equations, which describe, in particular, the processes in many issues of mechanics, is associated not only with the need to improve the analytical methods, but also with some fundamental issues of mathematical analysis, which include the possible ambiguity of the solution to a nonlinear differential equation.

To make it possible to solve analytically an inhomogeneous differential equation in accordance with the Augustin Louis Cauchy problem in the form of the sum of a general solution of a linear homogeneous equation, the right side of which is equal to zero, and a particular solution of an inhomogeneous equation, the hypothesis of linearization of the terms of the equation with respect to the desired function, its derivatives and argument can be applied. Linear (the linearized) inhomogeneous equation of ship rolling in regular waves has the form: $\ddot{\theta} + 2 \times \nu_4 \times \dot{\theta} + \omega_4^2 \times \theta = w_4 \times \omega_4^2 \times \alpha_W \times \cos(\sigma \times t)$.

The analytical solution of the linear inhomogeneous roll equation with obtaining the frequency characteristics of ship rolling (item 12.3) and the further use of frequency characteristics in determining the spectrum of roll in irregular waves are discussed above (item 12.4). Solutions of linear inhomogeneous equations of ship rolling on regular waves can be obtained in the form of rocking frequency characteristics, including the amplitude–frequency one, which are the functions, that are the unambiguous mathematical correspondences, in which each value of the argument (the frequency σ of the disturbing force or moment) corresponds only one value of the function: the ratio of the amplitudes of vertical heaving ζ_M to the wave amplitude r_W, or the roll θ_M to angle of its wave slope α_W found analytically by the complex amplitudes method and shown in the form of ship vertical heaving graphs (Fig. 12.3.4) and it roll (Fig. 12.3.3 a).

At the same time, solving a nonlinear equation can lead to ambiguity in the amplitude–frequency characteristic of rocking, as the example, when the value of the argument (the frequency σ of the disturbing effort) corresponds to two or three values of the function that is the solution to the nonlinear differential equation, which can be indicated on a schematic graph of the amplitude–frequency characteristic of ship rolling or heave, got by

solving the corresponding nonlinear equation (Fig. 12.8.1), if this would be possible to define.

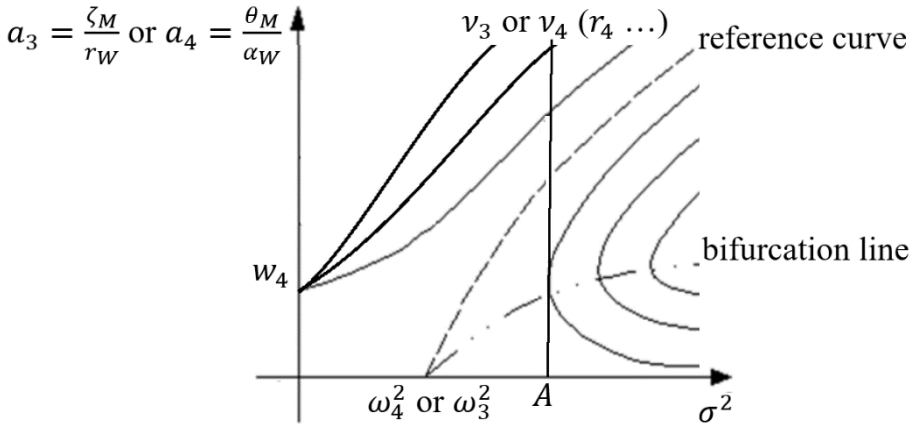


Fig. 12.8.1

The schematic graph shows that on considered amplitude–frequency characteristic of rocking, determined corresponding the nonlinear equation, along the x–axis to the right of point of the natural frequency of ship rocking on calm water (more precisely, the point of square of the natural frequency: ω_4^2 or ω_3^2) there is an ambiguity of solution of ship rocking nonlinear equation. For each value of the damping index (in the form of a decrement of rocking attenuation): ν_3 or ν_4 (that is r_3 or r_4 – in nonlinear rocking equations) along the abscissa axis on the graph σ^2 (the frequency square of the disturbing force or moment), a bifurcation boundary A, appears, to the right of which the nonlinear rocking equation has three solutions, and to the left – the one. That is, there is no single analytical solution to a nonlinear differential equation, but a solution, if it is required in the form of an amplitude–frequency characteristic, can be sought both by sections of the values of the argument and the function.

Since the solution to a differential equation must be a function, when substitute derivatives into the equation it must be transformed into an equality, and the function is a unique mapping of a set of variables into a set of its values, then the violation of the uniqueness of this mapping and violation of its continuity may run counter to mathematical analysis. Therefore, for nonlinear differential equations that describe processes in mechanical systems, solutions would be sought by methods of numerical integration in the areas of values of both the argument and the function in which the solutions to nonlinear equations exist.

Main reasons for the nonlinearity of the ship rocking equations are usually considered to be: the quadratic dependence of damping forces or moments on the translational or angular velocities of ship in water and the transcendental (non-algebraic) form of the dependences of forces or moments (the restoring or disturbing ones, which deviate ship from the equilibrium position), the form of which depends on characteristics of shape of the ship's hull, on translational or angular movements, for example, the nonlinear

dependence of the heeling moment acting on the ship on waves, which, like the dependence of the righting moment on the roll angle, could be assigned by the ship sustainability diagram. A methodological cause of the nonlinearity of the equations of controlled motion of a ship on water, in addition to the indicated nonlinearity of the dependencies of forces or moments acting on the ship, can also be considered the action of feedback, as the element of ship motion control system.

Dependence of the righting moment on heel angle θ, the ship displacement (its mass) Δ and the weight ($\Delta \times g$), on the initial transverse metacentric height h, which is the vertical distance between the center of ship gravity and the center of the curve along which the center of ship underwater volume (the point of application of the Archimedes' force) moves when heeling, this could be taken in a general approximation in the form of a sinusoid: $\Delta \times g \times h \times \sin(\theta)$.

Considering that and the similar assumption regarding the disturbing wave moment, that is the heeling moment from the action of waves, which is equal to:
$\Delta \times g \times h \times w_4 \times \sin(\alpha_W \times \cos(\sigma \times t))$, and also, taking into account the quadratic dependence of the damping moment: $\frac{3}{2 \times w_4 \times \alpha_W} \times \frac{\lambda_4}{(I_x + \mu_4)} \times \dot{\theta}^2$ (if one does not touch the derivation of the form of factor with the square of angular velocity of ship rotation relative to the abscissa axis Gx when roll), the nonlinear equation of ship rolling in regular waves, represented, like the roll of ship, by trigonometric functions having real number values, can have the form: $\ddot{\theta} + \frac{\dot{\theta}}{\sqrt{\dot{\theta}^2}} \times r_4 \times \dot{\theta}^2 + \omega_4^2 \times \sin(\theta) = w_4 \times \omega_4^2 \times \sin(\alpha_W \times \cos(\sigma \times t))$, where: $\frac{\dot{\theta}}{\sqrt{\dot{\theta}^2}}$ – the factor of damping moment term in the nonlinear roll equation, characterizing the sign of the moment (plus or minus) in the applied numerical integration algorithms in accordance with the direction of the angular velocity of rotation of the ship relative to the abscissa axis Gx during roll, if the damping moment is determined proportionally to its square; $r_4 \approx \frac{3}{w_4 \times \alpha_W} \times \frac{\lambda_4}{2 \times (I_x + \mu_4)} = 3 \times \nu_4 / (w_4 \times \alpha_W)$ – the damping coefficient recommended for a quadratic dependence of damping moment of ship roll; the indicators of the linear damping of roll and the natural frequency of roll of ship, as shown above (item 12.1), are equal: $\nu_4 = \frac{\lambda_4}{2 \times (I_x + \mu_4)}$ and $\omega_4 = \sqrt{\frac{\Delta \times g \times h}{(I_x + \mu_4)}}$.

To use the Euler's method when numerically integrating a differential equation of the second order or another order, if the function sought is smooth (the continuous function and differentiable) or it is smooth over the integration intervals (that is piecewise smooth), when finding the increments, one would consider expansion of the function in the Taylor's series: $f(x + dx) - f(x) = df = dx \times f'(x) + \frac{dx^2}{2} \times f''(x) + \cdots + \frac{dx^n}{n!} \times f^n(x) + \cdots$

To integrate numerically the nonlinear equation of ship rolling it is necessary to determine the initial conditions: $\theta = \theta_0$ and $\dot{\theta} = \dot{\theta}_0$ at $t_0 = 0$. The next values of the phase elements from their set during ship roll: θ, $\dot{\theta}$ and $\ddot{\theta}$ for the second–order equation can be found by numerical integration using the Euler's method in a sequence of steps, as:

$t_i = t_{i-1} + \Delta t,\ i = 1, 2, 3, \dots, n,$

$$\ddot{\theta}_{i-1} = w_4 \times \omega_4^2 \times \sin(\alpha_W \times \cos(\sigma \times t_{i-1})) - \frac{\dot{\theta}_{i-1}}{\sqrt{\dot{\theta}_{i-1}^2}} \times r_4 \times \dot{\theta}_{i-1}^2 - \omega_4^2 \times \sin(\theta_{i-1})$$

$$\dot{\theta}_i = \dot{\theta}_{i-1} + \Delta t \times \ddot{\theta}_{i-1} \text{ and } \theta_i = \theta_{i-1} + \Delta t \times \dot{\theta}_{i-1} + \frac{\Delta t^2}{2} \times \ddot{\theta}_{i-1}.$$

For numerical integration when determining data, the ratios of elements of regular sea waves discussed above (item 12.4) are used: the empirical dependence of the wave amplitude r_W on its length λ_W according to the Zimmerman's formula (Fig. 12.4.1 d): $r_W = 0.085 \times \lambda_W^{3/4}$, m; the geometric dependence of the wave slope angle is equal to the amplitude r_W of the wave ordinates and the wave length λ_W: $\alpha_W = k \times r_W = \frac{2\times\pi}{\lambda_W} \times r_W$, radian; the relationships between wave profile elements (Fig. 12.4.1 a and b) are: its period $\tau_W = 0.8 \times \sqrt{\lambda_W}$, s and the frequency $\sigma = \frac{7.85}{\sqrt{\lambda_W}}$, 1/s. After generalizing the above formulas to find the maximum wave slope angle α_W depending on the wave frequency σ one can use the formula: $\alpha_W = 0.0607 \times \pi \times \sigma^{1/2}$, radian.

The source data also uses the natural frequency of the roll of ship: $\omega_4 = 0.5625$ 1/s for the characteristics of the bunkering ship project for delivery of liquefied natural gas, considered as data for calculations (item 12.3): the displacement at summer load line draft: Δ=9 150 ton, the moment of inertia of ship relative to the abscissa axis Gx found above: $I_x \approx 457\ 500$, ton×m^2, the added moment of ship inertia relative to the abscissa axis Gx in water: $\mu_4 \approx 251\ 625$, ton×m^2, the initial transverse metacentric height assigned: h =2.5 m.

When set the initial conditions, it is necessary to take into account the compatibility of the phase characteristics of the angular displacement θ and the angular velocity $\dot{\theta}$ of ship during roll and the ratio of their amplitudes, since these roll elements are in a differential interdependence and the amplitude of the velocity is greater than the nominal value of the amplitude of the roll angles in proportion to the frequency (for initial conditions the proportionality coefficient is the natural frequency of the roll). Considering that the initial conditions are assumed to be equal: $\theta = 0.35$ radian and $\dot{\theta} = -0{,}20$ radian/s at $t_0 = 0$. Numerical integration of the nonlinear inhomogeneous roll equation in irregular waves is carried out with a step: Δt=0.5 s in the interval from 0 to 56 s.

For assigned initial conditions, the dependence of heel angle θ on time t and on the damping index r_4 in it considered interval of values: 4.20, 5.60, 6.99, 8.39, 9.79 and 11.19, and also on the frequency of oncoming waves from one side of ship: $\sigma = 0.5625$ 1/s,

which in this calculation is taken equal to the natural frequency of the ship's roll ω_4 (that is the conditions of ship rolling in resonance with the oncoming waves are considered), is shown in the form of a functional surface on the graph (Fig. 12.8.2).

The ship rolling in regular waves is a stationary process, the elements of which do not depend on the origin of time, just as regular waves themselves are a stationary process. Therefore, the use of the initial values of the elements of ship rolling (the amplitude and frequency) when integrating the differential equation introduces a certain convention – it is assumed that, due to undiscussed reasons, the ship rolling at the initial moment had elements different from those in regular waves.

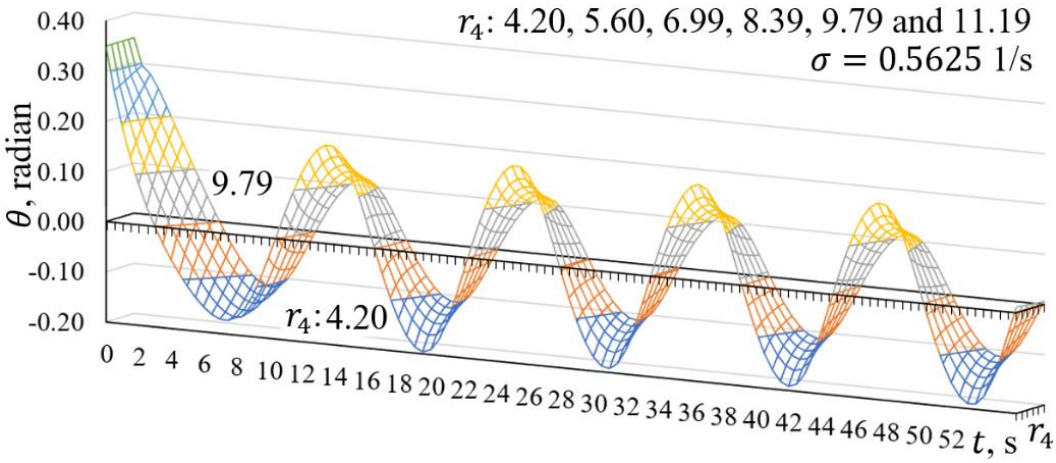


Fig. 12.8.2

For example, the ship before the initial moment performed a maneuver or a functional operation, which could be the cause for beginning of the roll serving as the causes of the initial values of elements of ship rolling in regular waves, which (the initial cause impact) attenuated due to damping in calm water. Subsequently the influence of the initial conditions decreases, and ship rolling elements come into compliance with the stationary process of ship rolling in regular waves (or the ship rolling elements attenuate in the case of rolling in calm water). The dependence of ship rolling angle θ on time t, on the damping index r_4, found by numerical integration of ship nonlinear rolling equation in regular waves under supposedly zero initial conditions: $\theta \sim 0$ and $\dot{\theta} \sim 0$ at $t_0 = 0$, is shown on the graphical functional surface (Fig. 12.8.3).

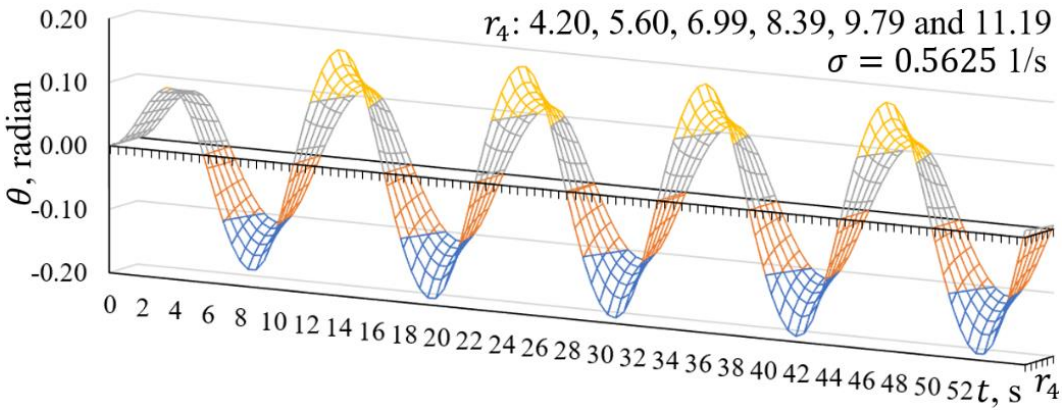


Fig. 12.8.3

The linear equation of ship rolling in regular waves has an analytical solution, which in accordance with the Cauchy problem includes the sum of the general solution of corresponding homogeneous equation and the particular solution of inhomogeneous equation, and the method of complex amplitudes can be used in solving it, as shown above (item 12.3). At the same time, the nonlinear equation of ship rolling on regular waves, discussed above with examples of its numerical integration (Fig. 12.8.2 and 12.8.3), is methodically more accurate, since trigonometric (the sinusoidal) forms of dependences of the righting and heeling moments are taken into account on the roll angle, and the dependence of the damping moment on the angular velocity during ship rolling is quadratic, which is closer to the laws of hydromechanics. Therefore, it is of interest to compare the results of the numerical integration of linear and nonlinear equations performed on a similar methodological basis under the same initial conditions in order to have judgment about the effect of simplification of equation (the linearization) on the resulting differences in solutions of the nonlinear and linear equations of ship rolling in regular waves.

The numerical integration of linear inhomogeneous ship rolling equation on regular waves is formally performed in the same way as the numerical solution of the nonlinear equation. For the assigned initial conditions: $\theta = \theta_0$ and $\dot{\theta} = \dot{\theta}_0$ at $t_0 = 0$ the values of ship rolling phase elements (θ, $\dot{\theta}$ and $\ddot{\theta}$) of the second–order equation can be found by numerical integration using the Euler's method in a sequence of steps, as:

$t_i = t_{i-1} + \Delta t,\ i = 1, 2, 3, \dots, n,$

$\ddot{\theta}_{i-1} = w_4 \times \omega_4^2 \times \alpha_W \times \cos(\sigma \times t) - 2 \times \nu_4 \times \dot{\theta}_{i-1} - \omega_4^2 \times \theta_{i-1},$

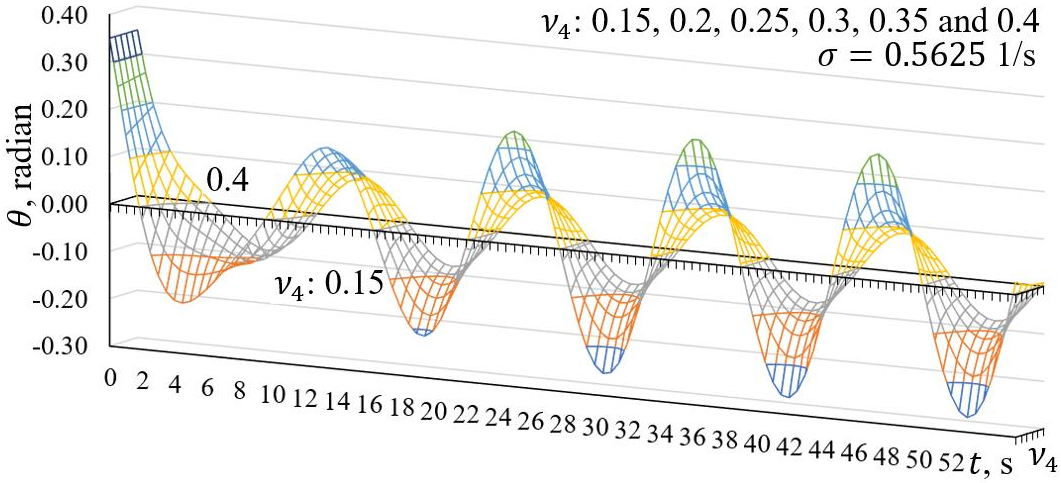
$\dot{\theta}_i = \dot{\theta}_{i-1} + \Delta t \times \ddot{\theta}_{i-1}$ and $\theta_i = \theta_{i-1} + \Delta t \times \dot{\theta}_{i-1} + \frac{\Delta t^2}{2} \times \ddot{\theta}_{i-1}.$

The initial conditions are taken equal: $\theta = 0.35$ radian and $\dot{\theta} = -0{,}20$ radian/s at $t_0 = 0$. The dependence on: the time t, the damping index ν_4 (the decrement) of ship rolling angle θ found by numerical integration, considering the linearization of the dependence of the damping moment on the angular velocity $\dot{\theta}$ in the range of values of the damping index ν_4: from 0.15 to 0.4 with a step of 0.05, as well as on the frequency of

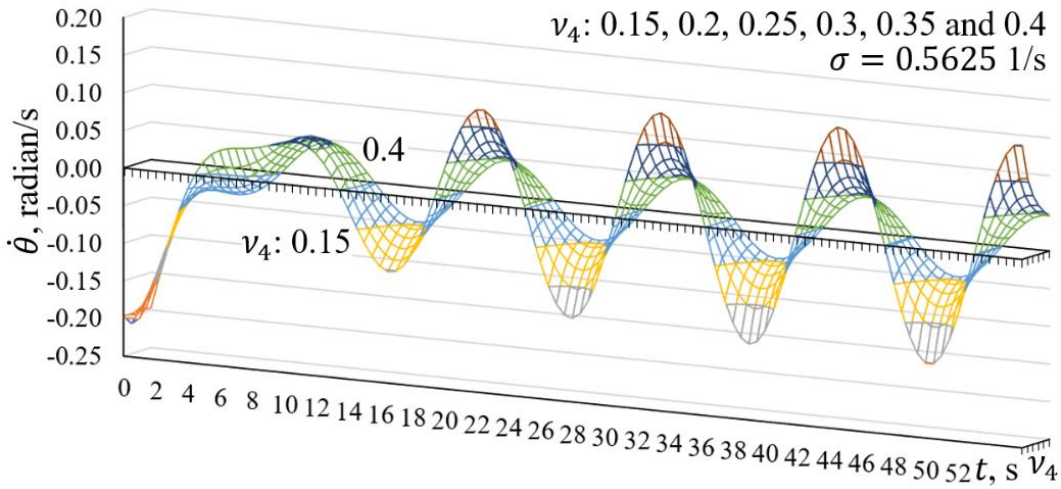
oncoming waves on the ship from the side: $\sigma = 0.5625$ 1/s, shown in the form of a functional surfaces on graphs of the roll angle θ relative to the abscissa Gx (Fig. 12.8.4 a) and angular velocity $\dot{\theta}$ (Fig. 12.8.4 b).

Comparing the results of the numerical integration of linear (Fig. 12.8.4 a) and nonlinear (Fig. 12.8.2) equations performed on a similar methodological basis under the same initial conditions, one can see that simplification of ship rolling equation on regular waves (the linearization of equation) leads to some differences in the results, which are less affected by the stationary process of ship roll, when the influence of the initial conditions ceases, and the nonlinearity of the equation is more manifested at the initial stage – when the initial elements of ship rolling, specified by the initial conditions, are attenuate.

In a steady rolling process of ship on regular waves, the results of integration of linear (more accurately, the linearized) and the nonlinear equations are similar in principle, therefore below for different wave elements (the frequencies): $\sigma = 0.40$ 1/s (Fig. 12.8.5 a), which is less than the resonant frequency equal to the natural frequency of the ship rolling ω_4 ($\sigma = 0.5625$ 1/s, Fig. 12.8.2 and Fig. 12.8.4 a), and at: $\sigma = 1.05$ 1/s (Fig. 12.8.5 b) the examples of linear equation integration are shown and it is assumed that the conclusions drawn from the analysis of linear equation could be transferred in general to the analysis of nonlinear one.



(a)



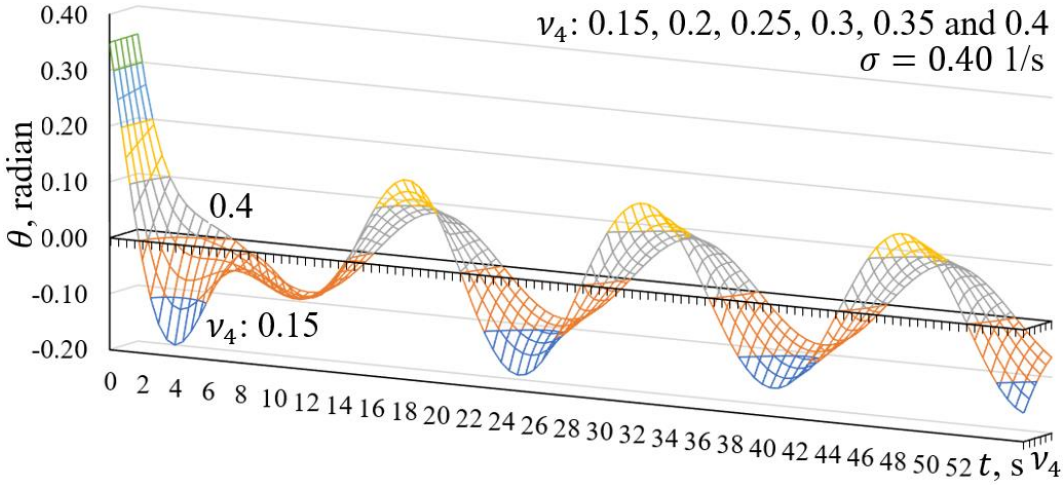
(b)

Fig. 12.8.4

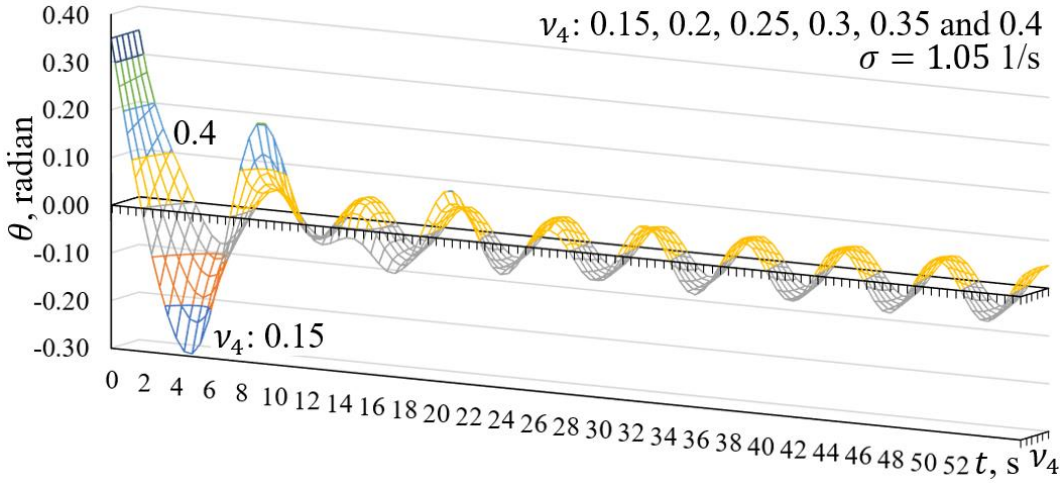
Greatest amplitude of ship rolling θ_M is characteristic when frequency of oncoming waves coincides with the ship natural frequency of rolling (Fig. 12.8.4 a, at: $\sigma = \omega_4$, when the roll dynamics coefficient $\left(\frac{\theta_M}{\alpha_W}\right)$ takes the greatest value, Fig. 12.3.3), although the amplitude of wave slope angles, in calculation by the formula: $\alpha_W = 0.0607 \times \pi \times \sigma^{1/2}$, increases with the frequency increase of oncoming waves, if the ship does not move in one direction or another parallel to the waves profile motion.

It is also of interest to illustrate the dialectical transition: with an increase in the frequency of waves oncoming on stationary ship, the shape of the curves, describing its rolling at a high frequency of waves and their small period (Fig. 12.8.6 a), is transformed to shape of the curves describing the ship rolling attenuated due to damping in calm water, that is at zero frequency of oncoming waves and their infinite period (Fig. 12.8.6 b).

Amplitude of ship rolling on regular waves θ_M decreases with a significant increase in wave frequency due to decrease of the dynamic coefficient $\left(\frac{\theta_M}{\alpha_W}\right)$ for wave frequencies significantly exceeding the natural frequency ω_4 of ship rolling, despite the fact that with increasing of wave frequency σ amplitudes of the wave slope angles α_W increase.



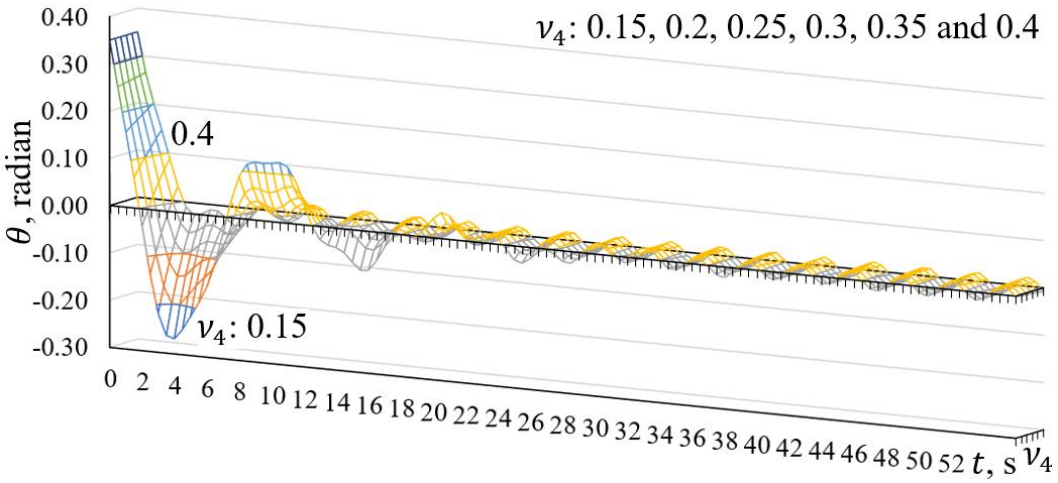
(a)



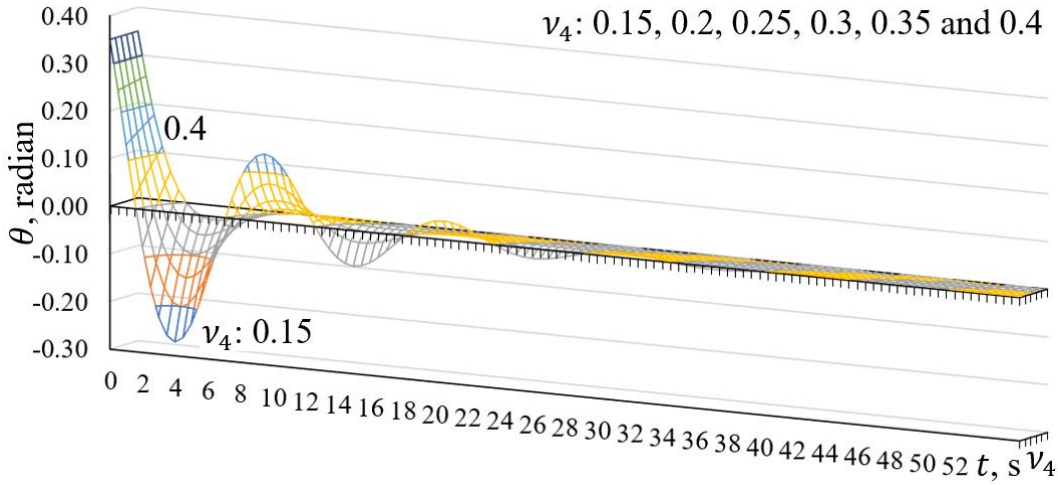
(b)

Fig. 12.8.5

One has to clarify that solutions of the inhomogeneous of ship rolling equation on irregular waves, which is the dependences of the rolling elements on time and on the initial conditions (Fig. 12.8.3 and 12.8.4 a), are results of the superposition of particular solutions (the solutions summation), to which ship rolling on regular waves with zero initial conditions (Fig. 12.8.3) and damped rolling of ship in the absence of waves and in the presence of initial conditions (Fig. 12.8.6 b) refers, which follows from the conditions of the Cauchy problem.



(a)



(b)

Fig. 12.8.6

Questions and comments:

12.8.1. Area of the numerical integration methods use for nonlinear differential equations solution, for which the analytical methods are unknown, as well as for solution linear equations in which the determination of antiderivatives is not required, and the values of the function sought are calculated directly using its derivatives.

12.8.2. The hypothesis of nonlinear differential equations linearization with respect to the function sought, its derivatives and argument, the linearized inhomogeneous equation of ship rolling on regular waves.

12.8.3. Amplitude–frequency and phase–frequency characteristics, which are solutions to linear equations of ship rocking on regular waves.

12.8.4. Possible ambiguity of a function (the multivalued or discontinuous mapping) serving as solution to a nonlinear differential equation, leading to the fact that this solution can be found by numerical integration over intervals in which the function that is this solution is unambiguous and continuous.

12.8.5. A change of the amplitude–frequency characteristic shape due to the nonlinearity of equation of the ship rolling on irregular waves, the bifurcation point – a break in the mathematical mapping (the function), which is solution to the nonlinear differential equation, and the emergence of ambiguity in this solution.

12.8.6. Finding the righting moment in the form of a sinusoidal dependence on the roll angle. The sinusoidal dependence of the disturbing wave moment, which is the heeling one under the action of waves slope angle.

12.8.7. Corresponding to hydromechanics the quadratic dependence of damping moment on the angular velocity of ship turning during rolling and the recommended relationship between the indicators of damping coefficients for the quadratic dependence of the damping moment on angular velocity and for the linear dependence of the moment on the angular speed – within the framework of the hypotheses of linearization of the dependences of moments on elements of ship motion when rocking.

12.8.8. The sequence of numerical integration by the Euler's method of a second–order differential equation, the identification of second derivative in explicit form, the determination of exponents and factors for the increments of function proportional to its first and second derivatives, the use of the Taylor's series to determine the factors for increments of a function during numerical integration by the Euler's method.

12.8.9. Wave characteristics, the ship data, initial conditions, the step duration and the integration interval used for numerical solution of differential equation of ship rolling on regular waves by the Euler's method.

12.8.10. Conventionality of initial values of ship motion elements in the numerical integration of the equations of it rolling on regular waves, which is a stationary process, which elements do not depend on the origin of time. The influence on the form of the solution of ship rolling equation on regular waves of initial values of ship rolling elements. A previous maneuver or functional operation of a ship as the cause of the initial values of the elements of its motion.

12.8.11. The methodological value of the linear rolling equation of a ship lies in the fact that its analytical solution is known, although the nonlinear rolling equation on regular waves has more accurate methodological prerequisites, which serve as ones for performing numerical integration of linear and nonlinear equations with the same data in order to make a judgment on influence of the equation linearization on their solutions deviations.

12.8.12. A greater influence of nonlinearity on some existing differences in the solutions of nonlinear and the linear equations of ship rolling on regular waves during the attenuation of set initial elements of rolling due to fact that nonlinearity affects the damping moment of ship rolling, depending on square of the angular velocity, less influence of the nonlinearity on elements of ship stationary rocking on regular waves.

12.8.13. Dialectical property of stationary process elements of ship rolling on regular waves (the roll angle, angular velocity and angular acceleration), which amplitudes decrease with a significant increase in wave frequency, and ship rolling elements tend to values corresponding to the rolling on calm water (when the frequency of the waves in the mathematical sense would be zero), that is to the damped motion of ship rolling, arising from the initial values of the motion elements.

12.8.14. The elements of ship rocking on regular waves found by numerical integration, which, in accordance with the Cauchy problem, are superposition of particular solutions, which include the stationary process of the ship's rolling in regular waves without the influence of the initial conditions and the damped rolling of the ship in the absence of waves and in the presence of the initial conditions.

12.9. Keeping the ship on course using controls. The particular solution of ship motion equations

Ship control systems with the feedback are characterized usually by linear transfer function, that is the relationship between the observed controlled element of ship motion, for example, the heading: $CR = \int_{t_0}^{t} \omega_z \times dt + CR_0$, or the angular velocity ω_z, or angular acceleration $\dot{\omega}_z$ and the control action: the moment M_z, turning the ship relative to the applicate axis Gz, when shifting the rudder, and the lateral force N_y, depending on the rudder shift of and the drift angles, which are terms in the system of equations (item 12.5):

$(\Delta + \mu_2) \times \dot{v}_y + (\Delta + \mu_1) \times v_x \times \omega_z = N_y,$

$(I_z + \mu_6) \times \dot{\omega}_z - (\mu_2 - \mu_1) \times v_x \times v_y = M_z,$

transformed to two equations, the second of which becomes isolated after the transformation, because it contains only one unknown function – the angular velocity of ship rotation ω_z relative to the applicate axis Gz and the second derivative $\ddot{\omega}_z$ of the angular velocity: $\ddot{\omega}_z + \omega_z \times \frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(I_z+\mu_6)\times(\Delta+\mu_2)} \times v_x^2 = \frac{\dot{M}_z}{I_z+\mu_6} + \frac{(\mu_2-\mu_1)\times v_x \times N_y}{(I_z+\mu_6)\times(\Delta+\mu_2)}$, and the first of the transformed equations is the dependence of derivative of the transverse translational velocity of ship $\dot{v}_y$, directed along the ordinate axis Gy depending on the previously found function of the angular velocity of ship rotation ω_z relative to the applicate axis Gz:

$\dot{v}_y = \frac{1}{\Delta+\mu_2} \times (N_y - (\Delta + \mu_1) \times v_x \times \omega_z).$

Therefore, second of equations in considered system can be solved in isolation as the sum of a general solution to the homogeneous equation ω_{Z1}, depending on the initial conditions, if the ship motion is not a stationary process that would not depend on initial conditions, such as, keeping the ship on course, which corresponds with particular solution

ω_{Z2} of the inhomogeneous equation: $\omega_Z = \omega_{Z1} + \omega_{Z2}$ in accordance with the Cauchy problem.

To solve another equation (the first in the system) and to find the transverse velocity function v_y one needs to set into it the function of angular velocity ω_z of ship rotation relative to the applicate axis Gz, found by integrating the second equation, and then to integrate the equation to find the transverse velocity function v_y.

To determine the elements of motion when keeping a ship on course it will be necessary to transfer the control efforts to left side of the inhomogeneous equation: the moment M_z and lateral force N_y on a rudder (of any type: on the regular rudder, on rotating nozzle or on rotating steering column, an Azipod or another), in particular, with the linear transfer function of the feedback control system in relation to the controlled element of motion (CR, ω_z or $\dot{\omega}_z$).

After this transformation, a zero will remain on the right side of the equation, therefore, the equation can be solved as formally homogeneous, because the terms that are transferred to the left side will be in linear dependencies on the phase elements – the antiderivative or derivatives of the function of the angular velocity ω_z sought of ship rotation relative to the applicate axis (Gz: CR, ω_z or $\dot{\omega}_z$), although the roots p of the characteristic equation in this case will cover, as well the transfer function linear with respect to the controlled element of the ship motion in addition to the translational velocity of ship v_x, taken usually as constant value, provided that the ship is kept on course, and in addition to own and associated inertia characteristics of ship in water (Δ, μ_1, μ_2, I_z and μ_6).

The proportionality coefficient between the controlled element of ship motion and the control action can impact the characteristic number p and the form of general solution of the homogeneous equation ω_{z1}, if one transforms the original inhomogeneous one to it considering the indicated linear dependence of the control efforts (N_y and M_z) on controlled element (CR, ω_z or $\dot{\omega}_z$) of ship motion.

In the event that root of the characteristic equation after transforming the inhomogeneous equation to the homogeneous form is an imaginary number at: $p^2 < 0$ and the solution, accordingly, will have trigonometric form, and the angular velocity ω_z of ship rotation relative to applicate axis Gz, determined by this general solution, as well as the course of ship CR, the angular acceleration $\dot{\omega}_z$, translational displacement along ordinate axis Gy: $y = \int_{t_0}^{t} \omega_z \times dt$, the translational transverse velocity v_y and the acceleration $\dot{v}_y$ along ordinate axis will be limited, changing relative to zero or initial values with the same constant period ($2 \times \pi/p$). The form of ship motion on water under these conditions is the self–oscillations, although, unlike ship motion without control, in this case the self–oscillations are result of the control system action, without which the trajectory of movement would be unstable.

The feedback control system, both the automatic one or the ship manual keeping on course, is characterized by the sensitivity threshold for input of controlled elements of ship motion (CR, ω_z or $\dot{\omega}_z$) and the performances speed limitations – due to the inertia of the mechanical parts of the control system and its delay. The sensitivity threshold and the performance speed limitations of control system can lead to ship motion along a periodic trajectory that is within the indicated limitations.

For purposes of the research, to simulate the controlled ship motion, the control efforts, which is the ship turning moment M_z and the lateral force N_y, can be specified without feedback in the form of analytical functions of time, including a transcendental function (not the algebraic), in the form of polynomial, the trigonometric series, another series or sequences, as well as to be specified physically, for example, by transfer functions of mechanical control system. In particular, the thrusters used in the past on sailing yachts to maintain course relative to the direction of the wind were the systems with a weather vane, mechanical transmission and rudder blade, and they did not include digital algorithmic devices in their design.

The control action, which is the time pattern of moment M_z and the lateral force on rudder blade N_y, depending on the rudder turn angle, for example, can be specified in the form of an analytical function, in particular, the sinusoidal one. In general, it is required that the control function takes limited values and is twice differentiable. Among the formally suitable functions, in addition to the sinusoidal, one could include: the sum of trigonometric functions that have different periods, for example, the Fourier series, etc.

When ship, which moves in water along x–axis Gx with constant velocity: $v_x = const$, is steered by rudder, assuming that the drift angle β_D is much less than rudder turn angle α_r (the shifting), the lateral force on rudder can be determined, as: $N_y(\alpha_r) = C_y(\alpha_r) \times \frac{\rho \times v_x^2}{2} \times S_r$, where $C_y(\alpha_r)$ – the dependence of the coefficient of transverse force (the lifting) arising on the rudder blade when it is turned on the angle of attack of the rudder blade α_r; $\rho \approx 1.025$ ton/m^3 – the mass density of sea water; S_r – the area of rudder blade, m^2. The angle of attack in fluid mechanics is usually understood as the angle between the direction of oncoming flow and the flow plane, with the rudder in a position at which the transverse force is zero.

The formula of G.F. Burago as amended by K.K. Fedyaevsky [18 and 28] is known, which in the most general approximation can be recommended for finding the coefficient of transverse force depending on the angle of attack α_r of the rudder located directly at the surface of the ship hull, and on hydrodynamic elongation of the rudder λ_r can be shown on the graph (Fig. 12.9.1): $C_y(\alpha_r) = 4.75 \times \alpha_r \times \frac{2.75 \times \lambda_r + 2 \times \alpha_r}{2.75 \times \lambda_r + 5.6} \times \cos \alpha_r$, assuming the angles of attack expected interval up to ones at which the separation of the boundary layer of water flow will occur on the side of the blade opposite to the oncoming water. For

rectangular rudder contour shape in plan, the hydrodynamic elongation λ_r can be defined as the ratio of the blade span to the chord of its profile.

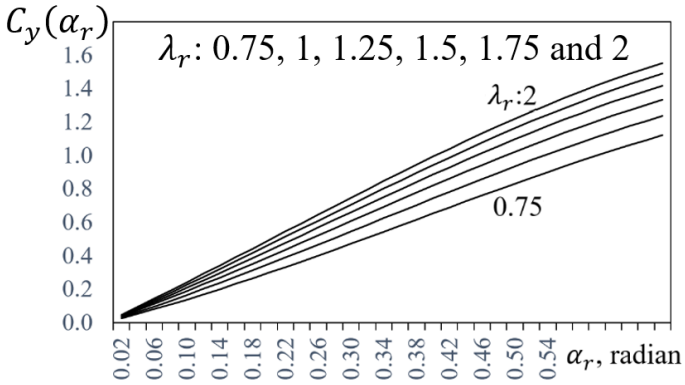


Fig. 12.9.1

Within the limits of the angles of attack with a continuous flow of water around the rudder for characteristic values of the hydrodynamic elongation of: $\lambda_r = 1.5 \div 2$ the approximate dependence of the coefficient of transverse (the lifting) force on the rudder on the angle of attack α_r of the hydrodynamic profile one can represent in the form: $C_y(\alpha_r) = (2.2 \div 2.25) \times \alpha_r$, here one needs to stick to the dimension of the angle α_r – the radian.

The largest value of the lifting force coefficient C_y^M (of the lateral force when turning the rudder) within the limits of linearity for flat rudders and for commonly used hydrodynamic profile rudders is usually from 1.1 to 1.7.

When set the data, the transverse force N_y on the rudder is determined, as: $N_y(\alpha_r) \approx 2.25 \times \alpha_r \times \frac{\rho \times v_x^2}{2} \times S_r$, and the moment M_z of transverse force on the rudder turning the ship is equal to: $M_z(\alpha_r) \approx N_y(\alpha_r) \times \left(-\frac{L}{2}\right) \approx -2.25 \times \alpha_r \times \frac{\rho \times v_x^2}{2} \times S_r \times \frac{L}{2}$.

Assumption that arm of the transverse force N_y, turning the ship, is equal with a minus sign (since rudder is located at the stern of ship) to half of ship waterline length $\left(-\frac{L}{2}\right)$, is valid within the assumption of ship symmetry relative to the midship section, near which the center of ship gravity and the center of volume of the immersed part of hull are located, and that the main part of the hydrodynamic forces, acting on a moving ship, consists of reactive ones, the resultant of which is applied near indicated points.

Blade area of the rudder S_r of a displacement ship is usually equal from 1/40 to 1/60 of the submerged area of ship diametrical section: $S_{DB} \approx (0.93 \div 1.0) \times L \times T$, where L and T – the length of waterline and the draught of loaded ship.

When the ship is kept on course by periodic shifting of the rudder, the drift of ship and the angle of flow bevel is small, while the hydrodynamic angle of attack of the rudder

α_r minus the angle of the oncoming flow (which is the angle of ship drift in the aft part, where the rudder is installed) is almost equal to the angle of it turn.

The formula for lateral force on the rudder N_y determining as function of the angle of rudder attack, equal at small drift angles to the angle of the rudder turn α_r, is more convenient for analysis and calculations than the direct dependence of the force on the angle of attack of rudder with respect to oncoming water flow, since the rudder angle can technically be controlled.

When calculating the elements of ship motion under the controls, the linear dependence of lateral force on the rudder on angle of attack is assumed, and, following the principle of superposition, the lateral force acting on the rudder of ship is vectorially decomposed into two collinear components, one of which corresponds to flow around the turned rudder without drift, and the second – the flow around rudder considering the drift, but without turning the rudder. The resulting force on rudder blade one can get if these collinear components are added. This force can coincide with one of the indicated components in direction, or with both components, if they are co–directed, as during a controlled turn of ship and its entry into the steady circulation mode.

The advantage of approach used when solving the equation of ship motion and when finding the function of ship rotation angular velocity ω_z under influence of the controls is that the linearity of the equations remains. The ship rudder is considered simultaneously as the control device and as a submerged part of the hull of ship, which is like a stern stabilizer.

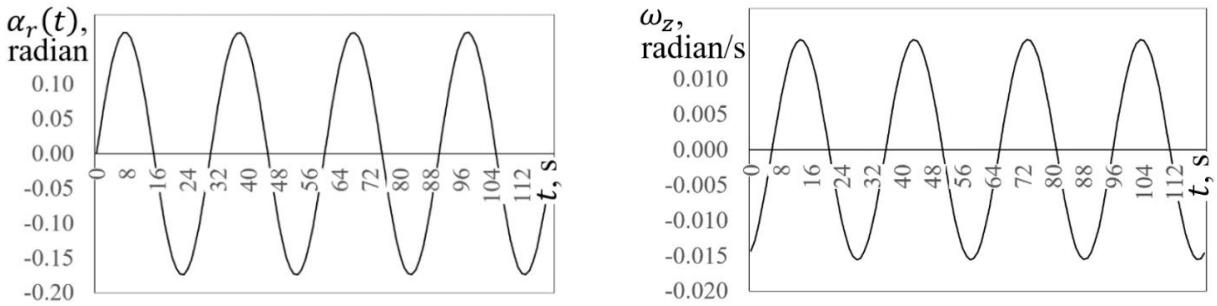
On the right side of ship motion in water equations, one of the mentioned components of the transverse force N_y – is used as a control effort – the one that depends on angle of the rudder turn, and the moment M_z arising under this component of the transverse force, and since the rudder of ship, located in the stern, moving with drift and without the rudder turn is the aft part of the hull of ship, then the other component of transverse force and the moment depending on this part of the force are attributed to the damping forces acting on the hull, which are included in the equation on the left in the form of functions on angular velocity ω_z and the translational velocity v_y of the transverse ship motion in the coordinate system associated with ship.

When set into the transformed second equation of the system (item 12.5) the formula for lateral force on the rudder blade: $N_y\big(\alpha_r(t)\big) = 2.25 \times \frac{\rho \times v_x^2}{2} \times S_r \times \alpha_r(t)$ and the formula for the derivative of moment of transverse force:

$\dot{M}_z\big(\alpha_r(t)\big) = -2.25 \times \frac{\rho \times v_x^2}{2} \times S_r \times \frac{L}{2} \times \dot{\alpha}_r(t)$ the considered equation takes the form:

$$\ddot{\omega}_z + \omega_z \times \frac{(\mu_2 - \mu_1) \times (\Delta + \mu_1)}{(I_z + \mu_6) \times (\Delta + \mu_2)} \times v_x^2 = \frac{2.25 \times \rho \times v_x^2 \times S_r}{2 \times (I_z + \mu_6)} \times \left(-\frac{L}{2} \times \dot{\alpha}_r(t) + \frac{(\mu_2 - \mu_1)}{(\Delta + \mu_2)} \times v_x \times \alpha_r(t) \right).$$

Exercise. The exercise assigns one to determine the functions of motion elements (ω_z, v_y and y) during periodic turns of the rudder: $\alpha_r(t) = 0.1745 \times \sin(0.209 \times t)$, radian (Fig. 12.9.2 a), which corresponds to the amplitude of angles of rudder turns: $\alpha_r^M = 10$ degrees, that is 0.1745 radian, and the full period of rudder turn 30 s of ship, having the following characteristics: the length along the waterline: $L = 46.2$ m, the breadth: $B = 10.5$ m, the average draught at the load line: T =4.37 m, the coefficient of overall completeness of hull: $\delta = 0.564$, the coefficient of completeness of the midship section: $C_\beta = 0.93$, the coefficient of completeness of waterline area: $C_\alpha = 0.72$, the coefficient of longitudinal sharpness: $C_\psi = \delta / C_\beta = 0.607$, ship operating velocity: $v_S = 12.7$ knots, the forward motion velocity of ship in the direction of abscissa axis Gx: $v_x = 0.514 \times v_S = 0.514 \times 12.7 = 6.53$ m/s, ship displacement at load line draught: $\Delta = 1228$ ton, the area of rudder blade: $S_r \approx L \times \frac{T}{50} = 46.2 \times \frac{4.37}{50} = 4.0$, m^2.



(a) (b)

Fig. 12.9.2

The added masses and moments of inertia of ship in water one can find using the empirical formulas discussed above (in the items of problems on ship rolling in water):

$\mu_1 \approx 0.25 \times \Delta = 0.25 \times 1228 = 307$, ton,

$\mu_2 \approx \mu_3 \approx 0.40 \times C_\alpha \times \frac{B}{T} \times \Delta = 0.40 \times 0.72 \times \frac{10.5}{4.37} \times 1228 = 850$, ton,

$I_y \approx I_z \approx 0.065 \times C_\alpha \times \Delta \times L^2 = 0.065 \times 0.72 \times 1228 \times 46.2^2 = 122\,667$, ton×m^2,

$\mu_6 \approx \mu_5 \approx 0.42 \times C_\psi^2 \times \frac{B}{T} \times I_x = 0.42 \times 0.607^2 \times \frac{10.5}{4.37} \times 122\,667 = 45\,610$, ton×m^2.

After the set of ship particulars used as data, the inhomogeneous equation takes the form: $\ddot{\omega}_z + \omega_z \times \frac{(850-307)\times(1228+307)}{(122\,667+45\,610)\times(1228+850)} \times 6.53^2 = \frac{2.25\times1.025\times6.53^2\times4.0}{2\times(122\,667+45\,610)} \times$

$\times (-0.209 \times \frac{46.2}{2} \times 0.1745 \times \cos(0.209 \times t) +$

$+ \frac{(850-307)}{(1228+850)} \times 6.53 \times 0.1745 \times \sin(0.209 \times t))$, or:

$\ddot{\omega}_z + \omega_z \times 0.1016 = -0.000832 \times \cos(0.209 \times t) + 0.000348 \times \sin(0.209 \times t)$.

The exercise contains determination of ship motion elements when the ship is kept on course by periodic shifting of the rudder according to sinusoidal pattern of angle of its

shifting. The simulated process is stationary, that is repeatable elements do not depend on the choice of initial conditions. In accordance with this, the general solution of a linear homogeneous equation is identical to zero: $\omega_{z1} \equiv 0$.

For this reason the sought solutions – the functions of the elements of ship motion: ω_z, v_y and y correspond to a particular solution of inhomogeneous equation: $\omega_z = \omega_{z2}$, which has the same form as the right side of the inhomogeneous equation:

$$\omega_{z2}(\dot{\alpha}_r, \alpha_r) = [\omega] \times \left(-\frac{L}{2} \times \dot{\alpha}_r(t) + \frac{(\mu_2 - \mu_1)}{(\Delta + \mu_2)} \times v_x \times \alpha_r(t) \right),$$

where $[\omega]$ – the constant to be determined when solving the equation.

One looks for a partial solution of the inhomogeneous equation in the form of its right–hand side, and finds the constant $[\omega]$ by double differentiating the partial solution and it set into the equation:

$\omega_z = [\omega] \times (-0.000832 \times \cos(0.209 \times t) + 0.000348 \times \sin(0.209 \times t))$,

because: $\ddot{\omega}_z = -0.209^2 \times \omega_z$, then: $[\omega] = \frac{1}{0.1016 - 0.209^2} = 17.27$,

$\omega_z = 17.27 \times (0.000348 \times \sin(0.209 \times t) - 0.000832 \times \cos(0.209 \times t))$, 1/s or:

$\omega_z = 0.00601 \times \sin(0.209 \times t) - 0.01437 \times \cos(0.209 \times t)$, 1/s (Fig. 12.9.2 b).

The projection onto the transverse associated with ship axis Gy of acceleration during maneuvering of the ship, which characteristics are considered as an example, is determined by solving the transformed first equation of the original system:

$(\Delta + \mu_2) \times \dot{v}_y + (\Delta + \mu_1) \times v_x \times \omega_z = N_y$,

or $\dot{v}_y = \frac{1}{(\Delta + \mu_2)} \times \left(2.25 \times \frac{\rho \times v_x^2}{2} \times S_r \times \alpha_r(t) - (\Delta + \mu_1) \times v_x \times \omega_z \right)$,

$$\dot{v}_y = \frac{1}{(1228 + 850)} \times (2.25 \times \frac{1.025 \times 6.53^2}{2} \times 4.0 \times 0.1745 \times \sin(0.209 \times t) -$$

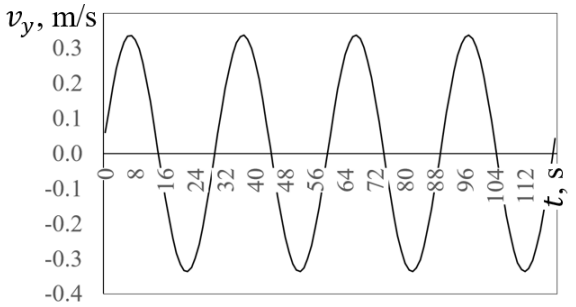
$$-(1228 + 307) \times 6.53 \times (0.00601 \times \sin(0.209 \times t) - 0.01437 \times \cos(0.209 \times t))),$$

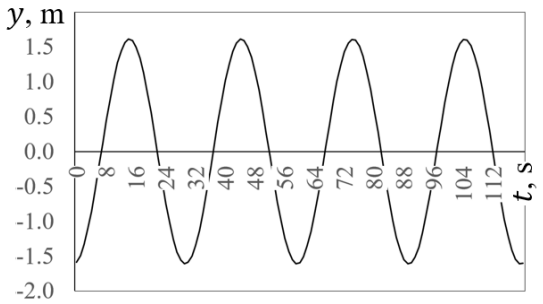
$\dot{v}_y = 0.06932 \times \cos(0.209 \times t) - 0.01247 \times \sin(0.209 \times t)$, m/s^2.

By integrating the resulting equation, one can find the transverse translational velocity v_y (Fig. 12.9.3 a) and the transverse displacement y (Fig. 12.9.3 b):

$v_y = 0.3317 \times \sin(0.209 \times t) + 0.0597 \times \cos(0.209 \times t)$,

$y = 0.285 \times \sin(0.209 \times t) - 1.587 \times \cos(0.209 \times t)$.





(a) (b)

Fig. 12.9.3

The kinematic conditions for a periodic stationary process are also observed when integrating: $v_y(0.209 \times t) = \frac{\dot{v}_y\left(0.209 \times t + \frac{\pi}{2}\right)}{0.209}$ and $y(0.209 \times t) = \frac{v_y\left(0.209 \times t + \frac{\pi}{2}\right)}{0.209}$.

The displacement y, which is an integral function of ship lateral velocity v_y, determined in the coordinate system $Gxyz$ associated with the ship and is subject to transfer to the fixed coordinate system $0\xi\eta\zeta$.

If it is necessary to determine the ship trajectory in water $\eta(\xi)$, which is the dependence of ship position along the ordinate axis on its position along the abscissa axis in fixed (the geographical) coordinate system $0\xi\eta\zeta$, then the translational motion along ordinate axis 0ξ of fixed coordinate system is defined, as:
$\xi(t) = 0.514 \times v_S \times \int_{t_0}^{t} \cos\left(\varphi_v(t)\right) \times dt$, and the lateral displacement is:
$\eta(t) = 0.514 \times v_S \times \int_{t_0}^{t} \sin\left(\varphi_v(t)\right) \times dt$, in this case, the velocity angle dependent on the drift β_D and the course CR is equal to: $\varphi_v = CR + \beta_D$.

Assuming that the drift β_D is small, the course change CR and the velocity angle φ_v (which is at $\cos\left(\varphi_v(t)\right) \sim 0$) for one period of rudder turning, for example, when keeping the ship on course with periodic small shifting of rudder, the translational motion of ship in water relative to a fixed coordinate system $0\xi\eta\zeta$ one can determine, as:
$\xi(t) = 0.514 \times v_S \times (t - t_0)$.

Questions and comments:

12.9.1. Linear transfer function of control systems with feedback between the controlled elements of ship motion, for example: the course, the angular velocity of turn or angular acceleration, on the one hand, and the control efforts: the moment that turns the ship relative to the applicate axis when shifting the rudder, and the transverse force depending on angle of rudder turn and on drift angle, on the other hand.

12.9.2. Transformation of the system of two equations of ship motion in water in the projections of efforts (the forces and moments) on the ordinate and applicate axes to two equations, one of which becomes isolated and suitable for finding the angular velocity of ship rotation relative to the applicate axis, and the other equation can be solved and find the transverse translational velocity after set the velocity of ship rotation found from the second equation into first one.

12.9.3. The transformation of a non–homogeneous equation, including the function of ship rotation angular velocity and this velocity second derivative to the formally

homogeneous form, by transferring to the left side the terms corresponding to control efforts with a linear feedback function by observed angular velocity of ship and by the elements that are in differential dependence with the angular velocity –heading of ship or the angular acceleration.

12.9.4. The sensitivity threshold for input of the controlled elements of ship motion in a feedback control system and the performances speed limitations – due to the inertia of the mechanical parts of control system and its delay, leading to ship motion along a trajectory that is periodic within the appropriate limits.

12.9.5. Mathematical forms of control efforts without feedback, suitable for simulating the controlled motion of ship for the research purposes: the analytical functions of time, trigonometric functions and the Fourier series, other series or sequences. The mechanical systems of ship motion control.

12.9.6. The dependence of transverse (the lifting) force on the rudder on angle of rudder turn and the dependence of moment turning the ship relative to the applicate axis, the assumption of ship small drift angle while keeping the ship on course. Dependence of the lifting force coefficient on angle of attack of the hydrodynamic rudder profile.

12.9.7. The assumption that the arm of the force turning ship is equal with a minus sign (since the rudder is located at the stern of the ship) to half of its length along the waterline within the framework of other assumption of the ship symmetry relative to midship section, near which the center of ship gravity is located and the center of the submerged part of ship hull volume.

12.9.8. Vector decomposition of the transverse force acting on turned rudder into two collinear components within framework of the assumptions of: the limited angles of the rudder attack, the small drift of ship, and that transverse (the lifting) force on rudder is linearly proportional to its turn angle.

12.9.9. Use of the superposition principle of the lateral force components acting on the rudder in order to consider the rudder simultaneously as device for control and as a the stern stabilizer, which allows maintaining the linearity of applied equations describing the elements of the steered ship motion.

12.9.10. Equality to zero of the general solution of homogeneous equation under zero initial conditions and determination of ship motion elements in the form of a particular solution of a non–homogeneous equation when the ship is kept on course.

12.9.11. The trajectory of ship in a fixed coordinate system as the dependence of its position along the ordinate axis on its position along the abscissa one. Finding the trajectory of ship motion by excluding the variable time parameter from the functions of longitudinal and transverse motion projections.

12.10. Finding the elements of ship motion when replacing the control function with its expansion into a trigonometric series

For the numerical solution of equations of ship motion under the controls, for example, when keeping ship on course, and for finding the elements of the ship trajectory, instead of periodic control function (not sinusoidal, and possibly non–analytical), or the function which with conditionally acceptable methodological accuracy could be considered as the periodic, its expansion into trigonometric Fourier series can be used, that in practical problems (with a limited number of series terms used for calculation) has the form: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \times \cos(n \times x) + b_n \times \sin(n \times x))$, where $f(x)$ – a function that can be expanded into a trigonometric series and is the sum of the terms of the series on the interval: $[-\pi, \pi]$, as well as periodic continuation of this function along the entire x–axis Gx; N – the number of series' terms used for calculation; a_0, a_n and b_n – the expansion coefficients in the trigonometric Fourier series, defined as:

$a_0 = \frac{1}{\pi} \times \int_{-\pi}^{\pi} f(x) \times dx$, $a_n = \frac{1}{\pi} \times \int_{-\pi}^{\pi} f(x) \times \cos(n \times x) \times dx$

and $b_n = \frac{1}{\pi} \times \int_{-\pi}^{\pi} f(x) \times \sin(n \times x) \times dx$.

If one compares the solutions of ship motion equations, found by numerical integration (by the Euler method, the Runge–Kutta one or by other numerical methods) and by expanding ship motion control function (in particular, the pattern of rudder shifting) into a trigonometric series, the terms of which can be integrated and differentiated, then the methodological difference of solutions is due to the influence of initial conditions when using numerical integration of equations, and the premise or assumption when using Fourier series expansion, that outside the series expansion interval the simulated function periodically repeats – this usually good matches the stationary processes features. The accuracy of solutions during numerical integration is affected by the step, and when using the expansion of the control function (the control efforts time pattern) into a trigonometric series, the accuracy is affected by the number of the series terms used.

With the performances speed limitations and the inertia of ship control system and within the sensitivity threshold of the feedback control, which is characterized by a linear transfer function, the ship, held on course by the control system, moves along a trajectory, the elements of which have a stationary periodic in nature (that is independent from the choice of the reference point), and keeping the ship on course can be done by the rudder periodic turns, in particular, the considered sinusoidal (item 12.9). The sinusoidal function of the ship keeping on course is methodologically convenient in the research because it is smooth, continuous, differentiable and integrable.

If the form of the periodic control function when keeping ship on course is not sinusoidal, then to find the elements of its motion the function can be expanded into a

trigonometric series, each term (the member of series), which can be integrated and differentiated, and the type of numerical solution is the found elements of ship motion, kept on course by turning the rudder in the form of this function (or by the periodic impacts of another control), but not in the form of a sinusoidal control function and, for example, in the form of periodic impulses.

If the functional dependence of the rudder turns on time is not specified analytically for simulation purposes, but, for example, by table, graph or the analog algorithm, then to determine the ship trajectory in water and the elements of it controlled motion, one can use the expansion of this functional dependence into a trigonometric series, which sum with acceptable accuracy can describe the rudder turn function. The accuracy depends on the number of terms used in the series expansion of the function.

One considers below the functional dependence of rudder shifting angle $\alpha_r(t)$, which is of pulse nature (Fig. 12.10.1) with a period T_r. For technical reasons, the angle of the rudder turns (or the impact of another control device) and other elements of the controlled motion of ship in water cannot be changed instantly, therefore the graph, in principle, should not have breaks, and the approximate substitute of impulse rudder turn pattern with a trigonometric series perhaps describes the rudder shifting process more accurate than its impulse pattern, which has the mathematical discontinuities, shown on the graph.

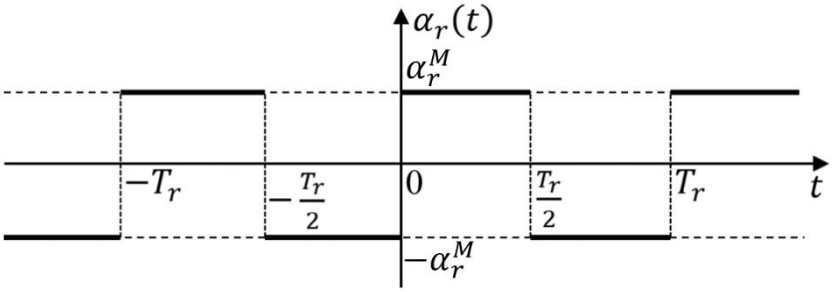


Fig. 12.10.1

Since for considered pulse pattern of the rudder shifting angle $\alpha_r(t)$ the function is odd (the function changes sign to the opposite when the sign of the time parameter $t$changes) and since the function is skew–symmetric, that is: $\alpha_r(t) = -\alpha_r\left(t \pm \frac{T_r}{2}\right)$, then the function of the rudder shifting $\alpha_r(t)$ expansion into a trigonometric series has the form: $\alpha_r(t) = \frac{4\times\alpha_r^M}{\pi} \times \sum_{n=1}^{N} \frac{1-(-1)^n}{2\times n} \times \sin\left(\frac{2\times\pi\times n}{T_r} \times t\right)$, at: $n = 1, 2, 3, \ldots, N$, and if for the calculation one takes five odd terms of the series by the factor: $\frac{1-(-1)^n}{2\times n}$, then one can approximately determine the impulse pattern of the rudder angle in the form of its decomposition into five odd terms: $n = 1, 3, 5, 7, 9$ of trigonometric series (Fig. 12.10.2), as:

$\alpha_r(t) \approx \frac{4\times\alpha_r^M}{\pi} \times (\sin(\omega_r \times t) + \frac{1}{3} \times \sin(3 \times \omega_r \times t) +$
$+\frac{1}{5} \times \sin(5 \times \omega_r \times t) + \frac{1}{7} \times \sin(7 \times \omega_r \times t) + \frac{1}{9} \times \sin(9 \times \omega_r \times t))$, where, the rudder pulse frequency is equal to: $\omega_r = \frac{2\times\pi}{T_r}$, depending on the period T_r.

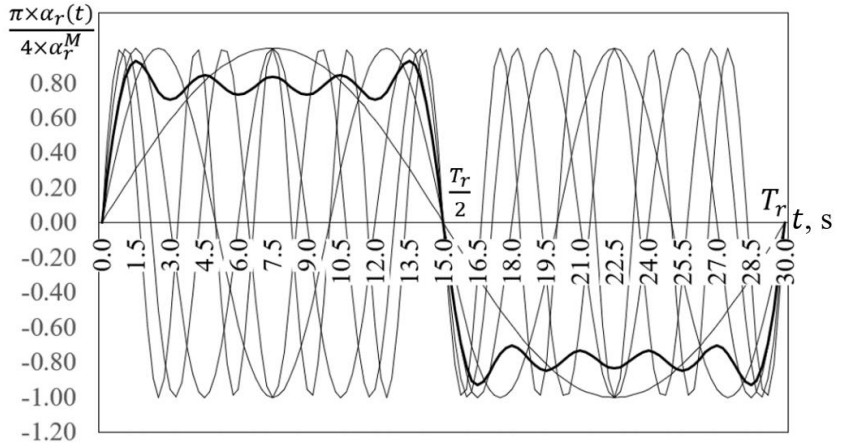


Fig. 12.10.2

The expansion term for lateral force function on rudder blade in a trigonometric series: $N_y\big(\alpha_r(t)\big) = 2.25 \times \rho \times v_x^2 \times S_r \times \frac{\alpha_r^M}{\pi} \times \sum_{n=1}^{N} \frac{1-(-1)^n}{n} \times \sin(\omega_r \times n \times t)$ one can define as: $N_{yn} = 2.25 \times \rho \times v_x^2 \times S_r \times \frac{\alpha_r^M}{\pi} \times \frac{1-(-1)^n}{n} \times \sin(\omega_r \times n \times t)$, or $N_{yn} = C_n \times \sin(\omega_r \times n \times t)$, where: $C_n = 2.25 \times \rho \times v_x^2 \times S_r \times \frac{\alpha_r^M}{\pi} \times \frac{1-(-1)^n}{n}$ – the multiplier of the n–th term of the trigonometric expansion of function of the lateral force on rudder independent on period of the rudder turn impulse T_r and on frequency of rudder shifting ω_r.

Expansion term of trigonometric series for function of the moment under the transverse force on the rudder, turning the ship relative to the point of application of reactive forces in the center of gravity, approximately correlated with the origin of coordinates under the assumption of symmetry of the ship with respect to coordinate planes, with the rudder of the ship located at the aft end and with moment arm: $-\frac{L}{2}$, is equal to:

$M_{zn} = -\frac{L}{2} \times C_n \times \sin(\omega_r \times n \times t)$,

that is the derivative equals: $\dot{M}_{zn} = -\frac{L}{2} \times \omega_r \times n \times C_n \times \cos(\omega_r \times n \times t)$.

The terms of expansion into trigonometric series of the functions of ship motion elements: the angular velocity ω_{zn} of ship rotation relative to associated axis Gz, the transverse translational acceleration $\dot{v}_{yn}$, the transverse translational velocity v_{yn} and the transverse displacement y, transferred to the fixed coordinate system $0\xi\eta\zeta$ one can find by integrating the differential equations, which elements are the terms of expansion into

trigonometric series of functions: the lateral force on the rudder blade and the moment under the lateral force. Each term of the expansion of a function into a series corresponding to one or another element of ship motion is a particular solution of an inhomogeneous equation with the right–hand side specified in the pattern of the term of expansion of this function into a trigonometric series.

For stationary processes, the elements of which do not depend on the origin of time reference, such as ship keeping on course by periodic, including pulsed turns of the rudder, the solution to differential equation for keeping a ship is the set (the linear combination) of all its particular solutions, found in the form of all terms of expansion into trigonometric series of functions of ship control efforts, for example, the terms of expansion of impulse periodic functions.

Considering the transformed second equation of the equations system (item 12.5): $\dot{\omega}_z + \omega_z \times \frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(I_z+\mu_6)\times(\Delta+\mu_2)} \times v_x^2 = \frac{\dot{M}_z}{I_z+\mu_6} + \frac{(\mu_2-\mu_1)\times v_x \times N_y}{(I_z+\mu_6)\times(\Delta+\mu_2)}$ and the expansions into trigonometric series of functional dependence on time of the lateral force on the rudder blade $N_y\big(\alpha_r(t)\big)$ and the dependence of the rudder angle $\alpha_r(t)$, which is of pulsed nature, n–th term of ship angular velocity function of the angular velocity expansion ω_z into a series in the form of the term ω_{zn} and the input coefficients (p_n, q_n and v_{vx}) one can find in the form: $\omega_{zn} = p_n \times (q_n \times \cos(\omega_r \times n \times t) + v_{vx} \times \sin(\omega_r \times n \times t))$,

where: $p_n = \frac{C_n}{\frac{(\mu_2-\mu_1)\times(\Delta+\mu_1)}{(\Delta+\mu_2)}\times v_x^2 - (I_z+\mu_6)\times(\omega_r \times n)^2}$, $q_n = -\omega_r \times n \times \frac{L}{2}$ and $v_{vx} = \frac{(\mu_2-\mu_1)}{(\Delta+\mu_2)} \times v_x$.

Exercise. The exercise assigns one to determine the functions of ship motion elements (ω_z, v_y and y) during periodic pulse shifting of the rudder: $\alpha_r(t) = 0.1745$, radian (Fig. 12.10.2), which corresponds to the amplitude of rudder turns angle: $\alpha_r^M = 10$ degrees and the full period of rudder shifting: T_r =30 s of ship, which characteristics of are specified above (item 12.9): the translational velocity in direction of abscissa axis Gx: $v_x = 6.53$ m/s; the waterline length is: $L = 46.2$ m; the displacement of the ship at load line draught equals: Δ= 1 228 ton; the rudder blade area is: $S_r = 4.0$ m^2; the added masses and moments of inertia, found using the empirical formulas: $\mu_1 = 307$ ton, $\mu_2 \approx \mu_3 = 850$ ton, $I_y \approx I_z = 122\,667$ ton×m^2 and $\mu_6 \approx \mu_5 = 45\,610$ ton×m^2.

The frequency of rudder shifting impulses when keeping ship on course is:

$\omega_r = \frac{2\times\pi}{T_r} = \frac{2\times\pi}{30}$ =0.2094, 1/s.

Calculation is carried out in Microsoft EXCEL program.

The constants are equal: $v_{vx} = \frac{(\mu_2-\mu_1)}{(\Delta+\mu_2)} \times v_x = \frac{(850-307)}{(1\,228+850)} \times 6.53 = 1.706$, m/s;

$C_n = 21.85$; q_n: –4.84, –14.51, –24.19, –33.87 and –43.54 when using five odd terms of the trigonometric series expansion: $n = 1, 3, 5, 7, 9$; p_n: 0.00225, –0.00044, –0.00013, –0.00006 and –0.00004.

The function of the angular rate of turn ω_z of ship under the influence of periodic impulse rudder shifting, found in the form of a trigonometric series as the set (the linear combination) of particular solutions ω_{zn} got by integrating of the terms of expansion into the trigonometric series of impulse periodic odd rudder turn function $\alpha_r(t)$, for five odd terms of the series has the form (Fig. 12.10.3):

$\omega_z \approx \omega_{z1} + \frac{1}{3} \times \omega_{z3} + \frac{1}{5} \times \omega_{z5} + \frac{1}{7} \times \omega_{z7} + \frac{1}{9} \times \omega_{z9}$.

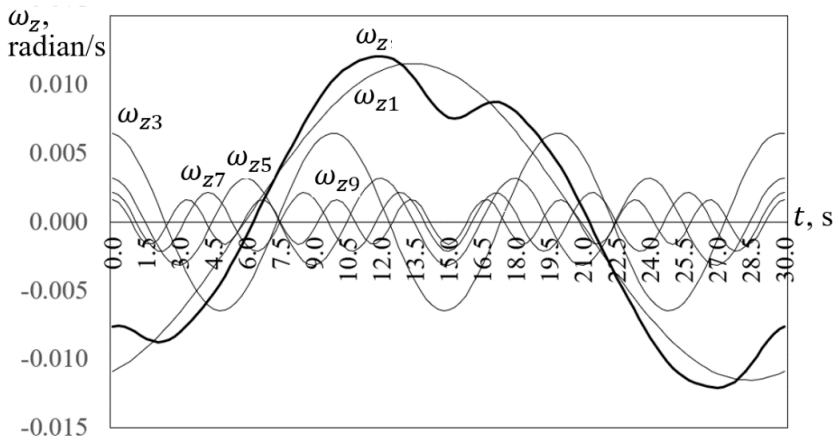


Fig. 12.10.3

The derivative $\dot{v}_y$ of ship transverse velocity (directed along the ordinate axis Gy) one can approximately find in the form of a linear combination of the odd terms of its series expansion, which, if one takes five terms of the series ($n = 1, 3, 5, 7, 9$), has the form: $\dot{v}_y \approx \dot{v}_{y1} + \frac{1}{3} \times \dot{v}_{y3} + \frac{1}{5} \times \dot{v}_{y5} + \frac{1}{7} \times \dot{v}_{y7} + \frac{1}{9} \times \dot{v}_{y9}$.

Each odd term of ship transverse velocity expansion derivative $\dot{v}_{yn}$, considering the transformed first equation of the system (item 12.5), when set into it the found function of the angular velocity of ship rotation ω_z relative to the applicate axis Gz:

$\dot{v}_y = \frac{1}{\Delta + \mu_2} \times (N_y - (\Delta + \mu_1) \times v_x \times \omega_z)$, the corresponding term in the expansion of the function of ship turn angular velocity ω_{zn} into the series has the form:

$\dot{v}_{yn} = \frac{1}{(\Delta + \mu_2)} \times (C_n \times \sin(\omega_r \times n \times t) -$

$-(\Delta + \mu_1) \times v_x \times p_n \times (q_n \times \cos(\omega_r \times n \times t) + \nu_{vx} \times \sin(\omega_r \times n \times t)))$.

To determine the transverse velocity function v_y, which is antiderivative with respect to its derivative $\dot{v}_y$, found in as linear combination of a certain number of terms of the trigonometric series, and to find the transverse displacement y of ship motion, which is antiderivative with respect to the transverse velocity function v_y, it is necessary to consider the characteristic sinusoidal functions, their phase correspondences with derivatives and antiderivatives with argument interval $(\pm\pi/2)$:

$v_{yn}(\omega_r \times n \times t) = \frac{1}{\omega_r \times n} \times \dot{v}_{yn}(\omega_r \times n \times t - \pi/2)$,

$v_y \approx v_{y1} + \frac{1}{3} \times v_{y3} + \frac{1}{5} \times v_{y5} + \frac{1}{7} \times v_{y7} + \frac{1}{9} \times v_{y9}$, (Fig. 12.10.4).

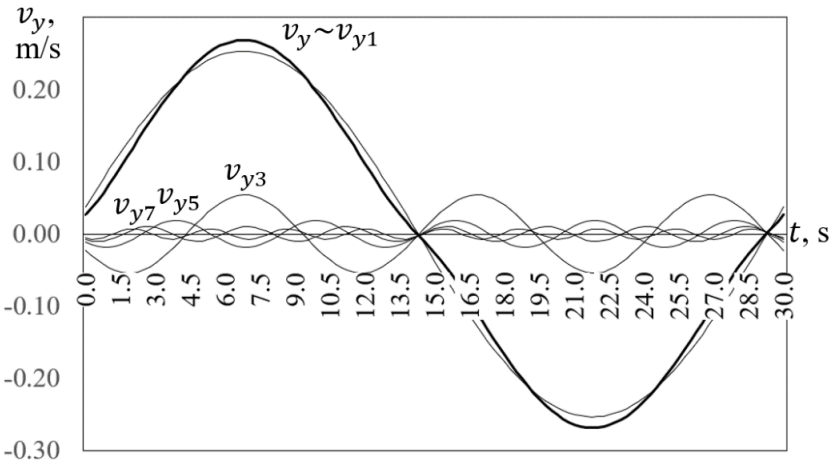


Fig. 12.10.4

In turn, the transverse displacement y of ship motion and the individual terms of the expansion into trigonometric series of the transverse displacement y_n are on a scale: $\frac{1}{\omega_r \times n}$ and in phase correspondence: $-\pi/2$ with respect to the transverse velocity v_y of ship kept on course and the individual terms of the transverse velocity v_{yn} expansion in a series are on a scale and in phase correspondence too. These also means the ratio of the transverse displacement y of ship motion (and each term y_n of motion displacement in the series) on a scale: $\frac{1}{(\omega_r \times n)^2}$ and the phase shift: $-\pi$ to the derivative of the transverse velocity $\dot{v}_y$ (and to each term of the series $\dot{v}_{yn}$):

$y_n(\omega_r \times n \times t) = \frac{1}{\omega_r \times n} \times v_{yn}(\omega_r \times n \times t - \pi/2) = \frac{1}{(\omega_r \times n)^2} \times \dot{v}_{yn}(\omega_r \times n \times t - \pi)$,

that is: $v_{yn} = \frac{1}{(\Delta + \mu_2)} \times \frac{1}{\omega_r \times n} \times (C_n \times \sin(\omega_r \times n \times t - \pi/2) - (\Delta + \mu_1) \times v_x \times p_n \times$

$\times (q_n \times \cos(\omega_r \times n \times t - \pi/2) + \nu_{vx} \times \sin(\omega_r \times n \times t - \pi/2)))$

and $y_n = \frac{1}{(\Delta + \mu_2)} \times \frac{1}{(\omega_r \times n)^2} \times (C_n \times \sin(\omega_r \times n \times t - \pi) - (\Delta + \mu_1) \times v_x \times p_n \times$

$\times (q_n \times \cos(\omega_r \times n \times t - \pi) + \nu_{vx} \times \sin(\omega_r \times n \times t - \pi)))$,

or on a scale: $-\frac{1}{(\omega_r \times n)^2}$ and without phase shift:

$y_n(\omega_r \times n \times t) = -\frac{1}{(\Delta + \mu_2)} \times \frac{1}{(\omega_r \times n)^2} \times \dot{v}_{yn}(\omega_r \times n \times t)$,

$y \approx y_1 + \frac{1}{3} \times y_3 + \frac{1}{5} \times y_5 + \frac{1}{7} \times y_7 + \frac{1}{9} \times y_9$, (Fig. 12.10.5).

From the graphs of the transverse velocity of ship v_y (Fig. 12.10.4) and the ship motion transverse displacement y (Fig. 12.10.5) it is clear that when expanding the trigonometric series of a pulsed periodic shifting of the rudder (Fig. 12.10.1 and 12.10.2, or with a pulse periodic impacting on ship of another control devices) and when finding the

elements of ship motion (v_y and y) in the form of linear combinations of particular solutions for the stationary process of ship keeping on course, the found patterns of the elements of ship motion are, in principle, similar to considered forms of the elements of motion (Fig. 12.9.3 a and b), characteristic for sinusoidal control pattern (the control efforts dependence on time, Fig. 12.9.2 a), with small differences of motion elements values.

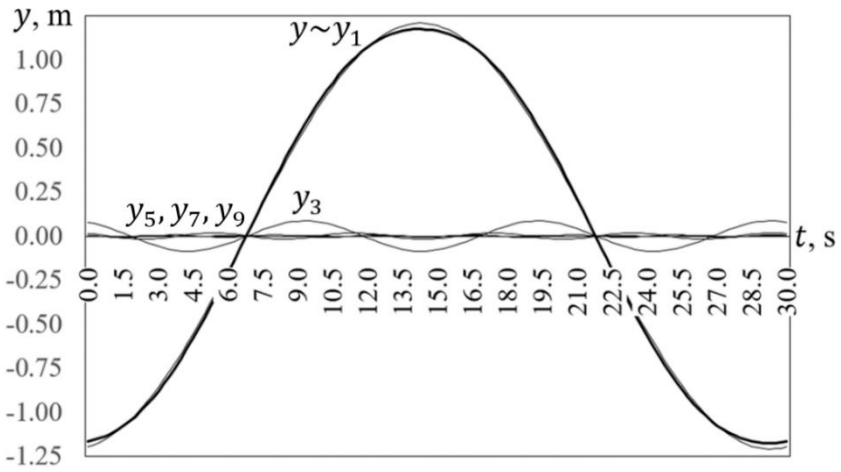


Fig. 12.10.5

The formal similarity of results (the patterns of changes in motion elements) for the differences of data set (the different sinusoidal and pulsed periodic forms of the dependences of the rudder angle or other control action on time) is due to the fact that for trigonometric functions sine and cosine influence is reduced for higher frequencies: $\omega_r \times n$ in the spectrum of impulse function expansion of the rudder shifting angle into a trigonometric series of ship keeping on course motion elements (v_y and y), when keeping on course, correspond to the scale factors arising during integration: $\frac{1}{\omega_r \times n}$ and $\frac{1}{(\omega_r \times n)^2}$, depending on the number of the trigonometric series' terms, as well as to the coefficients of expansion of functions in trigonometric series: $\frac{1}{n}$, inversely proportional to the number of term.

When the deviation in an observed ship motion element periodically exceeds the sensitivity threshold, the digital feedback control system is more likely to keep the ship on course in the form of pulsed rudder turns at a certain shifting angle than by the sinusoidal analytical pattern of rudder turns, although the trajectory of the ship held on course is a less degree depends on the difference the specified patterns of the rudder turn (or of the action of other controls).

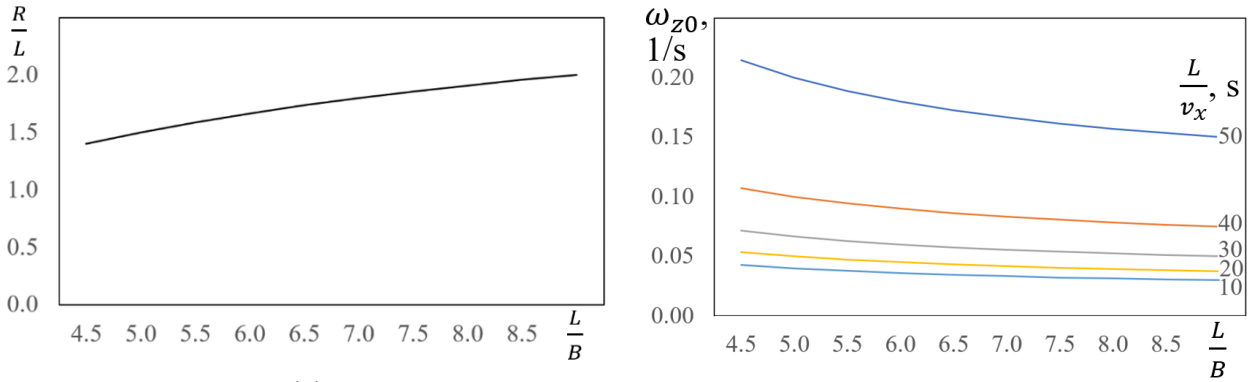
In the event of loss of control of ship motion, which is kept on course by turns of the rudder and by feedback control system, its departure from the course (which is the automatic start of the maneuver after the cessation of keeping on course) occurs mainly under the influence of external forces applied to the hull of ship, and in to a lesser extent, under the intentional effort on the part of the controls, although the controls, in addition

to ceasing the ship hold on course, functionally serve as a trigger for the direction of maneuver, therefore, the characteristic elements of the ship's maneuver, such as the radius of the steady circulation or the tactical radius (considering the dimensions of ship on the circulation outside the circle) and other elements of the ship's maneuver depend on the characteristics of physical ship to a greater extent than on the characteristics of its controls, and also the elements depend on the energy of the moving ship (that is, on it velocity).

A sea–going ship is usually unstable on course without the course holding (12.5), and when solving the equations of ship motion, the roots of the characteristic equation usually have real number values. That is, a ship without control usually tends to enter into a circulation maneuver, and its motion along the original direction can occur in the form of self–oscillations by inertia in the direction of the side after the turn, if the longitudinal velocity of ship (along the abscissa axis Gx, associated with ship) will attenuate.

At the same time, if the ship velocity is maintained along the abscissa axis Gx, associated with ship, for example, if the propulsors are working, then the ship, which is unstable without control on a straight path, will move steadily in a circle (the steady circulation mode), the radius of which is: $R \approx v_x/\omega_{z0}$ at a certain velocity v_x and depends on ship characteristics. The theoretical determination of radius of the steady circulation is complicated by the nonlinearity of the conditions. There are empirical formulas that were developed by different authors [13], in particular, A.M. Basin, G.A. Firsov, L.Ya. Pershits, G.E. Pavlenko, K.K. Fedyaevsky, etc. for an approximate determination of the steady circulation radius of a displacement ship moving at its characteristic velocity, the following empirical formula is known: $R = 3 \times L \times \frac{L-B}{L+3\times B}$, m, or this formula for determining the radius of steady circulation by relative value has the form: $\frac{R}{L} = 3 \times \frac{L/B-1}{L/B+3}$.

The value of relative radius of steady circulation $\frac{R}{L}$ for the range of values of relative length of ship $\frac{L}{B}$ in it range from 4.5 to 9.0 is shown in the graph (Fig. 12.10.6 a).



(a) (b)

Fig. 12.10.6

The highest angular velocity, which could be considered the limiting magnitude of ship rotation velocity relative to the applicate axis, which corresponds to the real number roots of considered characteristic equation ($p^2 > 0$ at: $\mu_2 > \mu_1$, item 12.5), can be determined by transforming the empirical formula depending on relative velocity of ship (the duration of ship passage of its length $\frac{L}{v_x}$, s) and is (Fig. 12.10.6 b): $\omega_{z0} = \frac{1}{\left(\frac{L}{v_x}\right)} \times \frac{L/B+3}{L/B-1}$, 1/s.

Questions and comments:

12.10.1. The use of decomposition of the ship's control function (the rudder shifting or using another control device) to determine the elements of ship motion under the control, for example, the elements of the movement of ship when keeping it on course, taking into account the premise or assumption of time repeatability of control function (the control efforts of the rudder shifting angles patterns) outside the limits interval of control function expansion into a trigonometric series.

12.10.2. Mathematical form of the trigonometric Fourier series and determination of the coefficients of expansion into a trigonometric series. The odd functions. The skew–symmetric functions. Expansion of an odd and skew–symmetric periodic function into a trigonometric series.

12.10.3. The use of expansion of function into a trigonometric series, which terms can be integrated and differentiated, for the numerical integration of equations which include the function expanded into this series.

12.10.4. Methodological difference between solutions of differential equations found by methods of numerical integration and dependent on initial conditions and the solutions got by expanding parts of the equation (namely, the right side) into a trigonometric series within the framework of the premise or assumption that beyond the interval of series expansion the simulated function is periodically repeated, which usually matches good the functions of the stationary processes.

12.10.5. Elements of the trajectory that are stationary in nature (independent of the reference time), of a ship held on course by periodic shifting of the rudder by a feedback control system with a linear transfer function and with limited performances speed.

12.10.6. Finding the elements of motion of a ship using the trigonometric series expansion of the function of periodic pulse turns of the rudder, which linear transfer function, the sensitivity threshold and the performances speed limit, simulating the feedback control system, can be specified analytically, in a table, graph, or by analog algorithm, etc.

12.10.7. The more accurate perhaps than the theoretical impulse model of the rudder periodic shifting, which has mathematical discontinuities, is its approximate description by trigonometric series, which terms are smooth, continuous and differentiable functions.

12.10.8. Terms of expansion into trigonometric series of functions: the transverse force on rudder and the moment from the transverse force, turning the ship relative to the point of application of reactive forces to the center of gravity, approximately correlated with the origin of coordinates under assumption of ship symmetry with respect to coordinates planes.

12.10.9. Determining the terms of the expansion into trigonometric series of the functions: the angular velocity of ship turn, the transverse translational acceleration, the transverse translational velocity and ship motion transverse displacement by integrating the differential equations, the elements of which are the terms of expansion into trigonometric series of the functions: the lateral force on ship rudder blade and the moment under lateral force on the blade.

12.10.10. The form of terms of the expansion into trigonometric series of functions of ship control efforts, which particular solutions of the differential equation for keeping a ship on course are sought, constituting of solutions set to the equation (their linear combination).

12.10.11. The form of term of the expansion into a trigonometric series of the function of the angular rate of turn of ship held on course by the periodic impulse shifting of the rudder, and the form of the coefficients included in the term of the expansion.

12.10.12. Finding terms of the expansion into a trigonometric series of functions: the translational transverse acceleration of a ship held on course, the translational transverse velocity and transverse displacement by integrating differential equations considering the sinusoidal functions with derivatives and antiderivatives phase correspondences with the argument interval $(\pm\pi/2)$.

12.10.13. Integral elements of motion of a ship, kept on course by impulse shifting of the rudder: the lateral velocity and the ship motion lateral displacement, found in the form of linear combinations of their particular solutions, similar mainly to the elements of ship motion characteristic under sinusoidal pattern of control efforts.

12.10.14. The overwhelming influence of the first odd term of the impulse expansion into a trigonometric series and the reduction of higher frequencies influence in the spectrum of the impulse function of the control effort expansion due to the decreasing scale factors when integrating trigonometric functions, depending on the number of series term, and the decreasing coefficients of functions expansion in these series inversely proportional to the number of series term.

12.10.15. The usual instability of ship without active keeping on course, which is visible when solving two equations of flat ship motion in water (in the horizontal plane), if the characteristic equation has real number roots, the tendency of a ship to maneuver a turn and move towards the side in the mode of self–oscillation, or to enter in stable

circulation if the longitudinal velocity of ship (along the x–axis associated with the ship) is maintained by propulsors operation.

12.10.16. Determination by the empirical formula of ship steady circulation radius (the circular motion) and in the mode of steady circulation, finding the maximum angular velocity of the ship rotation, which can serve as the limit of the theoretical values of the ship angular velocity, got by solving the equations of ship motion, with real number roots of characteristic equation.

C o n c l u s i o n s

The book contains methodological material on mechanics and elasticity theory and is intended for use in training in the engineering departments of universities of specialists for industries, in particular, for the shipbuilding industry, marine engineering, for the maintenance of mechanisms and equipment in maritime transport and the ship repair, in the industries related to mastering of the marine resources, including for the multilevel professional training, for the retraining and advanced training of specialists who graduated from universities, and contains the methodological material focused on the use of mathematical methods and the approaches for practical implementation of calculations in mechanics to train of the graduate students. The book will also be useful to practicing specialists of enterprises engaged in the ships construction and the marine engineering production and maintenance of these facilities. To learn the mechanics and theory of elasticity in this book the required basic level of the students' knowledge corresponds to the mathematics and physics programs of the junior year in universities engineering departments.

Unlike well–known methodological books on mechanics and the theory of materials elasticity, this textbook is focused on methods of the calculations performing and the mathematical modeling, including those based on operations with matrices and vectors, on solution of differential equations, determining the integrals, decisions of systems of the equations and on the transformation of digital information. In particular, the book shows the use of the finite element method in calculating of deformations and stresses in structures and in their discrete models, using the EXCEL program and the Mathcad computer algebra system. In addition to considering of mechanics and mathematical methods in the form of the relevant problems solving, the book includes applicable theoretical sections of mechanics and elasticity theory. To learn the book no other methodological material is required necessary other than what is contained in it, although the mastering of considered approaches use for the calculations may precede further extensive study of certain areas of mechanics.

The book contains the following methodological chapters. F i r s t chapter discusses the following concepts: the forces, moments of forces, systems of forces and the pairs of forces, the coordinate systems, conditions of equilibrium of forces and of the moments, connections in the mechanical systems used in the structures, the vector transformations of forces and moments, the replacement of rigid embedding with a pair of forces, the degrees of mechanical systems freedom and their static indefinability, the determination of support reactions, the balance of instantly variable mechanical systems and the degrees of freedom, etc.

S e c o n d chapter explains the methods for static calculations of frames and trusses, the reactions finding of their supports and the diagrams of acting forces determining, the

use of the three–moment equation to determine forces in a continuous multi–span beam, calculating the beams (or other mechanical systems) under influence of a moving load and determining the influence lines of the load application point of acting force, the equilibrium conditions for system of converging forces, determination of forces acting in the elements of statically definable mechanical systems using the methods of sections and of the cutting out nodes, the methods for determining forces in the rods of flat trusses and frames, the use of Ritter's method to determine the acting forces, determination of the applied forces resultant and point of application of the resultant, including the definition of the distributed load resultant, the Varignon's theorem on moment of the resultant force, the equilibrium of spatial system of forces and bringing the system of forces to its simplest form, the balance of forces in elements of a statically indefinable frame, the condition of deformations compatibility of rod elements.

Determination of reactions and acting forces in statically indefinable mechanical systems is discussed in t h i r d chapter, which provides the method for static indefinability degree determining, as well as methods of reactions of supports finding, the forces and mechanical displacements by loading and deforming of statically indefinable mechanical systems. For this purpose, the formation and use of systems of canonical equations of the methods of forces and the displacements, the choice of main systems for these methods – statically definable systems in which the static indefinability of specified initial system is replaced by unknown forces or displacements, the determination of coefficients and independent terms in equations of the method of forces and displacements, use of the Mohr's integral method to find coefficients and independent terms in equations, as well as the methods use of Vereshchagin, Simpson and the trapezoids to determine the Mohr's integral, the finding of coefficients and free terms of canonical equations of the displacements method with the use of standard solutions to find reactions and the bending moments corresponded the unit displacements in statically indefinable beams, practical use of the canonical equations for the forces determining in statically indefinable systems, when the loads of mechanical system are caused by temperature deformations, and the displacements depended on the subsidence of supports.

F o u r t h chapter contains the methodic material explaining the use of computer algebra system Mathcad and the Microsoft EXCEL program for mathematical operations on matrices and vectors, for determining the displacements of nodes of mechanical system with use of Mohr's integral, for finding the reactions of supports, the forces and displacements using the finite elements method. For these method use the mathematical operations with matrices and vectors are used, which can be conveniently processed in the Mathcad and Microsoft EXCEL for calculations of mechanical systems. Application of these programs to find forces and displacements in statically indefinable systems are determined using the Mohr's integral method, the initial data of the geometric characteristics of the

mechanical systems and the load are set, which when using the finite element method is converted to a nodal load, the stiffness matrices of finite elements and the discrete model of mechanical system are drawn up, the mathematical operations with matrices to redefine the forces in the local and global coordinate systems, to calculate forces in the nodes of flat truss using the method of cutting nodes, etc.

Methods of the stability evaluation under compression of elastic mechanical systems and their elements and the influence of structural material plasticity on the mechanical stability are performed in f i f t h chapter. The Euler's formula is used for finding the critical force at which a compressible elastic element of mechanical system loses stability, the influence of material plasticity on the element stability and the limits of applicability of the Euler's formula for the critical forces finding are considered, as well as the influence of reducing the permissible stress within the material elasticity on value of critical force upon loss stability of compressed mechanical element. Fifth chapter also shows the effect on the stiffness and the compression stability of the characteristics of sections of mechanical elements, when these sections are specified geometrically or in the form of typical standard profile.

S i x t h chapter discusses methods of the shear deformations and stresses calculation, as well as determining the diagrams of forces and stresses during the shear and torsion of the mechanical system elements. Methods are given for solving problems of the shear stresses determining and evaluation of strength of fasteners experiencing mechanical shear and evaluation the strength of shaft under the influence of torque.

S e v e n t h chapter contains methods of evaluation of three–dimensional stress state of elastic material of mechanical systems, which is associated with hypotheses of material destruction and the loaded elements of mechanical systems strength criteria, including the methods of the stresses determining on inclined areas of sections of elements of mechanical systems, finding the main areas and main stresses in a three–dimensional stress state of a material, the Hooke's law for three–dimensional stress state, the estimation of specific potential energy during deformation of an elastic material, the methods for assessing of strength based on hypotheses of plasticity and fracture of an elastic material of mechanical system in a stress–strain state, as well as the verification methods of material strength under a complex stress state – with the combined action on the mechanical system of bending and shear. When calculating the overall longitudinal strength of ship for selected critical load state, the critical cross–sections are sought, their geometric characteristics are calculated, the diagrams of the effective normal and tangential stresses are determined and the critical points in these sections experiencing complex stress state are found, the equivalent stresses are determined at the critical points, the hypothesis of the strength of material under a complex stress state is chosen and the strength of mechanical system is checked according to the permissible stresses.

Methods of kinematic calculations of the elements of motion of material point (the concentrated load), the rigid body of mechanism, considered in e i g h t h chapter, include: determination of the coordinate system and the method of specifying the trajectory of motion of a material point, including finding the elements of the trajectory of motion of a point, the methods for determining instantaneous centers of velocity and acceleration during the motion of rigid body, the use of the Mathcad computer algebra system to determine the elements of the trajectory of a material point, the methods of kinematic calculation of a plane mechanism.

N i n t h chapter explains the methods for solving direct and inverse dynamics problems about the motion of a rigid body or mechanical system under the influence of forces, as well as the methods for determining forces by values of the elements of motion of these objects. When solving direct and inverse dynamics problems in relation to rigid body or the mechanism, the indicators of forces, moments, accelerations (the translational and angular), the momentum, moment of momentum, the energy and work are evaluated and used. To find these indicators the resistance forces (sliding and the rolling friction), the inertia forces during accelerated motion or the complex motion (combined translational and rotational, i.e. when the Coriolis' acceleration occurs) are determined. Active forces on a body or on an element of mechanical system are considered under the condition of their equilibrium in accordance with the d'Alembert's principle.

T e n t h chapter considers the issues of indexes use of the amount of translational and rotational motion, the momentum and moment of momentum, the energy of mechanical system and the inertia characteristics of it elements, including the theorems: on the motion of the center of mass, on the Momentum Change Theorem, the Moment of Momentum Change Theorem, the Kinetic Energy Change Theorem, used in the practical issues when solving the equation of a material point (the concentrated load) motion under the influence of external forces, the weight and friction – the direct problem of dynamics, when determining the forces acting on rotating mechanism with an eccentricity of the center of mass – the inverse problem of dynamics. In these and other problems, the inertia characteristics of bodies and elements of mechanical systems can be used, depending on the distribution of their masses. Therefore, for finding the coordinates of the center of mass the methods are also in the chapter, including ones using the Mathcad computer algebra system, and a method for determining moments of inertia with the Huygens–Steiner theorem.

E l e v e n t h chapter discusses methods for determining the characteristics of mechanical vibrations under the influence of external forces, the weight, inertia and elasticity, and also shows the influence of mass and elasticity (the stiffness) of elements of the mechanical systems on characteristics of their vibrations, the influence of parallel and series connection of springs on their vibration characteristics. For a multi–mass mechanical

system the method is shown for determining the natural characteristics of free vibrations, the characteristics of the spectrum of natural frequencies and finding the main modes of the vibration, as well as use of the Mathcad computer algebra system to solve this issue.

T w e l f t h chapter discusses the patterns of motion of ship as mechanical system and the methods for determining the elements of ship motion. For this purpose the classification of forces and moments acting on the ship is considered, taking into account the features of this classification when determining the elements of the ship rocking and elements of it motion without control and with control – when keeping the ship on course. To solve these problems, the theoretical and numerical methods are used. When determining the own and added characteristics of inertia of ship in water the empirical formulas can be used. The twelfth chapter provides information from hydromechanics, used as data in determining the elements of motion of a ship as mechanical system. Methods for determining of rocking elements are shown in calm water, in regular and in the irregular waves, including the determination of the spectral characteristics of ship rolling and heaving, as well as a numerical solution of the nonlinear equation of ship rolling in regular waves and the comparison of results got by solving of nonlinear equation and the linearized one, which can be found by the numerical methods and analytically, for example, by the complex amplitudes method. The straight–line motion of ship without control is usually unstable, and the ship, left to its own motion, tends to enter into the maneuver. To keep the ship on course, the feedback control system can be used, and to determine the pattern of the trajectory the particular solutions of ship motion equations are sought, and the form of these solutions depends on pattern of control function. Due to the limited performance speed and sensitivity threshold of the control system the trajectory of ship held on course is usually not rectilinear, but the limited and periodic, and with pulsed periodic control function the integration of expansions of this function into a trigonometric series of Fourier can be used to find the trajectory elements.

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